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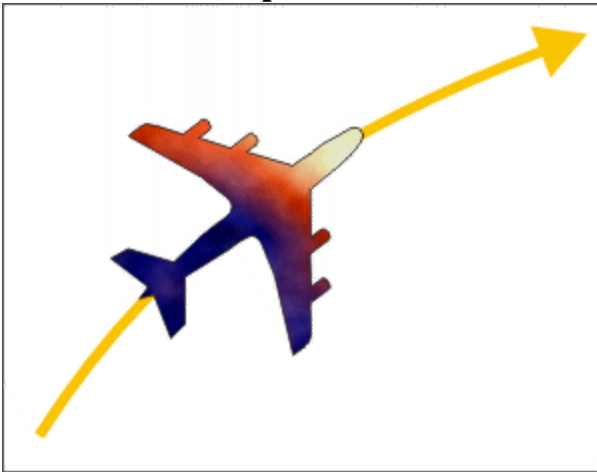
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(application)

Motion

Motion is a universal attribute of all objects. Our current understanding estimates that everything in this universe is constantly moving. Contrary to our common belief based on our existence on earth, rest is simply a perception in a given frame of reference.

Motion is a state, which indicates change of position. Surprisingly, everything in this world is constantly moving and nothing is stationary. The apparent state of rest, as we shall learn, is a notional experience confined to a particular system of reference. A building, for example, is at rest in Earth's reference, but it is a moving body for other moving systems like train, motor, airplane, moon, sun etc.

Motion of an airplane



The position of plane with respect to the earth keeps changing with time.

Motion

Motion of a body refers to the change in its position with respect to time in a given frame of reference.

A frame of reference is a mechanism to describe space from the perspective of an observer. In other words, it is a system of measurement for locating positions of the bodies in space with respect to an observer (reference).

Since, frame of reference is a system of measurement of positions in space as measured by the observer, frame of reference is said to be attached to the observer. For this reason, terms “frame of reference” and “observer” are interchangeably used to describe motion.

In our daily life, we recognize motion of an object with respect to ourselves and other stationary objects. If the object maintains its position with respect to the stationary objects, we say that the object is at rest; else the object is moving with respect to the stationary objects. Here, we conceive all objects moving with earth without changing their positions on earth surface as stationary objects in the earth’s frame of reference. Evidently, all bodies not changing position with respect to a specific observer is stationary in the frame of reference attached with the observer.

We require an observer to identify motion

Motion has no meaning without a reference system.

An object or a body under motion, as a matter of fact, is incapable of identifying its own motion. It would be surprising for some to know that we live on this earth in a so called stationary state without ever being aware that we are moving around sun at a very high speed - at a speed faster than the fastest airplane that the man kind has developed. The earth is moving around sun at a speed of about 30 km/s (≈ 30000 m/s ≈ 100000 km/hr) – a speed about 1000 times greater than the motoring speed and 100 times greater than the aircraft’s speed.

Likewise, when we travel on aircraft, we are hardly aware of the speed of the aircraft. The state of fellow passengers and parts of the aircraft are all moving at the same speed, giving the impression that passengers are simply sitting in a stationary cabin. The turbulence that the passengers experience occasionally is a consequence of external force and is not indicative of the motion of the aircraft.

It is the external objects and entities which indicate that aircraft is actually moving. It is the passing clouds and changing landscape below, which make us think that aircraft is actually moving. The very fact that we land at

geographically distant location at the end of travel in a short time, confirms that aircraft was actually cruising at a very high speed.

The requirement of an observer in both identifying and quantifying motion brings about new dimensions to the understanding of motion. Notably, the motion of a body and its measurement is found to be influenced by the state of motion of the observer itself and hence by the state of motion of the attached frame of reference. As such, a given motion is evaluated differently by different observers (system of references).

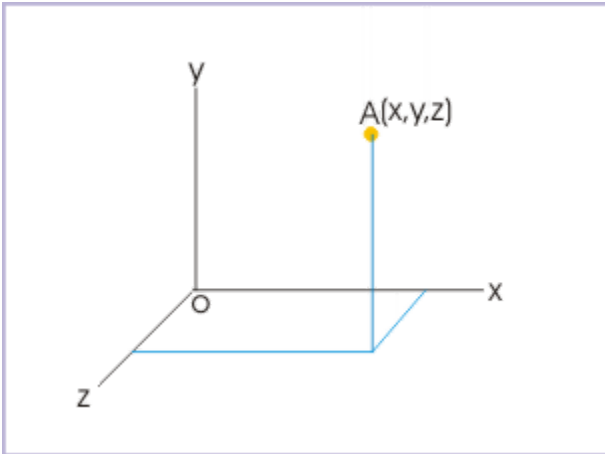
Two observers in the same state of motion, such as two persons standing on the platform, perceive the motion of a passing train in exactly same manner. On the other hand, the passenger in a speeding train finds that the other train crossing it on the parallel track in opposite direction has the combined speed of the two trains ($v_1 + v_2$). The observer on the ground, however, find them running at their individual speeds v_1 and v_2 .

From the discussion above, it is clear that motion of an object is an attribute, which can **not** be stated in absolute term; but it is a kind of attribute that results from the interaction of the motions of the both object and observer (frame of reference).

Frame of reference and observer

Frame of reference is a mathematical construct to specify position or location of a point object in space. Basically, frame of reference is a coordinate system. There are plenty of coordinate systems in use, but the Cartesian coordinate system, comprising of three mutually perpendicular axes, is most common. A point in three dimensional space is defined by three values of coordinates i.e. x,y and z in the Cartesian system as shown in the figure below. We shall learn about few more useful coordinate systems in next module titled "[Coordinate systems in physics](#)".

Frame of reference

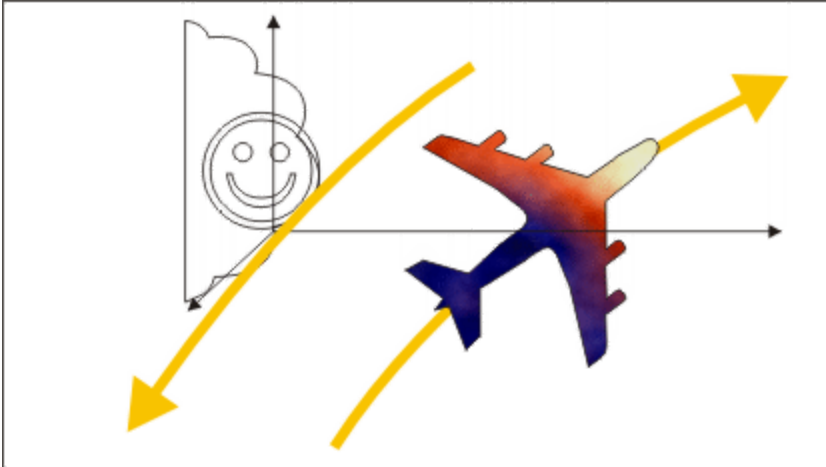


A point in three dimensional space is defined by three values of coordinates

We need to be specific in our understanding of the role of the observer and its relation with frame of reference. Observation of motion is considered an human endeavor. But motion of an object is described in reference of both human and non-human bodies like clouds, rivers, poles, moon etc. From the point of view of the study of motion, we treat reference bodies capable to make observations, which is essentially a human like function. As such, it is helpful to imagine ourselves attached to the reference system, making observations. It is essentially a notional endeavor to consider that the measurements are what an observer in that frame of reference would make, had the observer with the capability to measure was actually present there.

Earth is our natural abode and we identify all non-moving ground observers equivalent and at rest with the earth. For other moving systems, we need to specify position and determine motion by virtually (in imagination) transposing ourselves to the frame of reference we are considering.

Measurement from a moving reference



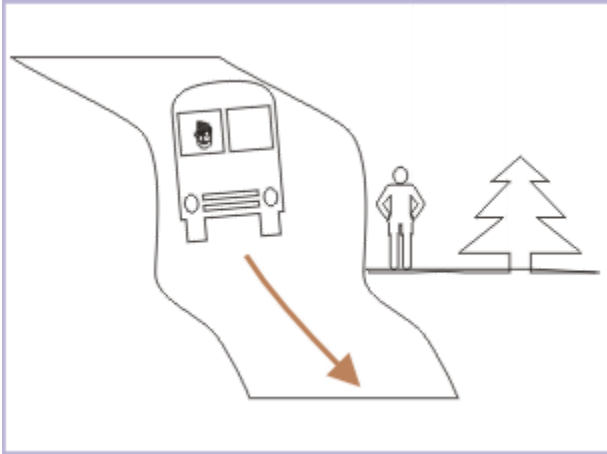
Take the case of observations about the motion of an aircraft made by two observers one at a ground and another attached to the cloud moving at certain speed. For the observer on the ground, the aircraft is moving at a speed of ,say, 1000 km/hr.

Further, let the reference system attached to the cloud itself is moving, say, at the speed of 50 km/hr, in a direction opposite to that of the aircraft as seen by the person on the ground. Now, locating ourselves in the frame of reference of the cloud, we can visualize that the aircraft is changing its position more rapidly than as observed by the observer on the ground i.e. at the combined speed and would be seen flying by the observer on the cloud at the speed of $1000 + 50 = 1050$ km/hr.

We need to change our mind set

The scientific measurement requires that we change our mindset about perceiving motion and its scientific meaning. To our trained mind, it is difficult to accept that a stationary building standing at a place for the last 20 years is actually moving for an observer, who is moving towards it. Going by the definition of motion, the position of the building in the coordinate system of an approaching observer is changing with time. Actually, the building is moving for all moving bodies. What it means that the study of motion requires a new scientific approach about perceiving motion. It also means that the scientific meaning of motion is not limited to its interpretation from the perspective of earth or an observer attached to it.

Motion of a tree



Motion of the tree as seen by the person driving the truck

Consider the motion of a tree as seen from a person driving a truck ([See Figure above](#)). The tree is undeniably stationary for a person standing on the ground. The coordinates of the tree in the frame of reference attached to the truck, however, is changing with time. As the truck moves ahead, the coordinates of the tree is increasing in the opposite direction to that of the truck. The tree, thus, is moving backwards for the truck driver – though we may find it hard to believe as the tree has not changed its position on the ground and is stationary. This deep rooted perception negating scientific hard fact is the outcome of our conventional mindset based on our life long perception of the bodies grounded to the earth.

Is there an absolute frame of reference?

Let us consider following :

In nature, we find that smaller entities are contained within bigger entities, which themselves are moving. For example, a passenger is part of a train, which in turn is part of the earth, which in turn is part of the solar system and so on. This aspect of containership of an object in another moving object is chained from smaller to bigger bodies. We simply do not know

which one of these is the ultimate container and the one, which is not moving.

These aspects of motion as described in the above paragraph leads to the following conclusions about frame of reference :

"There is no such thing like a “mother of all frames of reference” or the ultimate container, which can be considered at rest. As such, no measurement of motion can be considered absolute. All measurements of motion are, therefore, relative."

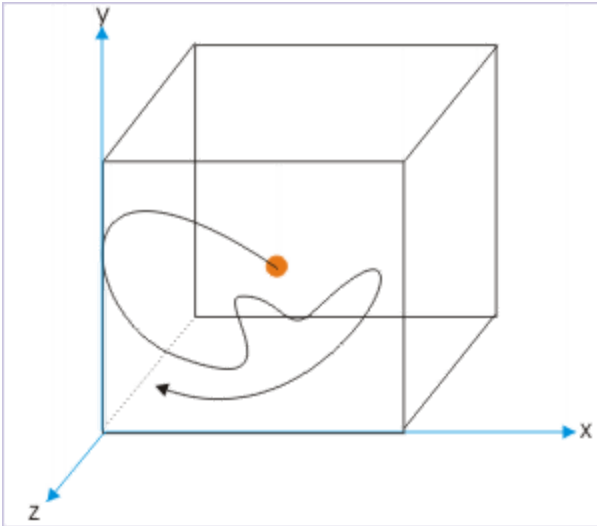
Motion types

Nature displays motions of many types. Bodies move in a truly complex manner. Real time motion is mostly complex as bodies are subjected to various forces. These motions are not simple straight line motions. Consider a bird's flight for example. Its motion is neither in the straight line nor in a plane. The bird flies in a three dimensional space with all sorts of variations involving direction and speed. A boat crossing a river, on the other hand, roughly moves in the plane of water surface.

Motion in one dimension is rare. This is surprising, because the natural tendency of all bodies is to maintain its motion in both magnitude and direction. This is what Newton's first law of motion tells us. Logically all bodies should move along a straight line at a constant speed unless it is acted upon by an external force. The fact of life is that objects are subjected to verities of forces during their motion and hence either they deviate from straight line motion or change speed.

Since, real time bodies are mostly non-linear or varying in speed or varying in both speed and direction, we may conclude that bodies are always acted upon by some force. The most common and omnipresent forces in our daily life are the gravitation and friction (electrical) forces. Since force is not the subject of discussion here, we shall skip any further elaboration on the role of force. But, the point is made : bodies generally move in complex manner as they are subjected to different forces.

Real time motion

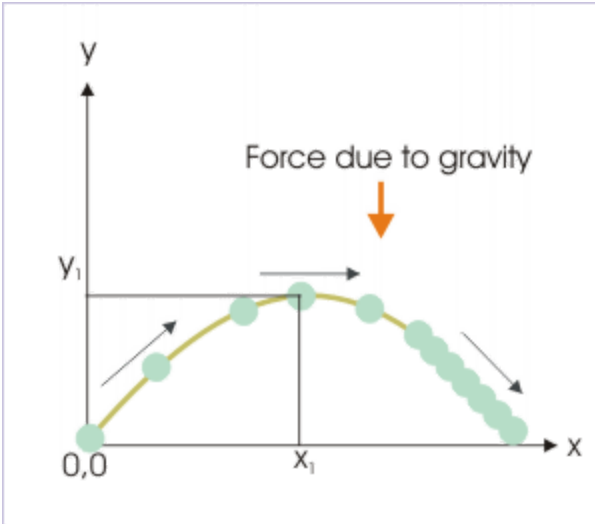


A gas molecule in a container moves randomly under electrostatic interaction with other molecules

Nevertheless, study of motion in one dimension is basic to the understanding of more complex scenarios of motion. The very nature of physical laws relating to motion allows us to study motion by treating motions in different directions separately and then combining the motions in accordance with vector rules to get the overall picture.

A general classification of motion is done in the context of the dimensions of the motion. A motion in space, comprising of three dimensions, is called three dimensional motion. In this case, all three coordinates are changing as the time passes by. While, in two dimensional motion, any two of the three dimensions of the position are changing with time. The parabolic path described by a ball thrown at certain angle to the horizon is an example of the two dimensional motion ([See Figure](#)). A ball thrown at an angle with horizon is described in terms of two coordinates x and y .

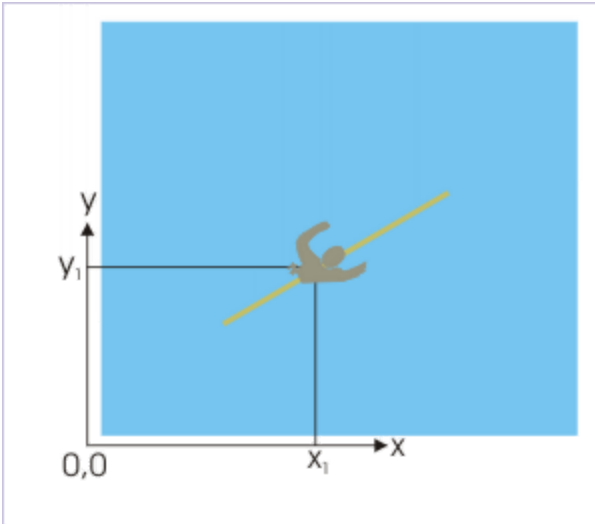
Two dimensional motion



A ball thrown at an angle with horizon is described in terms of two coordinates x and y

One dimensional motion, on the other hand, is described using any one of the three coordinates; remaining two coordinates remain constant throughout the motion. Generally, we believe that one dimensional motion is equivalent to linear motion. This is not further from truth either. A linear motion in a given frame of reference, however, need not always be one dimensional. Consider the motion of a person swimming along a straight line on a calm water surface. Note here that position of the person at any given instant in the coordinate system is actually given by a pair of coordinate (x,y) values ([See Figure below](#)).

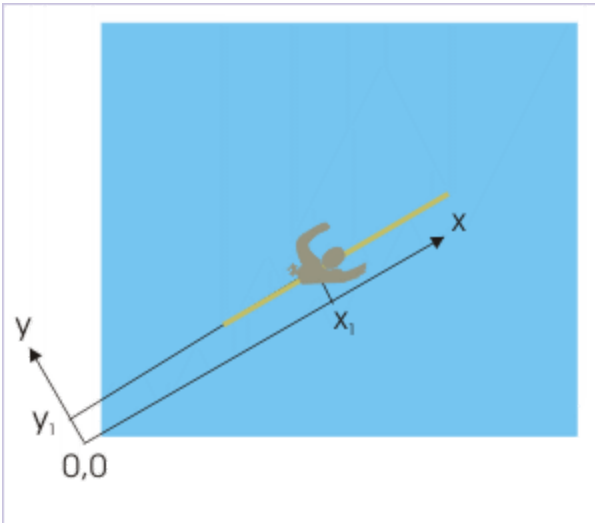
Linear motion



Description of motion requires
two coordinate values

There is a caveat though. We can always rotate the pair of axes such that one of it lies parallel to the path of motion as shown in the figure. One of the coordinates, y_1 is constant through out the motion. Only the x -coordinate is changing and as such motion can be described in terms of x -coordinate alone. It follows then that all linear motion can essentially be treated as one dimensional motion.

Linear and one dimensional motion



Choice of appropriate coordinate system renders linear motion as one dimensional motion.

Kinematics

Kinematics refers to the study of motion of natural bodies. The bodies that we see and deal with in real life are three dimensional objects and essentially not a point object.

A point object would occupy a point (without any dimension) in space. The real bodies, on the other hand, are entities with dimensions, having length, breadth and height. This introduces certain amount of complexity in so far as describing motion. First of all, a real body can not be specified by a single set of coordinates. This is one aspect of the problem. The second equally important aspect is that different parts of the bodies may have path trajectories different to each other.

When a body moves **with** rotation (rolling while moving), the path trajectories of different parts of the bodies are different; on the other hand, when the body moves **without** rotation (slipping/ sliding), the path trajectories of the different parts of the bodies are parallel to each other.

In the second case, the motion of all points within the body is equivalent as far as translational motion of the body is concerned and hence, such bodies may be said to move like a point object. It is, therefore, possible to treat the body under consideration to be equivalent to a point so long rotation is not involved.

For this reason, study of kinematics consists of studies of :

1. Translational kinematics
2. Rotational kinematics

A motion can be pure translational or pure rotational or a combination of the two types of motion.

The translational motion allows us to treat a real time body as a point object. Hence, we freely refer to bodies, objects and particles in one and the same sense that all of them are point entities, whose position can be represented by a single set of coordinates. We should keep this in mind while studying translational motion of a body and treating the same as point.

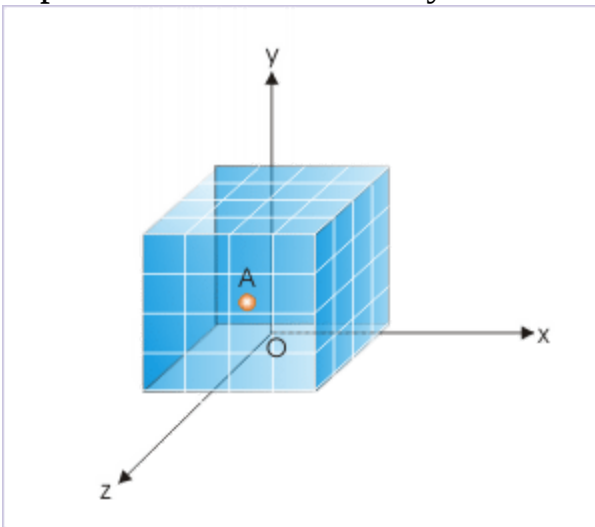
Coordinate systems in physics

Coordinate system is a mathematical construct to measure distance and direction in relation to a system of rigid bodies. The connection to rigid bodies is a crucial consideration for drawing a coordinate system, with out which coordinate system has no meaning in physics.

Coordinate system is a system of measurement of distance and direction with respect to rigid bodies. Structurally, it comprises of coordinates and a reference point, usually the origin of the coordinate system. The coordinates primarily serve the purpose of reference for the direction of motion, while origin serves the purpose of reference for the magnitude of motion.

Measurements of magnitude and direction allow us to locate a position of a point in terms of measurable quantities like linear distances or angles or their combinations. With these measurements, it is possible to locate a point in the spatial extent of the coordinate system. The point may lie anywhere in the spatial (volumetric) extent defined by the rectangular axes as shown in the figure. (Note : The point, in the figure, is shown as small sphere for visual emphasis only)

A point in the coordinate system



A distance in the coordinate system is measured with a standard rigid linear length like that of a “meter” or a “foot”. A distance of 5 meters, for example, is 5 times the length of the standard length of a meter. On the other hand, an angle is defined as a ratio of lengths and is dimensional-less.

Hence, measurement of direction is indirectly equivalent to the measurement of distances only.

The coordinate system represents the system of rigid body like earth, which is embodied by an observer, making measurements. Since measurements are implemented by the observer, they (the measurements in the coordinate system) represent distance and direction as seen by the observer. It is, therefore, clearly implied that measurements in the coordinates system are specific to the state of motion of the coordinate system.

In a plane language, we can say that the description of motion is specific to a system of rigid bodies, which involves measurement of distance and direction. The measurements are done, using standards of length, by an observer, who is at rest with the system of rigid bodies. The observer makes use of a coordinate system attached to the system of rigid bodies and uses the same as reference to make measurements.

It is apparent that the terms “system of rigid bodies”, “observer” and “coordinate system” etc. are similar in meaning; all of which conveys a system of reference for carrying out measurements to describe motion. We sum up the discussion thus far as :

1. Measurements of distance, direction and location in a coordinate system are specific to the system of rigid bodies, which serve as reference for both magnitude and direction.
2. Like point, distance and other aspects of motion, the concept of space is specific to the reference represented by coordinate system. It is, therefore, suggested that use of word “space” independent of coordinate system should be avoided and if used it must be kept in mind that it represents volumetric extent of a specific coordinate system. The concept of space, if used without caution, leads to an inaccurate understanding of the laws of nature.
3. Once the meanings of terms are clear, “the system of reference” or “frame of reference” or “rigid body system” or “observer” or “coordinate system” may be used interchangeably to denote a unique system for determination of motional quantities and the representation of a motion.

Coordinate system types

Coordinate system types determine position of a point with measurements of distance or angle or combination of them. A spatial point requires three measurements in each of these coordinate types. It must, however, be noted that the descriptions of a point in any of these systems are equivalent. Different coordinate types are mere convenience of appropriateness for a given situation. Three major coordinate systems used in the study of physics are :

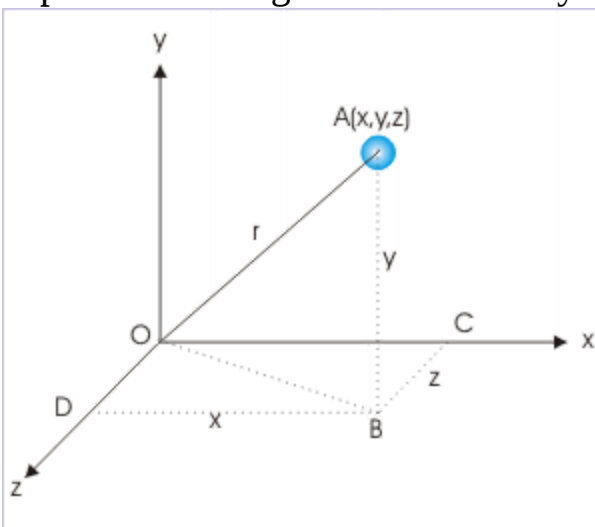
- Rectangular (Cartesian)
- Spherical
- Cylindrical

Rectangular (Cartesian) coordinate system is the most convenient as it is easy to visualize and associate with our perception of motion in daily life. Spherical and cylindrical systems are specifically designed to describe motions, which follow spherical or cylindrical curvatures.

Rectangular (Cartesian) coordinate system

The measurements of distances along three mutually perpendicular directions, designated as x, y and z , completely define a point $A(x, y, z)$.

A point in rectangular coordinate system



A point is specified with three
coordinate values

From geometric consideration of triangle OAB,

$$r = \sqrt{OB^2 + AB^2}$$

From geometric consideration of triangle OBD,

$$OB^2 = \sqrt{BD^2 + OD^2}$$

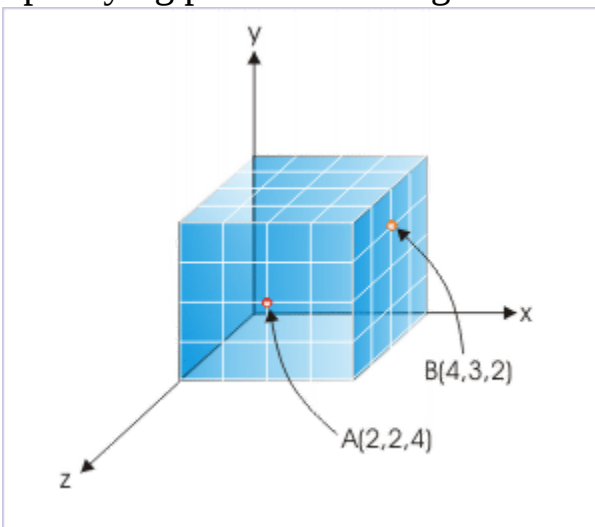
Combining above two relations, we have :

$$\Rightarrow r = \sqrt{BD^2 + OD^2 + AB^2}$$

$$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

The numbers are assigned to a point in the sequence x, y, z as shown for the
points A and B.

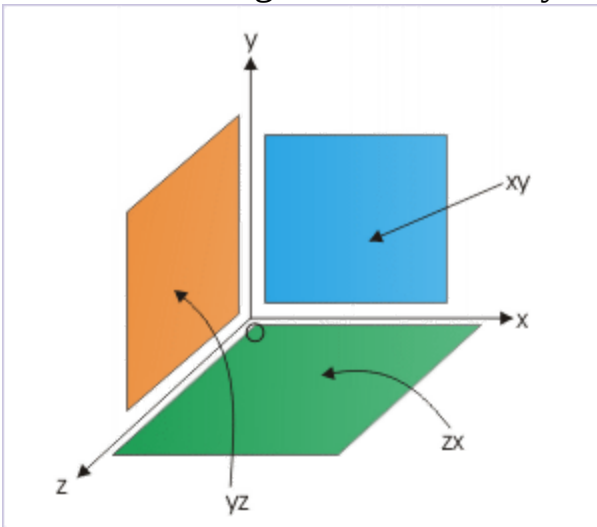
Specifying points in rectangular coordinate system



A point is specified with
coordinate values

Rectangular coordinate system can also be viewed as volume composed of three rectangular surfaces. The three surfaces are designated as a pair of axial designations like “xy” plane. We may infer that the “xy” plane is defined by two lines (x and y axes) at right angle. Thus, there are “xy”, “yz” and “zx” rectangular planes that make up the space (volumetric extent) of the coordinate system (See figure).

Planes in rectangular coordinate system



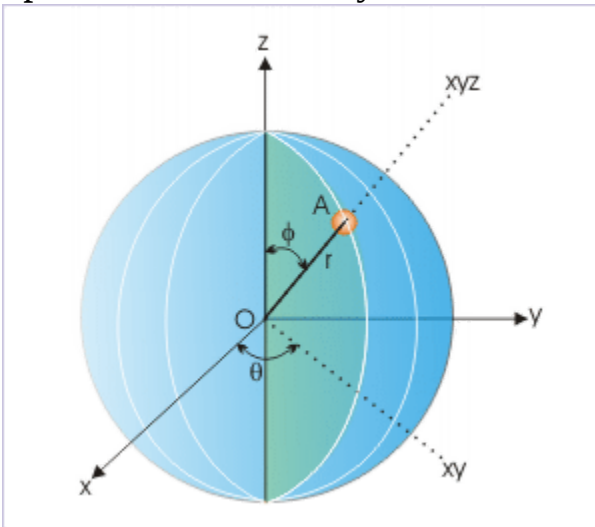
Three mutually perpendicular
planes define domain of
rectangular system

The motion need not be extended in all three directions, but may be limited to two or one dimensions. A circular motion, for example, can be represented in any of the three planes, whereby only two axes with an origin will be required to describe motion. A linear motion, on the other hand, will require representation in one dimension only.

Spherical coordinate system

A three dimensional point “A” in spherical coordinate system is considered to be located on a sphere of a radius “ r ”. The point lies on a particular cross section (or plane) containing origin of the coordinate system. This cross section makes an angle “ θ ” from the “ zx ” plane (also known as longitude angle). Once the plane is identified, the angle, ϕ , that the line joining origin O to the point A, makes with “ z ” axis, uniquely defines the point A (r, θ, ϕ).

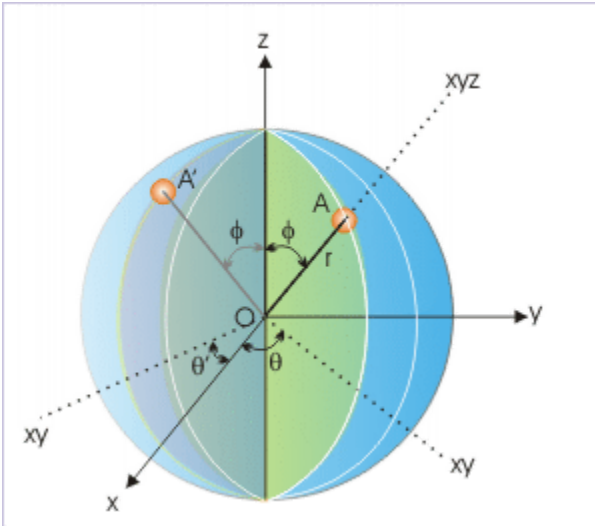
Spherical coordinate system



A point is specified with three coordinate values

It must be realized here that we need to designate three values r , θ and ϕ to uniquely define the point A. If we do not specify θ , the point could then lie any of the infinite numbers of possible cross section through the sphere like A'(See Figure below).

Spherical coordinate system



A point is specified with three coordinate values

From geometric consideration of spherical coordinate system :

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$x = r \sin \varphi \cos \theta$$

$$y = r \sin \varphi \sin \theta$$

$$z = r \cos \varphi$$

$$\tan \varphi = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\tan \theta = \frac{y}{x}$$

These relations can be easily obtained, if we know to determine projection of a directional quantity like position vector. For example, the projection of "r" in "xy" plane is "r sin φ". In turn, projection of "r sin φ" along x-axis is ""r sin φ cos θ". Hence,

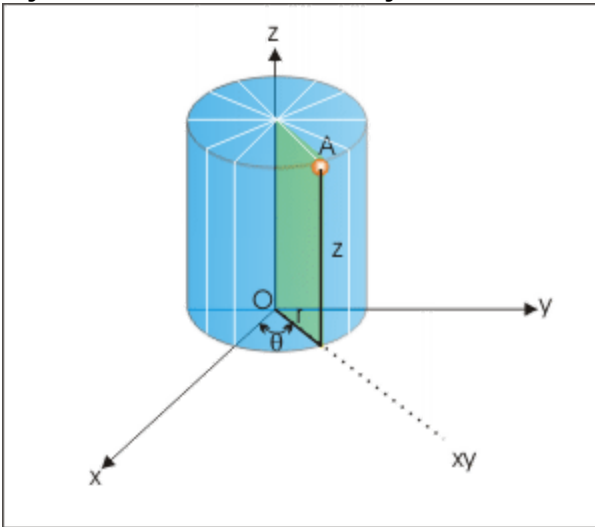
$$x = r \sin \varphi \cos \theta$$

In the similar fashion, we can determine other relations.

Cylindrical coordinate system

A three dimensional point “A” in cylindrical coordinate system is considered to be located on a cylinder of a radius “r”. The point lies on a particular cross section (or plane) containing origin of the coordinate system. This cross section makes an angle “ θ ” from the “zx” plane. Once the plane is identified, the height, z, parallel to vertical axis “z” uniquely defines the point A(r, θ , z)

Cylindrical coordinate system



A point is specified with three coordinate values

$$r = \sqrt{x^2 + y^2}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$\tan \theta = \frac{y}{x}$$

Distance

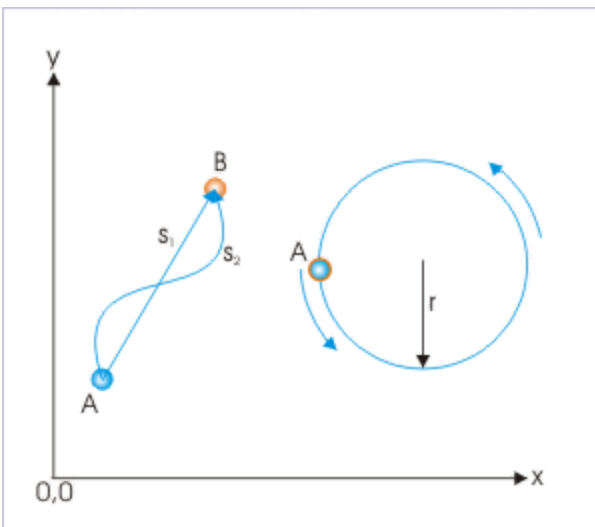
Motion involves two types of measurements : one, which depends on the end points (displacement) and the other, which depends on all points (distance) of motion.

Distance represents the magnitude of motion in terms of the "length" of the path, covered by an object during its motion. The terms "distance" and "distance covered" are interchangeably used to represent the same length along the path of motion and are considered equivalent terms. Initial and final positions of the object are mere start and end points of measurement and are not sufficient to determine distance. It must be understood that the distance is measured by the length covered, which may not necessarily be along the straight line joining initial and final positions. The path of the motion between two positions is an important consideration for determining distance. One of the paths between two points is the shortest path, which may or may not be followed during the motion.

Distance

Distance is the length of path followed during a motion.

Distance



Distance depends on the choice of path between two points

In the diagram shown above, s_1 , represents the shortest distance between points A and B. Evidently,

$$s_2 \geq s_1$$

The concept of distance is associated with the magnitude of movement of an object during the motion. It does not matter if the object goes further away or suddenly moves in a different direction or reverses its path. The magnitude of movement keeps adding up so long the object moves. This notion of distance implies that distance is not linked with any directional attribute. The distance is, thus, a scalar quantity of motion, which is cumulative in nature.

An object may even return to its original position over a period of time without any “net” change in position; the distance, however, will not be zero. To understand this aspect of distance, let us consider a point object that follows a circular path starting from point A and returns to the initial position as shown in the figure above. Though, there is no change in the position over the period of motion; but the object, in the meantime, covers a circular path, whose length is equal to its perimeter i.e. $2\pi r$.

Generally, we choose the symbol 's' to denote distance. A distance is also represented in the form of “ Δs ” as the distance covered in a given time interval Δt . The symbol “ Δ ” pronounced as “del” signifies the change in the quantity before which it appears.

Distance is a scalar quantity but with a special feature. It does not take negative value unlike some other scalar quantities like “charge”, which can assume both positive and negative values. The very fact that the distance keeps increasing regardless of the direction, implies that distance for a body in motion is always positive. Mathematically :

$$s > 0$$

Since distance is the measurement of length, its dimensional formula is [L] and its SI measurement unit is “meter”.

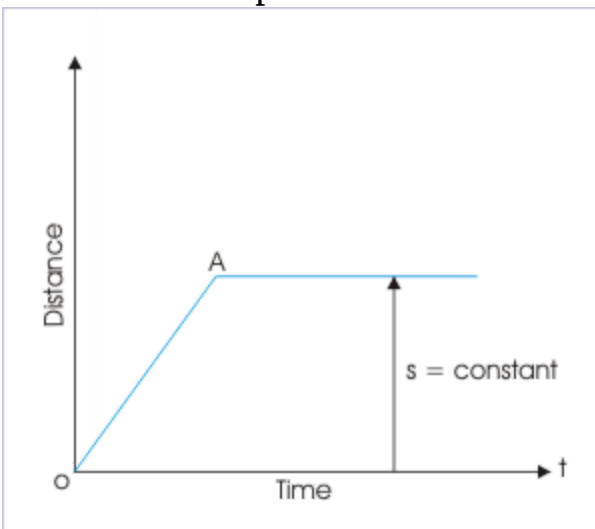
Distance – time plot

Distance – time plot is a simple plot of two scalar quantities along two axes. However, the nature of distance imposes certain restrictions, which characterize "distance - time" plot.

The nature of "distance – time" plot, with reference to its characteristics, is summarized here :

1. Distance is a positive scalar quantity. As such, "the distance – time" plot is a curve in the first quadrant of the two dimensional plot.
2. As distance keeps increasing during a motion, the slope of the curve is always positive.
3. When the object undergoing motion stops, then the plot becomes straight line parallel to time axis so that distance is constant as shown in the figure here.

Distance - time plot



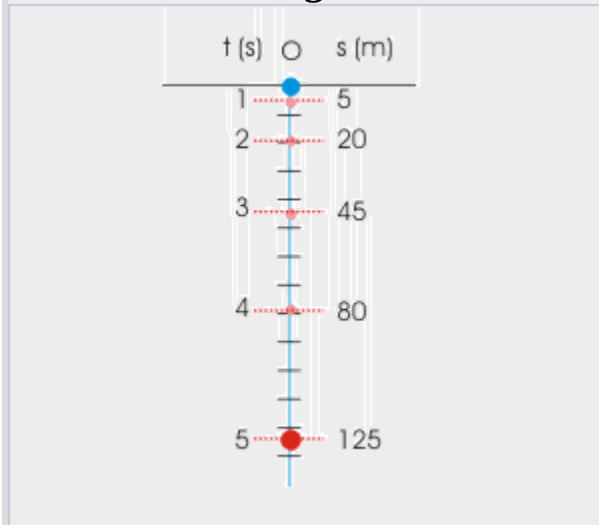
One important implication of the positive slope of the "distance - time" plot is that the curve never drops below a level at any moment of time. Besides, it must be noted that "distance - time" plot is handy in determining "instantaneous speed", but we choose to conclude the discussion of "distance - time" plot as these aspects are separately covered in subsequent module.

Example:

Distance – time plot

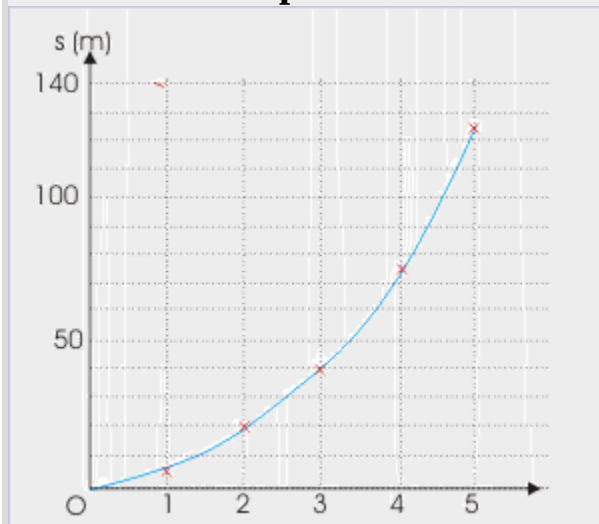
Question : A ball falling from an height ‘h’ strikes the ground. The distance covered during the fall at the end of each second is shown in the figure for the first 5 seconds. Draw distance – time plot for the motion during this period. Also, discuss the nature of the curve.

Motion of a falling ball



Solution : We have experienced that a free falling object falls with increasing speed under the influence of gravity. The distance covered in successive time intervals increases with time. The magnitudes of distance covered in successive seconds given in the plot illustrate this point. In the plot between distance and time as shown, the origin of the reference (coordinate system) is chosen to coincide with initial point of the motion.

Distance – time plot



From the plot, it is clear that the ball covers more distance as it nears the ground. The "distance- time" curve during fall is, thus, flatter near start point and steeper near earth surface. Can you guess the nature of plot when a ball is thrown up against gravity?

Exercise:

Problem:

A ball falling from an height 'h' strikes a hard horizontal surface with increasing speed. On each rebound, the height reached by the ball is half of the height it fell from. Draw "distance – time" plot for the motion covering two consecutive strikes, emphasizing the nature of curve (ignore actual calculation). Also determine the total distance covered during the motion.

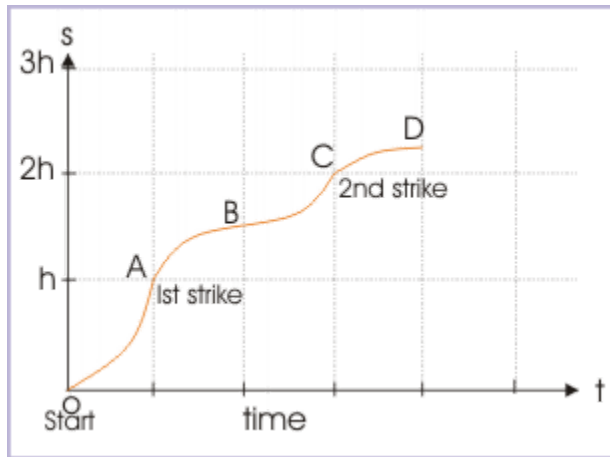
Solution:

Here we first estimate the manner in which distance is covered under gravity as the ball falls or rises.

The distance- time curve during fall is flatter near start point and steeper near earth surface. On the other hand, we can estimate that the distance- time curve, during rise, is steeper near the earth surface (covers more distance due to greater speed) and flatter as it reaches the maximum height, when speed of the ball becomes zero.

The "distance – time" plot of the motion of the ball, showing the nature of curve during motion, is :

Distance – time plot

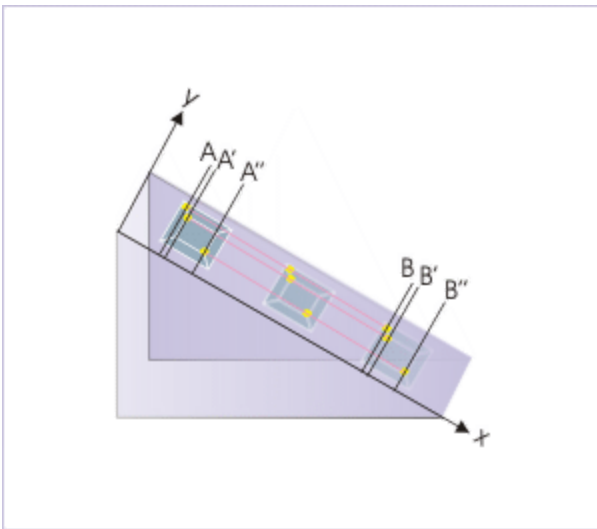


The origin of plot (O) coincides with the initial position of the ball ($t = 0$). Before striking the surface for the first time (A), it travels a distance of ' h '. On rebound, it rises to a height of ' $h/2$ ' (B on plot). Total distance is ' $h + h/2 = 3h/2$ '. Again falling from a height of ' $h/2$ ', it strikes the surface, covering a distance of ' $h/2$ '. The total distance from the start to the second strike (C on plot) is ' $3h/2 + h/2 = 2h$ '.

Position

Coordinate system enables us to specify a point in its defined volumetric space. We must recognize that a point is a concept without dimensions; whereas the objects or bodies under motion themselves are not points. The real bodies, however, approximates a point in translational motion, when paths followed by the particles, composing the body are parallel to each other (See Figure). As we are concerned with the geometry of the path of motion in kinematics, it is, therefore, reasonable to treat real bodies as “point like” mass for description of translational motion.

Translational motion



Particles follow parallel paths

We conceptualize a particle in order to facilitate the geometric description of motion. A particle is considered to be dimensionless, but having a mass. This hypothetical construct provides the basis for the logical correspondence of point with the position occupied by a particle.

Without any loss of purpose, we can designate motion to begin at A or A' or A'' corresponding to final positions B or B' or B'' respectively as shown in the figure above.

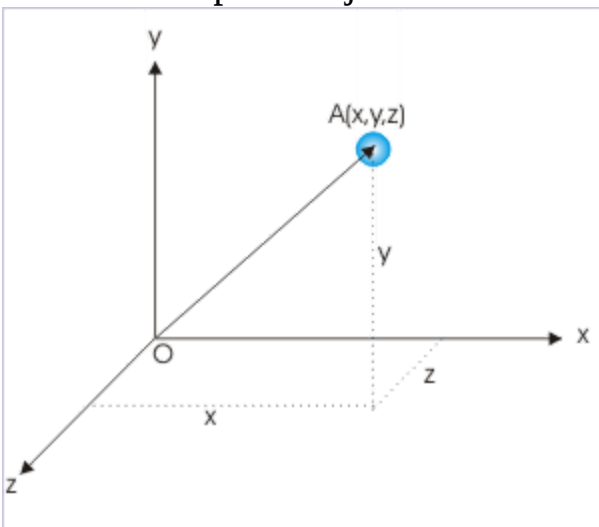
For the reasons as outlined above, we shall freely use the terms “body” or “object” or “particle” in one and the same way as far as description of translational motion is concerned. Here, pure translation conveys the meaning that the object is under motion without rotation, like sliding of a block on a smooth inclined plane.

Position

Position

The position of a particle is a point in the defined volumetric space of the coordinate system.

Position of a point object



Position of a point is specified
by three coordinate values

The position of a point like object, in three dimensional coordinate space, is defined by three values of coordinates i.e. x, y and z in Cartesian coordinate system as shown in the figure above.

It is evident that the relative position of a point with respect to a fixed point such as the origin of the system “O” has directional property. The position

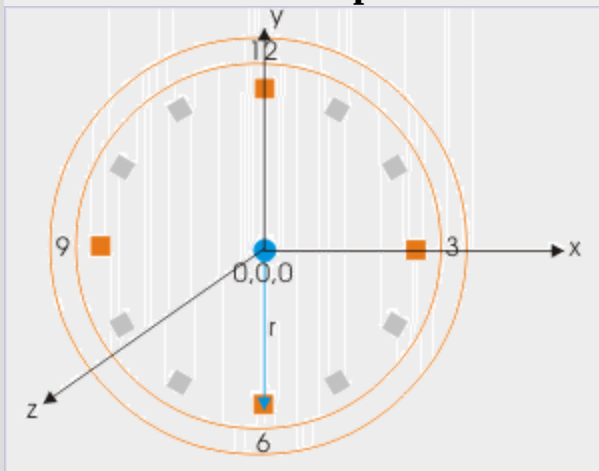
of the object, for example, can lie either to the left or to the right of the origin or at a certain angle from the positive x - direction. As such the position of an object is associated with directional attribute with respect to a frame of reference (coordinate system).

Example:

Coordinates

Problem : The length of the second's hand of a round wall clock is 'r' meters. Specify the coordinates of the tip of the second's hand corresponding to the markings 3,6,9 and 12 (Consider the center of the clock as the origin of the coordinate system.).

Coordinates of the tip of the second's hand



The origin coincides with the center

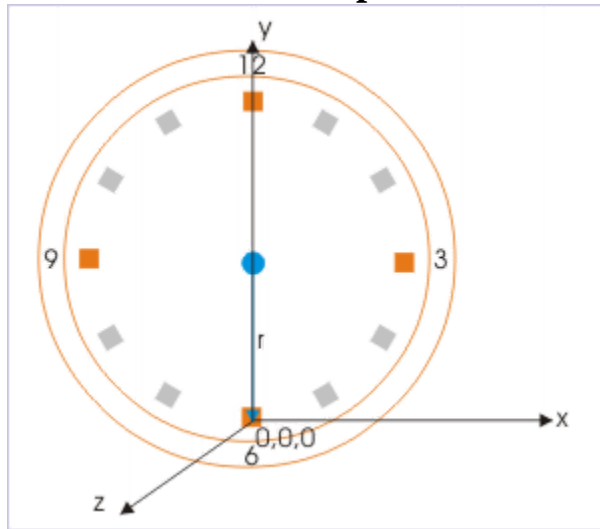
Solution : The coordinates of the tip of the second's hand is given by the coordinates :

3 : $r, 0, 0$ 6 : $0, -r, 0$ 9 : $-r, 0, 0$ 12 : $0, r, 0$

Exercise:

Problem:

What would be the coordinates of the markings 3, 6, 9 and 12 in the earlier example, if the origin coincides with the marking 6 on the clock?

Coordinates of the tip of the second's hand

Origin coincides with the marking 6 O' clock

Solution:

The coordinates of the tip of the second's hand is given by the coordinates :

3 : $r, r, 0$ 6 : $0, 0, 0$ 9 : $-r, r, 0$ 12 : $0, 2r, 0$

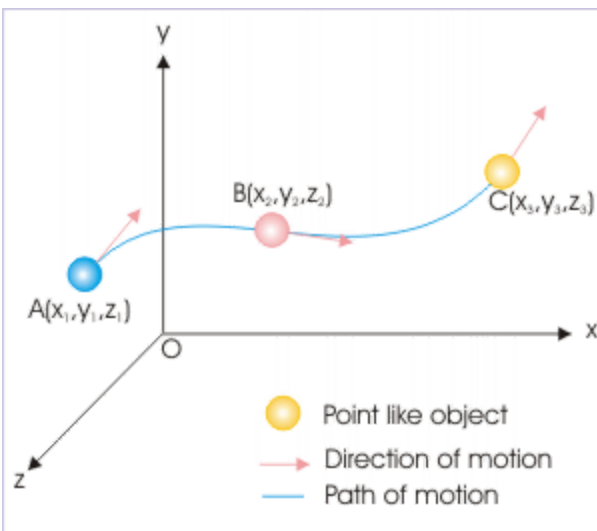
The above exercises point to an interesting feature of the frame of reference: that the specification of position of the object (values of coordinates) depends on the choice of origin of the given frame of reference. We have already seen that description of motion depends on the state of observer i.e. the attached system of reference. This additional

dependence on the choice of origin of the reference would have further complicated the issue, but for the linear distance between any two points in a given system of reference, is found to be independent of the choice of the origin. For example, the linear distance between the markings 6 and 12 is '2r', irrespective of the choice of the origin.

Plotting motion

Position of a point in the volumetric space is a three dimensional description. A plot showing positions of an object during a motion is an actual description of the motion in so far as the curve shows the path of the motion and its length gives the distance covered. A typical three dimensional motion is depicted as in the figure below :

Motion in three dimension



The plot is the path followed by the object during motion.

In the figure, the point like object is deliberately shown not as a point, but with finite dimensions. This has been done in order to emphasize that an object of finite dimensions can be treated as point when the motion is purely translational.

The three dimensional description of positions of an object during motion is reduced to be two or one dimensional description for the planar and linear motions respectively. In two or one dimensional motion, the remaining coordinates are constant. In all cases, however, the plot of the positions is meaningful in following two respects :

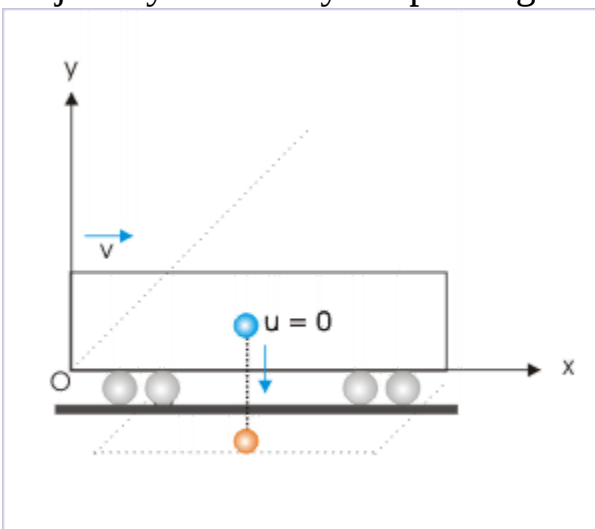
- The length of the curve (i.e. plot) is equal to the distance.
- A tangent in forward direction at a point on the curve gives the direction of motion at that point

Description of motion

Position is the basic element used to describe motion. Scalar properties of motion like distance and speed are expressed in terms of position as a function of time. As the time passes, the positions of the motion follow a path, known as the trajectory of the motion. It must be emphasized here that the path of motion (trajectory) is unique to a frame of reference and so is the description of the motion.

To illustrate the point, let us consider that a person is traveling on a train, which is moving with the velocity $\mathbf{v}$ along a straight track. At a particular moment, the person releases a small pebble. The pebble drops to the ground along the vertical direction as seen by the person.

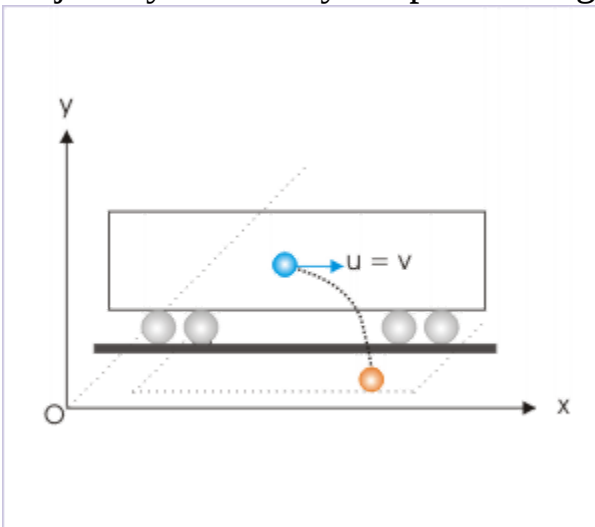
Trajectory as seen by the passenger



Trajectory is a straight line.

The same incident, however, is seen by an observer on the ground as if the pebble followed a parabolic path (See Figure blow). It emerges then that the path or the trajectory of the motion is also a relative attribute, like other attributes of the motion (speed and velocity). The coordinate system of the passenger in the train is moving with the velocity of train (v) with respect to the earth and the path of the pebble is a straight line. For the person on the ground, however, the coordinate system is stationary with respect to earth. In this frame, the pebble has a horizontal velocity, which results in a parabolic trajectory.

Trajectory as seen by the person on ground



Trajectory is a parabolic curve.

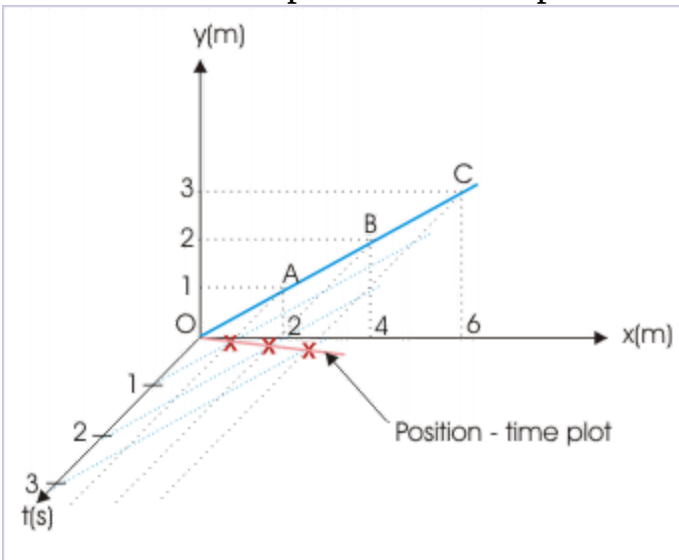
Without overemphasizing, we must acknowledge that the concept of path or trajectory is essentially specific to the frame of reference or the coordinate system attached to it. Interestingly, we must be aware that this particular observation happens to be the starting point for the development of special theory of relativity by Einstein (see his original transcript on the subject of relativity).

Position – time plot

The position in three dimensional motion involves specification in terms of three coordinates. This requirement poses a serious problem, when we want to investigate positions of the object with respect to time. In order to draw such a graph, we would need three axes for describing position and one axis for plotting time. This means that a position – time plot for a three dimensional motion would need four (4) axes !

A two dimensional position – time plot, however, is a possibility, but its drawing is highly complicated for representation on a two dimensional paper or screen. A simple example consisting of a linear motion in the x-y plane is plotted against time on z – axis (See Figure).

Two dimensional position – time plot



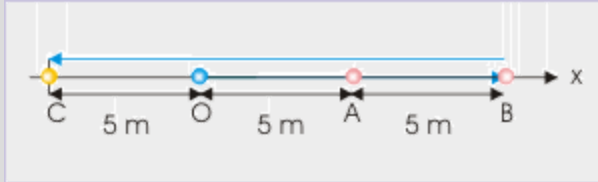
As a matter of fact, it is only the one dimensional motion, whose position – time plot can be plotted conveniently on a plane. In one dimensional motion, the point object can either be to the left or to the right of the origin in the direction of reference line. Thus, drawing position against time is a straight forward exercise as it involves plotting positions with appropriate sign.

Example:

Coordinates

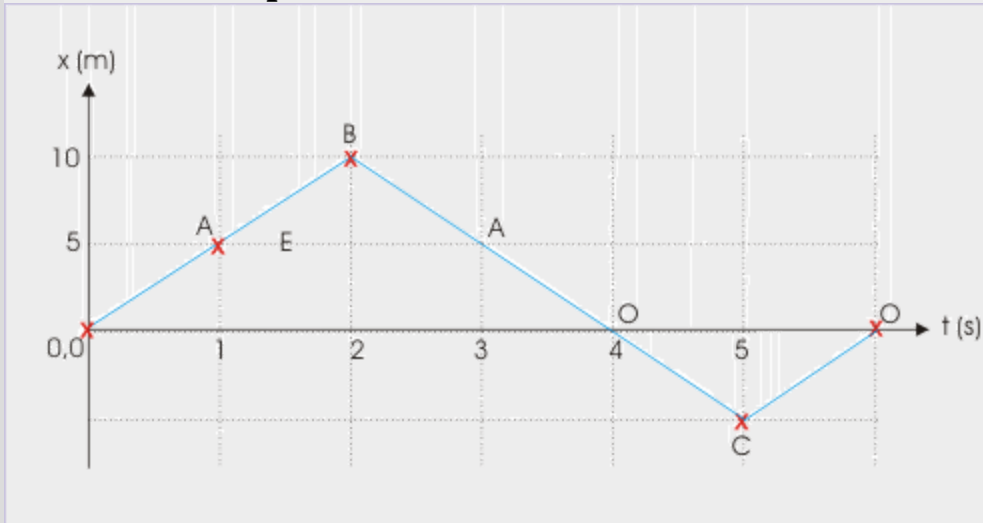
Problem : A ball moves along a straight line from O to A to B to C to O along x-axis as shown in the figure. The ball covers each of the distance of 5 m in one second. Plot the position – time graph.

Motion along a straight line



Solution : The coordinates of the ball are 0, 5, 10, -5 and 0 at points O, A, B, C and O (on return) respectively. The position – time plot of the motion is as given below :

Position – time plot in one dimension



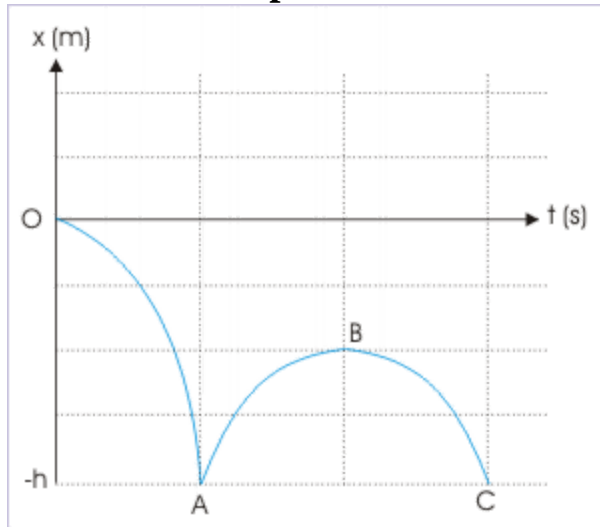
Exercise:

Problem:

A ball falling from a height 'h' strikes a hard horizontal surface with increasing speed. On each rebound, the height reached by the ball is half of the height it fell from. Draw position – time plot for the motion covering two consecutive strikes, emphasizing the nature of curve (ignore actual calculation).

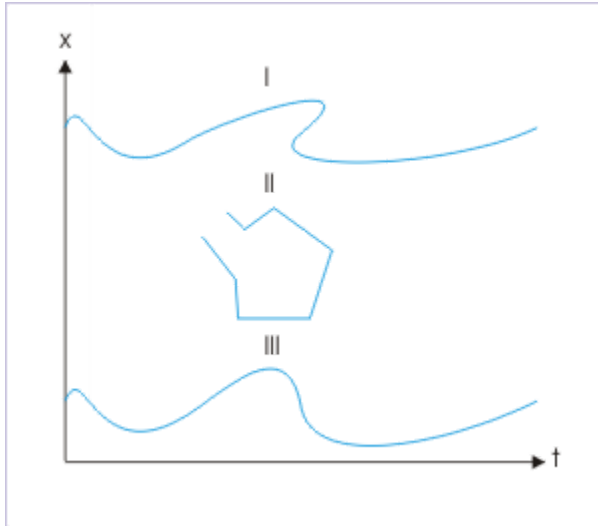
Solution:

Now, as the ball falls towards the surface, it covers path at a quicker pace. As such, the position changes more rapidly as the ball approaches the surface. The curve (i.e. plot) is, therefore, steeper towards the surface. On the return upward journey, the ball covers lesser distance as it reaches the maximum height. Hence, the position – time curve (i.e. plot) is flatter towards the point of maximum height.

Position – time plot in one dimension**Exercise:****Problem:**

The figure below shows three position – time plots of a motion of a particle along x-axis. Giving reasons, identify the valid plot(s) among them. For the valid plot(s), determine following :

Position – time plots in one dimension

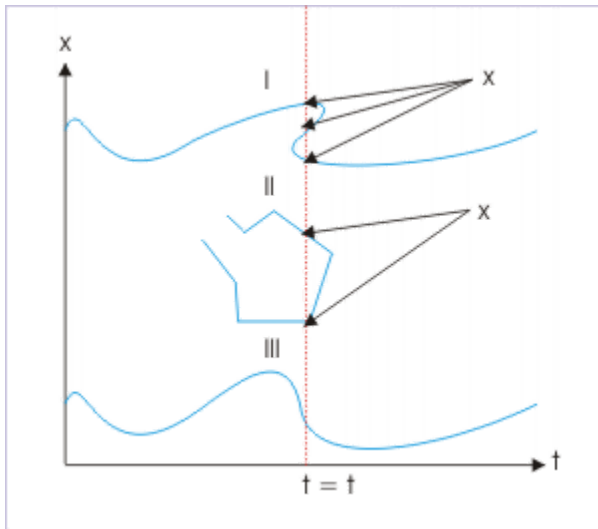


1. How many times the particle has come to rest?
2. Does the particle reverse its direction during motion?

Solution:

Validity of plots : In the portion of plot I, we can draw a vertical line that intersects the curve at three points. It means that the particle is present at three positions simultaneously, which is not possible. Plot II is also not valid for the same reason. Besides, it consists of a vertical portion, which would mean presence of the particle at infinite numbers of positions at the same instant. Plot III, on the other hand, is free from these anomalies and is the only valid curve representing motion of a particle along x – axis (See Figure).

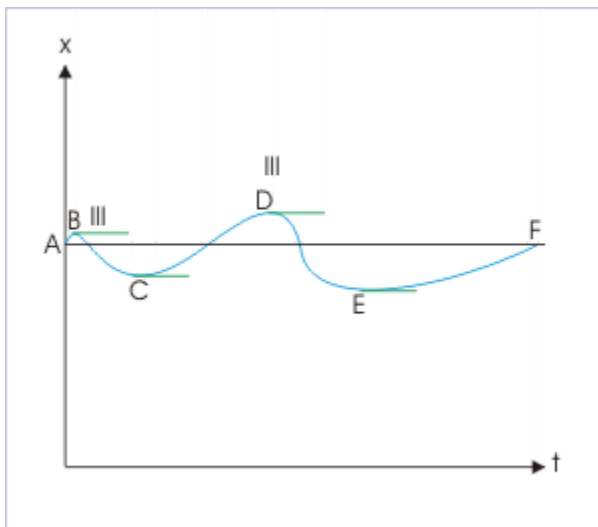
Valid position – time plot in one dimension



A particle can not be present at more than one position at a given instant.

When the particle comes to rest, there is no change in the value of “ x ”. This, in turn, means that tangent to the curve at points of rest is parallel to x -axis. By inspection, we find that tangent to the curve is parallel to x -axis at four points (B,C,D and E) on the curve shown in the figure below. Hence, the particle comes to rest four times during the motion.

Positions of rest



At rest, the tangent to the path is
parallel to time axis.

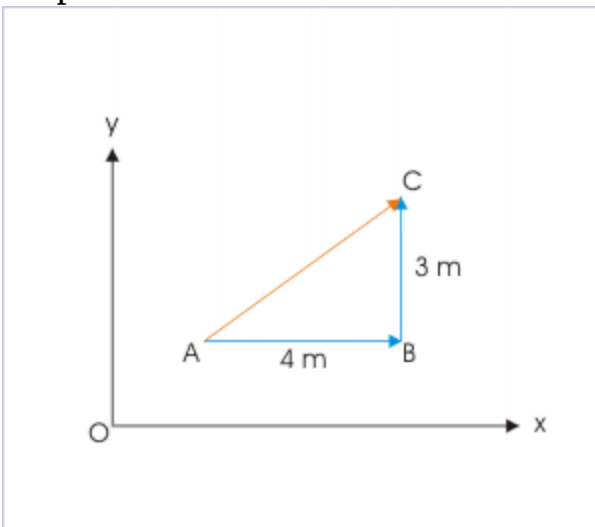
Vectors

A number of key fundamental physical concepts relate to quantities, which display directional property. Scalar algebra is not suited to deal with such quantities. The mathematical construct called vector is designed to represent quantities with directional property. A vector, as we shall see, encapsulates the idea of “direction” together with “magnitude”.

In order to elucidate directional aspect of a vector, let us consider a simple example of the motion of a person from point A to point B and from point B to point C, covering a distance of 4 and 3 meters respectively as shown in the [Figure](#) . Evidently, AC represents the linear distance between the initial and the final positions. This linear distance, however, is not equal to the sum of the linear distances of individual motion represented by segments AB and BC (4 + 3 = 7 m) i.e.

$$AC \neq AB + BC$$

Displacement



Scalar inequality

However, we need to express the end result of the movement appropriately as the sum of two individual movements. The inequality of the scalar

equation as above is basically due to the fact that the motion represented by these two segments also possess directional attributes; the first segment is directed along the positive x – axis, where as the second segment of motion is directed along the positive y –axis. Combining their magnitudes is not sufficient as the two motions are perpendicular to each other. We require a mechanism to combine directions as well.

The solution of the problem lies in treating individual distance with a new term "displacement" – a vector quantity, which is equal to “linear distance plus direction”. Such a conceptualization of a directional quantity allows us to express the final displacement as the sum of two individual displacements in vector form :

$$\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$$

The magnitude of displacement is obtained by applying Pythagoras theorem :

$$AC \sqrt{(AB^2 + BC^2)} = \sqrt{(4^2 + 3^2)} = 5 \text{ m}$$

It is clear from the example above that vector construct is actually devised in a manner so that physical reality having directional property is appropriately described. This "fit to requirement" aspect of vector construct for physical phenomena having direction is core consideration in defining vectors and laying down rules for vector operation.

A classical example, illustrating the “fit to requirement” aspect of vector, is the product of two vectors. A product, in general, should evaluate in one manner to yield one value. However, there are natural quantities, which are product of two vectors, but evaluate to either scalar (example : work) or vector (example : torque) quantities. Thus, we need to define the product of vectors in two ways : one that yields scalar value and the other that yields vector value. For this reason product of two vectors is either defined as dot product to give a scalar value or defined as cross product to give vector value. This scheme enables us to appropriately handle the situations as the case may be.

$$W = \mathbf{F} \cdot \Delta \mathbf{r} \quad \dots\dots\dots \text{Scalar dot product}$$

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \quad \dots\dots\dots \text{Vector cross product}$$

Mathematical concept of vector is basically secular in nature and general in application. This means that mathematical treatment of vectors is without reference to any specific physical quantity or phenomena. In other words, we can employ vector and its methods to all quantities, which possess directional attribute, in a uniform and consistent manner. For example two vectors would be added in accordance with vector addition rule irrespective of whether vectors involved represent displacement, force, torque or some other vector quantities.

The moot point of discussion here is that vector has been devised to suit the requirement of natural process and not the other way around that natural process suits vector construct as defined in vector mathematics.

What is a vector?

Vector

Vector is a physical quantity, which has both magnitude and direction.

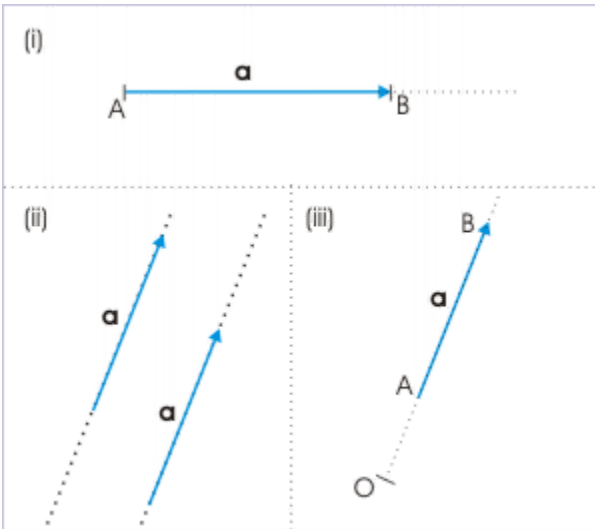
A vector is represented graphically by an arrow drawn on a scale as shown [Figure i](#). In order to process vectors using graphical methods, we need to draw all vectors on the same scale. The arrow head point in the direction of the vector.

A vector is notionally represented in a characteristic style. It is denoted as bold face type like “**a**” as shown [Figure \(i\)](#) or with a small arrow over the symbol like “ $\vec{a}$ ” or with a small bar as in “ $\bar{a}$ ”. The magnitude of a vector quantity is referred by simple identifier like “a” or as the absolute value of the vector as “ $|\mathbf{a}|$ ”.

Two vectors of equal magnitude and direction are equal vectors ([Figure \(ii\)](#)). As such, a vector can be laterally shifted as long as its direction remains same ([Figure \(ii\)](#)). Also, vectors can be shifted along its line of application represented by dotted line ([Figure \(iii\)](#)). The flexibility by virtue of

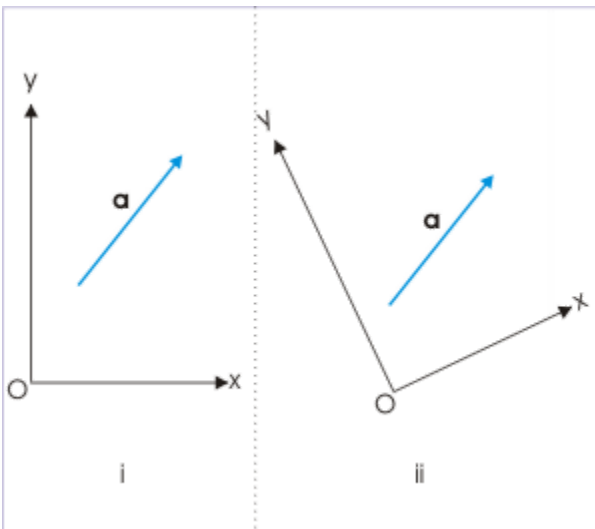
shifting vector allows a great deal of ease in determining vector's interaction with other scalar or vector quantities.

Vectors



It should be noted that graphical representation of vector is independent of the origin or axes of coordinate system except for few vectors like position vector (called localized vector), which is tied to the origin or a reference point by definition. With the exception of localized vector, a change in origin or orientation of axes or both does not affect vectors and vector operations like addition or multiplication (see figure below).

Vectors



The vector is not affected, when the coordinate is rotated or displaced as shown in the figure above. Both the orientation and positioning of origin i.e

reference point do not alter the vector representation. It remains what it is. This feature of vector operation is an added value as the study of physics in terms of vectors is simplified, being independent of the choice of coordinate system in a given reference.

Vector algebra

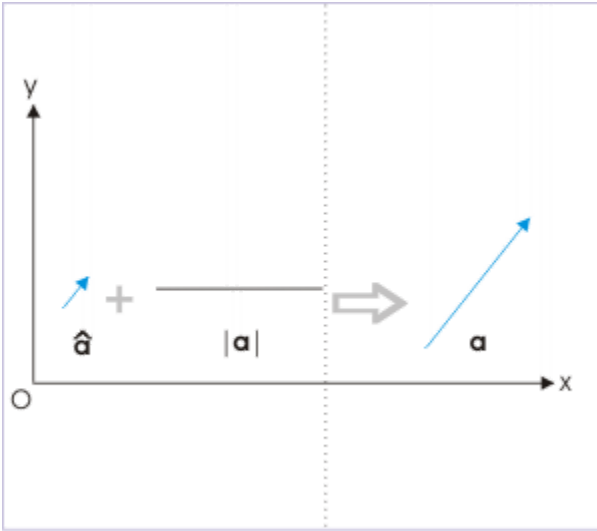
Graphical method is slightly meticulous and error prone as it involves drawing of vectors on scale and measurement of angles. In addition, it does not allow algebraic manipulation that otherwise would give a simple solution as in the case of scalar algebra. We can, however, extend algebraic techniques to vectors, provided vectors are represented on a rectangular coordinate system. The representation of a vector on a coordinate system uses the concept of unit vectors and scalar magnitudes. We shall discuss these aspects in a separate module titled [Components of a vector](#). Here, we briefly describe the concept of unit vector and technique to represent a vector in a particular direction.

Unit vector

Unit vector has a magnitude of one and is directed in a particular direction. It does not have dimension or unit like most other physical quantities. Thus, multiplying a scalar by unit vector converts the scalar quantity into a vector without changing its magnitude, but assigning it a direction ([Figure](#)).

$$\mathbf{a} = a\hat{a}$$

Vector representation with unit vector



This is an important relation as it allows determination of unit vector in the direction of any vector "**a**" as :

$$\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Conventionally, unit vectors along the rectangular axes is represented with bold type face symbols like : **i**, **j** and **k**, or with a cap heads like $\hat{i}$, $\hat{j}$ and $\hat{k}$. The unit vector along the axis denotes the direction of individual axis.

Using the concept of unit vector, we can denote a vector by multiplying the magnitude of the vector with unit vector in its direction.

$$\mathbf{a} = a\hat{a}$$

Following this technique, we can represent a vector along any axis in terms of scalar magnitude and axial unit vector like (for x-direction) :

$$\mathbf{a} = a\mathbf{i}$$

Other important vector terms

Null vector

Null vector is conceptualized for completing the development of vector algebra. We may encounter situations in which two equal but opposite vectors are added. What would be the result? Would it be a zero real number or a zero vector? It is expected that result of algebraic operation should be compatible with the requirement of vector. In order to meet this requirement, we define null vector, which has neither magnitude nor direction. In other words, we say that null vector is a vector whose all components in rectangular coordinate system are zero.

Strictly, we should denote null vector like other vectors using a bold faced letter or a letter with an overhead arrow. However, it may generally not be done. We take the exception to denote null vector by number “0” as this representation does not contradicts the defining requirement of null vector.

$$\mathbf{a} + \mathbf{b} = 0$$

Negative vector

Negative vector

A negative vector of a given vector is defined as the vector having same magnitude, but applied in the opposite direction to that of the given vector.

It follows that if $\mathbf{b}$ is the negative of vector $\mathbf{a}$, then

$$\mathbf{a} = -\mathbf{b}$$

$$\Rightarrow \mathbf{a} + \mathbf{b} = 0$$

$$\text{and } |\mathbf{a}| = |\mathbf{b}|$$

There is a subtle point to be made about negative scalar and vector quantities. A negative scalar quantity, sometimes, conveys the meaning of lesser value. For example, the temperature -5 K is a smaller temperature than any positive value. Also, a greater negative like – 100 K is less than the smaller negative like -50 K. However, a scalar like charge conveys different meaning. A negative charge of -10 μC is a bigger negative charge than – 5 μC . The interpretation of negative scalar is, thus, situational.

On the other hand, negative vector always indicates the sense of opposite direction. Also like charge, a greater negative vector is larger than smaller negative vector or a smaller positive vector. The magnitude of force $-10 \mathbf{i}$ N, for example is greater than $5 \mathbf{i}$ N, but directed in the opposite direction to that of the unit vector $\mathbf{i}$. In any case, negative vector does not convey the meaning of lesser or greater magnitude like the meaning of a scalar quantity in some cases.

Co-planar vectors

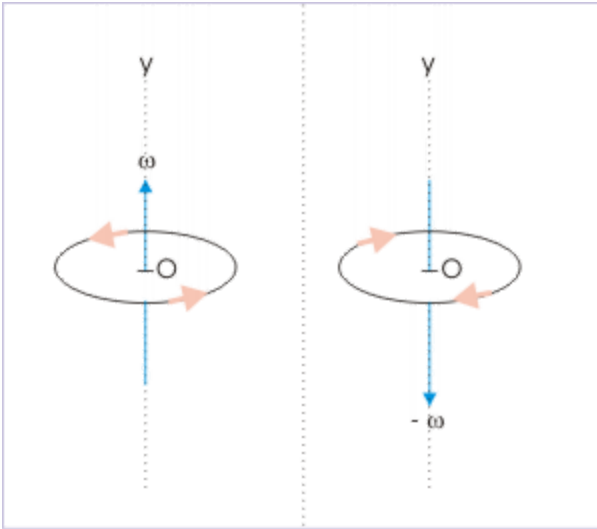
A pair of vectors determines a unique plane. The pair of vectors defining the plane and other vectors in that plane are called coplanar vectors.

Axial vector

Motion has two basic types : translational and rotational motions. The vector and scalar quantities, describing them are inherently different. Accordingly, there are two types of vectors to deal with quantities having direction. The system of vectors that we have referred so far is suitable for describing translational motion and such vectors are called “rectangular” or “polar” vectors.

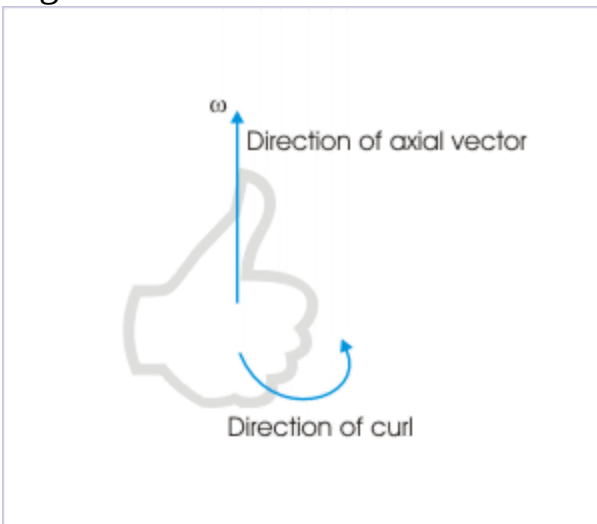
A different type of vector called axial vector is used to describe rotational motion. Its graphical representation is same as that of rectangular vector, but its interpretation is different. What it means that the axial vector is represented by a straight line with an arrow head as in the case of polar vector; but the physical interpretation of axial vector differs. An axial vector, say ω , is interpreted to act along the positive direction of the axis of rotation, while rotating anti-clockwise. A negative axial vector like, $-\omega$, is interpreted to act along the negative direction of axis of rotation, while rotating clockwise.

Axial vector



The [figure above](#) captures the concept of axial vector. It should be noted that the direction of the axial vector is essentially tied with the sense of rotation (clockwise or anti-clockwise). This linking of directions is stated with "Right hand (screw) rule". According to this rule ([see figure below](#)), if the stretched thumb of right hand points in the direction of axial vector, then the curl of the fist gives the direction of rotation. Its inverse is also true i.e if the curl of the right hand fist is placed in a manner to follow the direction of rotation, then the stretched thumb points in the direction of axial vector.

Right hand rule



Axial vector is generally shown to be perpendicular to a plane. In such cases, we use a shortened symbol to represent axial or even other vectors,

which are normal to the plane, by a "dot" or "cross" inscribed within a small circle. A "dot" inscribed within the circle indicates that the vector is pointing towards the viewer of the plane and a "cross" inscribed within the circle indicates that the vector is pointing away from the viewer of the plane.

Axial vector are also known as "pseudovectors". It is because axial vectors do not follow transformation of rectangular coordinate system. Vectors which follow coordinate transformation are called "true" or "polar" vectors. One important test to distinguish these two types of vector is that axial vector has a mirror image with negative sign unlike true vectors. Also, we shall learn about vector or cross product subsequently. This operation represent many important physical phenomena such as rotation and magnetic interaction etc. We should know that the vector resulting from cross product of true vectors is always axial i.e. pseudovectors vector like magnetic field, magnetic force, angular velocity, torque etc.

Why should we study vectors?

The basic concepts in physics – particularly the branch of mechanics - have a direct and inherently characterizing relationship with the concept of vector. The reason lies in the directional attribute of quantities, which is used to describe dynamical aspect of natural phenomena. Many of the physical terms and concepts are simply vectors like position vector, displacement vector etc. They are as a matter of fact defined directly in terms of vector like “it is a vector”.

The basic concept of “cause and effect” in mechanics (comprising of kinematics and dynamics), is predominantly based on the interpretation of direction in addition to magnitude. Thus, there is no way that we could accurately express these quantities and their relationship without vectors. There is, however, a general tendency (particular in the treatment designed for junior classes) to try to evade vectors and look around ways to deal with these inherently vector based concepts without using vectors! As expected this approach is a poor reflection of the natural process, where basic concepts are simply ingrained with the requirement of handling direction along with magnitude.

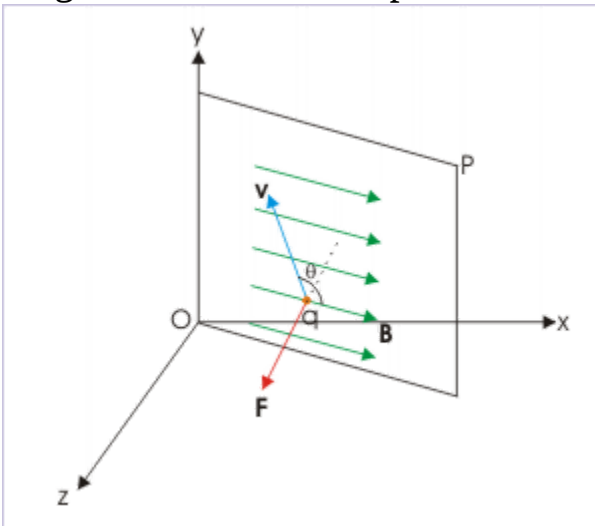
It is, therefore, imperative that we switch over from work around approach to vector approach to study physics as quickly as possible. Many a times, this scalar “work around” inculcates incorrect perception and understanding that may persist for long, unless corrected with an appropriate vector description.

The best approach, therefore, is to study vector in the backdrop of physical phenomena and use it with clarity and advantage in studying nature. For this reasons, our treatment of “vector physics” – so to say - in this course will strive to correlate vectors with appropriate physical quantities and concepts.

The most fundamental reason to study nature in terms of vectors, wherever direction is involved, is that vector representation is concise, explicit and accurate.

To score this point, let us consider an example of the magnetic force experienced by a charge, q , moving with a velocity $\mathbf{v}$ in a magnetic field, “ $\mathbf{B}$ ”. The magnetic force, $\mathbf{F}$, experienced by moving particle, is perpendicular to the plane, P , formed by the the velocity and the magnetic field vectors as shown in the [figure](#).

Magnetic force as cross product of vectors



The force is given in the vector form as :

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

This equation does not only define the magnetic force but also outlines the intricacies about the roles of the each of the constituent vectors. As per vector rule, we can infer from the vector equation that :

- The magnetic force (**F**) is perpendicular to the plane defined by vectors **v** and **B**.
- The direction of magnetic force i.e. which side of plane.
- The magnitude of magnetic force is " $qvB \sin\theta$ ", where θ is the smaller angle enclosed between the vectors **v** and **B**.

This example illustrates the compactness of vector form and completeness of the information it conveys. On the other hand, the equivalent scalar strategy to describe this phenomenon would involve establishing an empirical frame work like Fleming's left hand rule to determine direction. It would be required to visualize vectors along three mutually perpendicular directions represented by three fingers in a particular order and then apply Fleming rule to find the direction of the force. The magnitude of the product, on the other hand, would be given by $qvB \sin\theta$ as before.

The difference in two approaches is quite remarkable. The vector method provides a paragraph of information about the physical process, whereas a paragraph is to be followed to apply scalar method ! Further, the vector rules are uniform and consistent across vector operations, ensuring correctness of the description of physical process. On the other hand, there are different set of rules like Fleming left and Fleming right rules for two different physical processes.

The last word is that we must master the vectors rather than avoid them - particularly when the fundamentals of vectors to be studied are limited in extent.

Vector addition

Vectors operate with other scalar or vector quantities in a particular manner. Unlike scalar algebraic operation, vector operation draws on graphical representation to incorporate directional aspect.

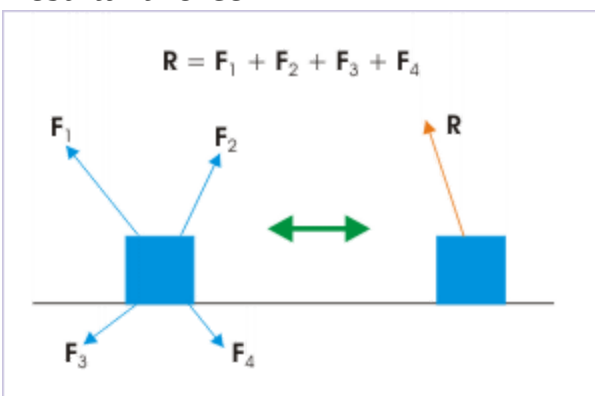
Vector addition is, however, limited to vectors only. We can not add a vector (a directional quantity) to a scalar (a non-directional quantity). Further, vector addition is dealt in three conceptually equivalent ways :

1. graphical methods
2. analytical methods
3. algebraic methods

In this module, we shall discuss first two methods. Third algebraic method will be discussed in a separate module titled [Components of a vector](#)

The resulting vector after addition is termed as sum or resultant vector. The resultant vector corresponds to the “resultant” or “net” effect of a physical quantities having directional attributes. The effect of a force system on a body, for example, is determined by the resultant force acting on it. The idea of resultant force, in this case, reflects that the resulting force (vector) has the same effect on the body as that of the forces (vectors), which are added.

Resultant force



It is important to emphasize here that vector rule of addition (graphical or algebraic) do not distinguish between vector types (whether displacement or

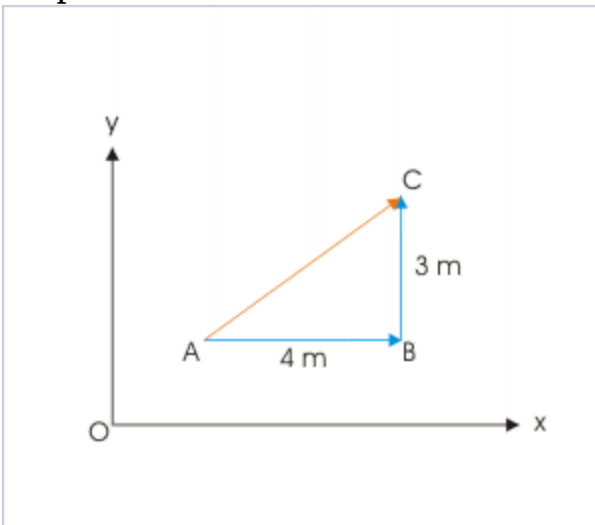
acceleration vector). This means that the rule of vector addition is general for all vector types.

It should be clearly understood that though rule of vector addition is general, which is applicable to all vector types in same manner, but vectors being added should be like vectors only. It is expected also. The requirement is similar to scalar algebra where 2 plus 3 is always 5, but we need to add similar quantity like 2 meters plus 3 meters is 5 meters. But, we can not add, for example, distance and temperature.

Vector addition : graphical method

Let us examine the example of displacement of a person in two different directions. The two displacement vectors, perpendicular to each other, are added to give the “resultant” vector. In this case, the closing side of the right triangle represents the sum (i.e. resultant) of individual displacements **AB** and **BC**.

Displacement



Equation:

$$\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$$

The method used to determine the sum in this particular case (in which, the closing side of the triangle represents the sum of the vectors in both

magnitude and direction) forms the basic consideration for various rules dedicated to implement vector addition.

Triangle law

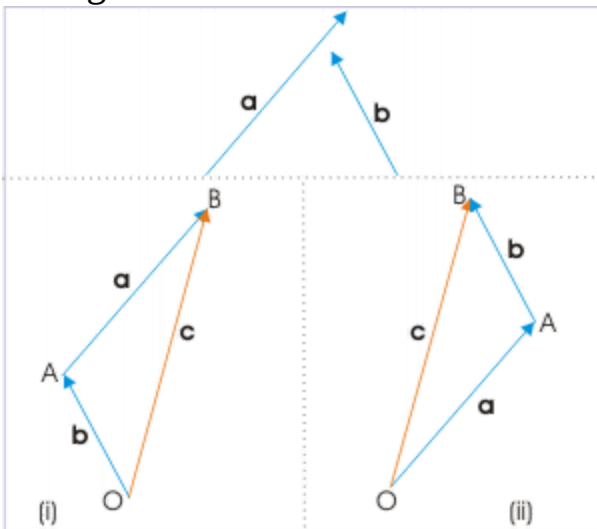
In most of the situations, we are involved with the addition of two vector quantities. Triangle law of vector addition is appropriate to deal with such situation.

Triangle law of vector addition

If two vectors are represented by two sides of a triangle in sequence, then third closing side of the triangle, in the opposite direction of the sequence, represents the sum (or resultant) of the two vectors in both magnitude and direction.

Here, the term “sequence” means that the vectors are placed such that tail of a vector begins at the arrow head of the vector placed before it.

Triangle law of vector addition



The triangle law does not restrict where to start i.e. with which vector to start. Also, it does not put conditions with regard to any specific direction for the sequence of vectors, like clockwise or anti-clockwise, to be maintained. In figure (i), the law is applied starting with vector, **b**; whereas

the law is applied starting with vector, **a**, in figure (ii). In either case, the resultant vector, **c**, is same in magnitude and direction.

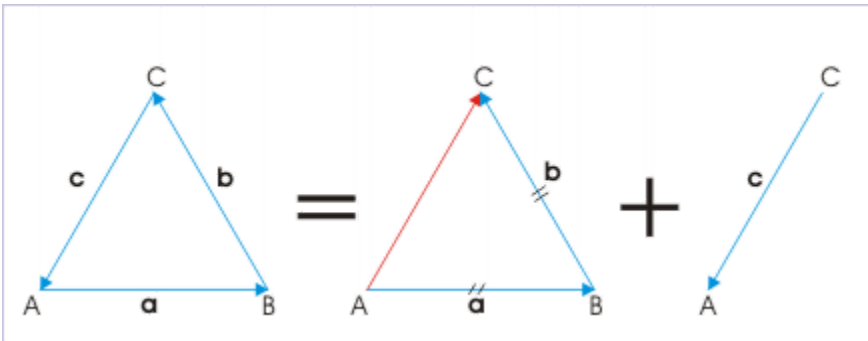
This is an important result as it conveys that vector addition is commutative in nature i.e. the process of vector addition is independent of the order of addition. This characteristic of vector addition is known as “commutative” property of vector addition and is expressed mathematically as :

Equation:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

If three vectors are represented by three sides of a triangle in sequence, then resultant vector is zero. In order to prove this, let us consider any two vectors in sequence like AB and BC as shown in the figure. According to triangle law of vector addition, the resultant vector is represented by the third closing side in the opposite direction. It means that :

Three vectors



Three vectors are represented by three sides in sequence.

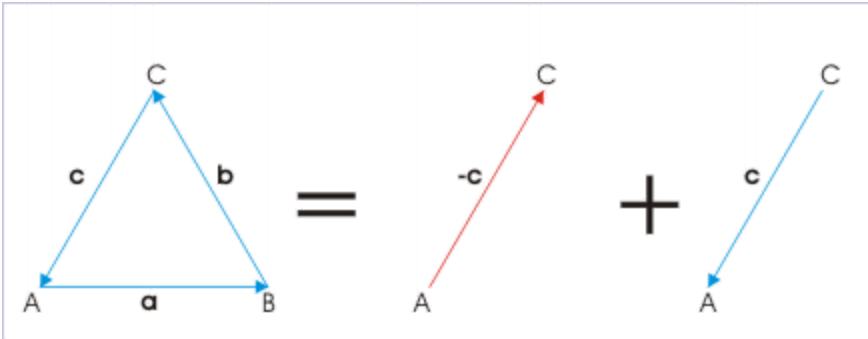
$$\Rightarrow \mathbf{AB} + \mathbf{BC} = \mathbf{AC}$$

Adding vector **CA** on either sides of the equation,

$$\Rightarrow \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = \mathbf{AC} + \mathbf{CA}$$

The right hand side of the equation is vector sum of two equal and opposite vectors, which evaluates to zero. Hence,

Three vectors



The resultant of three vectors represented by three sides is zero.

$$\Rightarrow \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$$

Note : If the vectors represented by the sides of a triangle are force vectors, then resultant force is zero. It means that three forces represented by the sides of a triangle in a sequence is a balanced force system.

Parallelogram law

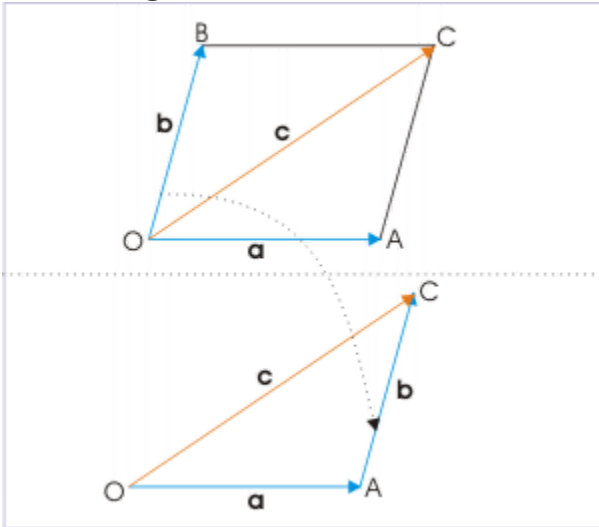
Parallelogram law, like triangle law, is applicable to two vectors.

Parallelogram law

If two vectors are represented by two adjacent sides of a parallelogram, then the diagonal of parallelogram through the common point represents the sum of the two vectors in both magnitude and direction.

Parallelogram law, as a matter of fact, is an alternate statement of triangle law of vector addition. A graphic representation of the parallelogram law and its interpretation in terms of the triangle is shown in the figure :

Parallelogram law

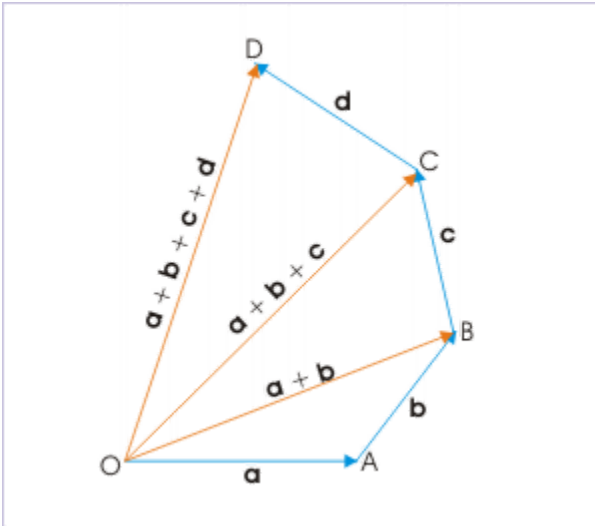


Converting parallelogram sketch to that of triangle law requires shifting vector, $\mathbf{b}$, from the position OB to position AC laterally as shown, while maintaining magnitude and direction.

Polygon law

The polygon law is an extension of earlier two laws of vector addition. It is successive application of triangle law to more than two vectors. A pair of vectors ($\mathbf{a}$, $\mathbf{b}$) is added in accordance with triangle law. The intermediate resultant vector ($\mathbf{a} + \mathbf{b}$) is then added to third vector ($\mathbf{c}$) again, successively till all vectors to be added have been exhausted.

Successive application of triangle law

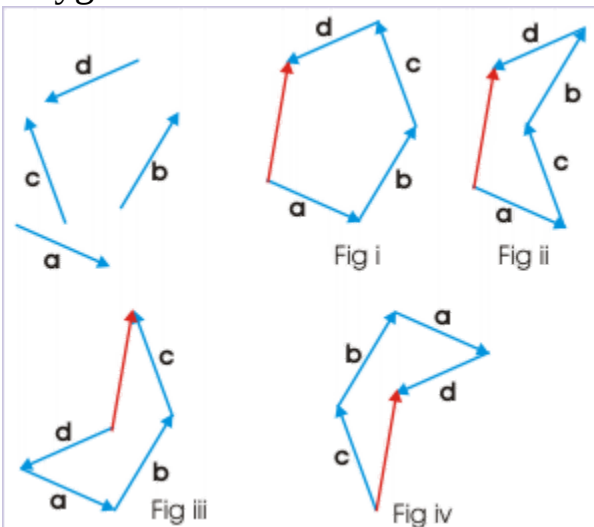


Polygon law

Polygon law of vector addition : If $(n-1)$ numbers of vectors are represented by $(n-1)$ sides of a polygon in sequence, then n^{th} side, closing the polygon in the opposite direction, represents the sum of the vectors in both magnitude and direction.

In the figure shown below, four vectors namely **a**, **b**, **c** and **d** are combined to give their sum. Starting with any vector, we add vectors in a manner that the subsequent vector begins at the arrow end of the preceding vector. The illustrations in figures i, iii and iv begin with vectors **a**, **d** and **c** respectively.

Polygon law



Matter of fact, polygon formation has great deal of flexibility. It may appear that we should elect vectors in increasing or decreasing order of direction (i.e. the angle the vector makes with reference to the direction of the first vector). But, this is not so. This point is demonstrated in figure (i) and (ii), in which the vectors **b** and **c** have simply been exchanged in their positions in the sequence without affecting the end result.

It means that the order of grouping of vectors for addition has no consequence on the result. This characteristic of vector addition is known as “associative” property of vector addition and is expressed mathematically as :

Equation:

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

Subtraction

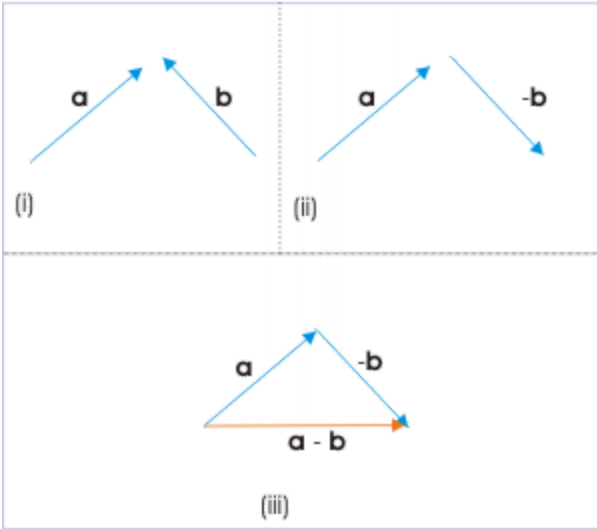
Subtraction is considered an addition process with one modification that the second vector (to be subtracted) is first reversed in direction and is then added to the first vector. To illustrate the process, let us consider the problem of subtracting vector, **b**, from , **a**. Using graphical techniques, we first reverse the direction of vector, **b**, and obtain the sum applying triangle or parallelogram law.

Symbolically,

Equation:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

Subtraction



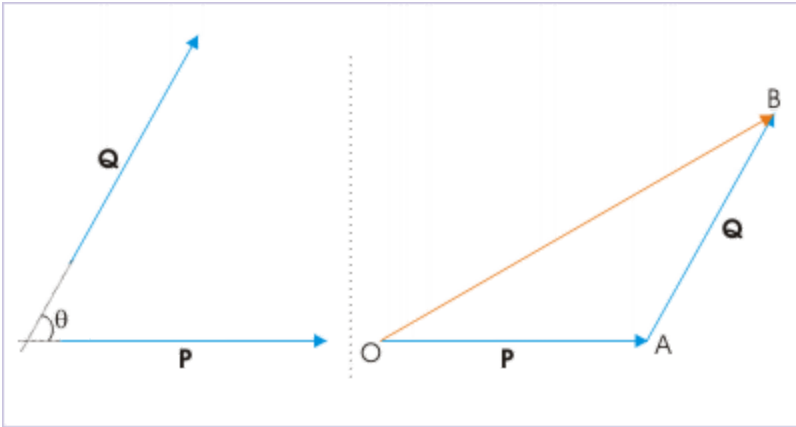
Similarly, we can implement subtraction using algebraic method by reversing sign of the vector being subtracted.

Vector addition : Analytical method

Vector method requires that all vectors be drawn true to the scale of magnitude and direction. The inherent limitation of the medium of drawing and measurement techniques, however, renders graphical method inaccurate. Analytical method, based on geometry, provides a solution in this regard. It allows us to accurately determine the sum or the resultant of the addition, provided accurate values of magnitudes and angles are supplied.

Here, we shall analyze vector addition in the form of triangle law to obtain the magnitude of the sum of the two vectors. Let P and Q be the two vectors to be added, which make an angle θ with each other. We arrange the vectors in such a manner that two adjacent sides OA and AB of the triangle OAB , represent two vectors P and Q respectively as shown in the figure.

Analytical method

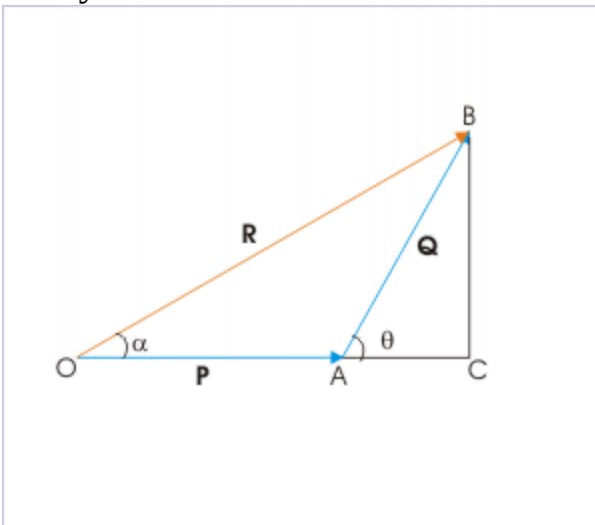


According to triangle law, the closing side OB represent sum of the vectors in both magnitude and direction.

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \mathbf{P} + \mathbf{Q}$$

In order to determine the magnitude, we drop a perpendicular BC on the extended line OC.

Analytical method



In $\triangle ACB$,

$$\mathbf{AC} = \mathbf{AB} \cos \theta = \mathbf{Q} \cos \theta$$

$$\mathbf{BC} = \mathbf{AB} \sin \theta = \mathbf{Q} \sin \theta$$

In right $\triangle OCB$, we have :

$$OB = \sqrt{OC^2 + BC^2} = \sqrt{\{(OA + AC)^2 + BC^2\}}$$

Substituting for AC and BC,

Equation:

$$\Rightarrow OB = \sqrt{(P + Q \cos \theta)^2 + Q^2 \sin^2 \theta}$$

$$\Rightarrow OB = \sqrt{P^2 + Q^2 \cos^2 \theta + 2PQ \cos \theta + Q^2 \sin^2 \theta}$$

$$\Rightarrow R = OB = \sqrt{P^2 + 2PQ \cos \theta + Q^2}$$

Let " α " be the angle that line OA makes with OC, then

$$\tan \alpha = \frac{BC}{OC} = \frac{Q \sin \theta}{P + Q \cos \theta}$$

The equations give the magnitude and direction of the sum of the vectors. The above equation reduces to a simpler form, when two vectors are perpendicular to each other. In that case, $\theta = 90^\circ$; $\sin \theta = \sin 90^\circ = 1$; $\cos \theta = \cos 90^\circ = 0$ and,

Equation:

$$\Rightarrow OB = \sqrt{P^2 + Q^2}$$

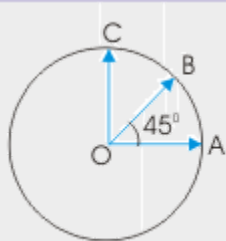
$$\Rightarrow \tan \alpha = \frac{Q}{P}$$

These results for vectors at right angle are exactly same as determined, using Pythagoras theorem.

Example:

Problem : Three radial vectors OA, OB and OC act at the center of a circle of radius "r" as shown in the figure. Find the magnitude of resultant vector.

Sum of three vectors



Three radial vectors OA, OB and OC act at the center of a circle of radius “r”.

Solution : It is evident that vectors are equal in magnitude and is equal to the radius of the circle. The magnitude of the resultant of horizontal and vertical vectors is :

$$R' = \sqrt{(r^2 + r^2)} = \sqrt{2}r$$

The resultant of horizontal and vertical vectors is along the bisector of angle i.e. along the remaining third vector OB. Hence, magnitude of resultant of all three vectors is :

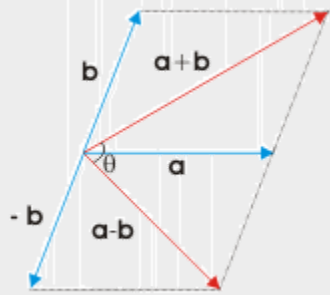
$$R' = OB + R' = r + \sqrt{2}r = (1 + \sqrt{2})r$$

Example:

Problem : At what angle does two vectors $\mathbf{a+b}$ and $\mathbf{a-b}$ act so that the resultant is $\sqrt{(3a^2 + b^2)}$.

Solution : The magnitude of resultant of two vectors is given by :

Angle



The angle between the sum and difference of vectors.

$$R = \sqrt{\left\{ (a+b)^2 + (a-b)^2 + 2(a+b)(a-b) \cos \theta \right\}}$$

Substituting the expression for magnitude of resultant as given,

$$\Rightarrow \sqrt{(3a^2 + b^2)} = \sqrt{\left\{ (a+b)^2 + (a-b)^2 + 2(a+b)(a-b) \cos \theta \right\}}$$

Squaring on both sides, we have :

$$\Rightarrow (3a^2 + b^2) = \left\{ (a+b)^2 + (a-b)^2 + 2(a+b)(a-b) \cos \theta \right\}$$

$$\Rightarrow \cos \theta = \frac{(a^2 - b^2)}{2(a^2 - b^2)} = \frac{1}{2} = \cos 60^\circ$$

$$\Rightarrow \theta = 60^\circ$$

Nature of vector addition

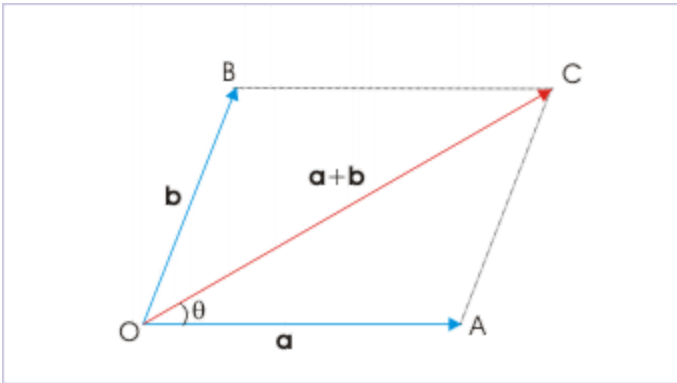
Vector sum and difference

The magnitude of sum of two vectors is either less than or equal to sum of the magnitudes of individual vectors. Symbolically, if **a** and **b** be two vectors, then

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

We know that vectors **a**, **b** and their sum **a+b** is represented by three side of a triangle OAC. Further we know that a side of triangle is always less than the sum of remaining two sides. It means that :

Two vectors



Sum of two vectors

$$\begin{aligned} OC &< OA + AC \\ OC &< OA + OB \\ \Rightarrow |\mathbf{a} + \mathbf{b}| &< |\mathbf{a}| + |\mathbf{b}| \end{aligned}$$

There is one possibility, however, that two vectors **a** and **b** are collinear and act in the same direction. In that case, magnitude of their resultant will be "equal to" the sum of the magnitudes of individual vector. This magnitude represents the maximum or greatest magnitude of two vectors being combined.

$$\begin{aligned} OC &= OA + OB \\ \Rightarrow |\mathbf{a} + \mathbf{b}| &= |\mathbf{a}| + |\mathbf{b}| \end{aligned}$$

Combining two results, we have :

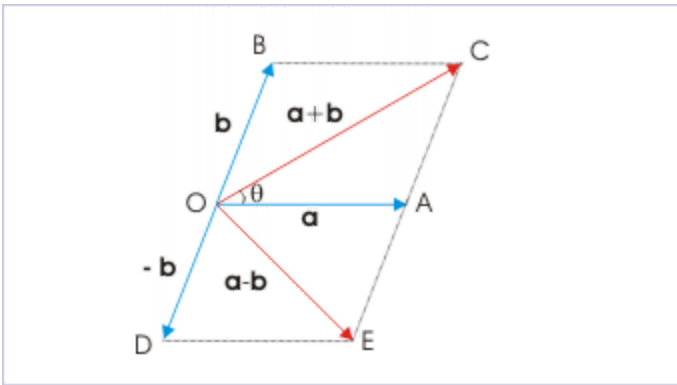
$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

On the other hand, the magnitude of difference of two vectors is either greater than or equal to difference of the magnitudes of individual vectors. Symbolically, if $\mathbf{a}$ and $\mathbf{b}$ be two vectors, then

$$|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$$

We know that vectors $\mathbf{a}$, $\mathbf{b}$ and their difference $\mathbf{a} - \mathbf{b}$ are represented by three side of a triangle OAE. Further we know that a side of triangle is always less than the sum of remaining two sides. It means that sum of two sides is greater than the third side :

Two vectors



Difference of two vectors

$$\begin{aligned} OE + AE &> OA \\ \Rightarrow OE &> OA - AE \\ \Rightarrow |\mathbf{a} - \mathbf{b}| &> |\mathbf{a}| - |\mathbf{b}| \end{aligned}$$

There is one possibility, however, that two vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear and act in the opposite directions. In that case, magnitude of their difference will be equal to the difference of the magnitudes of individual vector. This magnitude represents the minimum or least magnitude of two vectors being combined.

$$\Rightarrow \mathbf{OE} = \mathbf{OA} - \mathbf{AE}$$

$$\Rightarrow |\mathbf{a} - \mathbf{b}| = |\mathbf{a}| - |\mathbf{b}|$$

Combining two results, we have :

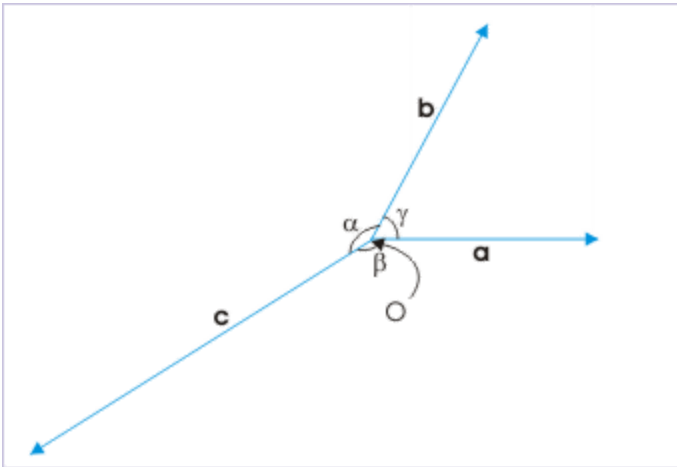
$$|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$$

Lami's theorem

Lami's theorem relates magnitude of three non-collinear vectors with the angles enclosed between pair of two vectors, provided resultant of three vectors is zero (null vector). This theorem is a manifestation of triangle law of addition. According to this theorem, if resultant of three vectors **a**, **b** and **c** is zero (null vector), then

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

Three vectors

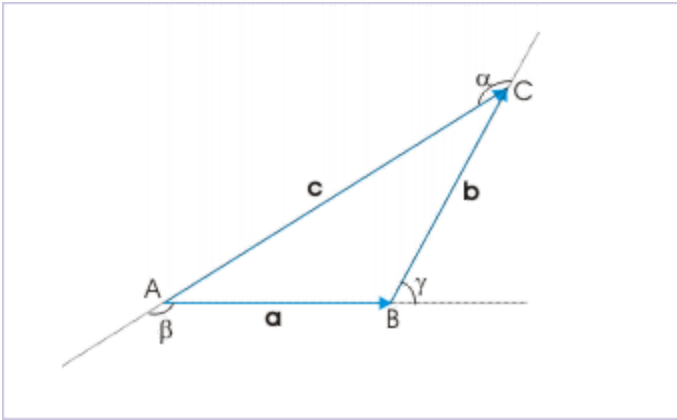


Three non-collinear vectors.

where α , β and γ be the angle between the remaining pairs of vectors.

We know that if the resultant of three vectors is zero, then they are represented by three sides of a triangle in magnitude and direction.

Three vectors



Three vectors are represented by three sides of a triangle.

Considering the magnitude of vectors and applying sine law of triangle, we have :

$$\begin{aligned}\frac{AB}{\sin BCA} &= \frac{BC}{\sin CAB} = \frac{CA}{\sin ABC} \\ \Rightarrow \frac{AB}{\sin(\pi-\alpha)} &= \frac{BC}{\sin(\pi-\beta)} = \frac{CA}{\sin(\pi-\gamma)} \\ \Rightarrow \frac{AB}{\sin \alpha} &= \frac{BC}{\sin \beta} = \frac{CA}{\sin \gamma}\end{aligned}$$

It is important to note that the ratio involves exterior (outside) angles – not the interior angles of the triangle. Also, the angle associated with the magnitude of a vector in the individual ratio is the included angle between the remaining vectors.

Exercises

Exercise:

Problem:

Two forces of 10 N and 25 N are applied on a body. Find the magnitude of maximum and minimum resultant force.

Solution:

Resultant force is maximum when force vectors act along the same direction. The magnitude of resultant force under this condition is :

$$R_{\max} = 10 + 25 = 35 \text{ N}$$

$$R_{\min} = 25 - 10 = 15 \text{ N}$$

Exercise:**Problem:**

Can a body subjected to three coplanar forces 5 N, 17 N and 9 N be in equilibrium?

Solution:

The resultant force, on a body in equilibrium, is zero. It means that three forces can be represented along three sides of a triangle. However, we know that sum of any two sides is greater than third side. In this case, we see that :

$$5 + 9 < 17$$

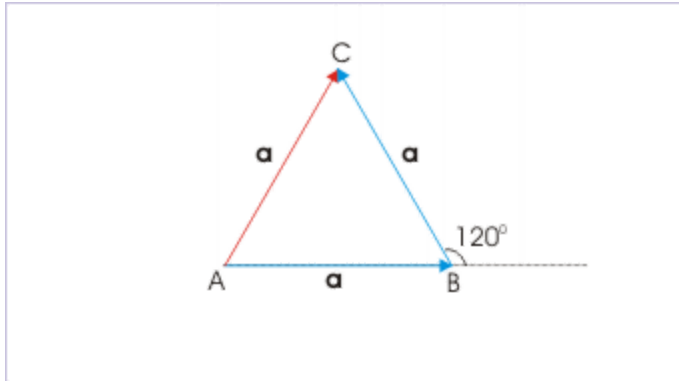
Clearly, three given forces can not be represented by three sides of a triangle. Thus, we conclude that the body is not in equilibrium.

Exercise:**Problem:**

Under what condition does the magnitude of the resultant of two vectors of equal magnitude, is equal in magnitude to either of two equal vectors?

Solution:

We know that resultant of two vectors is represented by the closing side of a triangle. If the triangle is equilateral then all three sides are equal. As such magnitude of the resultant of two vectors is equal to the magnitude of either of the two vectors.

Two vectors

Resultant of two vectors

Under this condition, vectors of equal magnitude make an angle of 120° between them.

Components of a vector

The concept of component of a vector is tied to the concept of vector sum. We have seen that the sum of two vectors represented by two sides of a triangle is given by a vector represented by the closing side (third) of the triangle in opposite direction. Importantly, we can analyze this process of summation of two vectors inversely. We can say that a single vector (represented by third side of the triangle) is equivalent to two vectors in two directions (represented by the remaining two sides).

We can generalize this inverse interpretation of summation process. We can say that a vector can always be considered equivalent to a pair of vectors. The law of triangle, therefore, provides a general frame work of resolution of a vector in two components in as many ways as we can draw triangle with one side represented by the vector in question. However, this general framework is not very useful. Resolution of vectors turns out to be meaningful, when we think resolution in terms of vectors at right angles. In that case, associated triangle is a right angle. The vector being resolved into components is represented by the hypotenuse and components are represented by two sides of the right angle triangle.

Resolution of a vector into components is an important concept for two reasons : (i) there are physical situations where we need to consider the effect of a physical vector quantity in specified direction. For example, we consider only the component of weight along an incline to analyze the motion of the block over it and (ii) the concept of components in the directions of rectangular axes, enable us to develop algebraic methods for vectors.

Resolution of a vector in two perpendicular components is an extremely useful technique having extraordinary implication. Mathematically, any vector can be represented by a pair of co-planar vectors in two perpendicular directions. It has, though, a deeper meaning with respect to physical phenomena and hence physical laws. Consider projectile motion for example. The motion of projectile in two dimensional plane is equivalent to two motions – one along the vertical and one along horizontal direction. We can accurately describe motion in any of these two directions independent of motion in the other direction! To some extent, this

independence of two component vector quantities from each other is a statement of physical law – which is currently considered as property of vector quantities and not a law by itself. But indeed, it is a law of great importance which we employ to study more complex physical phenomena and process.

Second important implication of resolution of a vector in two perpendicular directions is that we can represent a vector as two vectors acting along two axes of a rectangular coordinate system. This has far reaching consequence. It is the basis on which algebraic methods for vectors are developed – otherwise vector analysis is tied, by definition, to geometric construct and analysis. The representation of vectors along rectangular axes has the advantage that addition of vector is reduced to simple algebraic addition and subtraction as consideration along an axis becomes one dimensional. It is not difficult to realize the importance of the concept of vector components in two perpendicular directions. Resolution of vector quantities is the way we transform two and three dimensional phenomena into one dimensional phenomena along the axes.

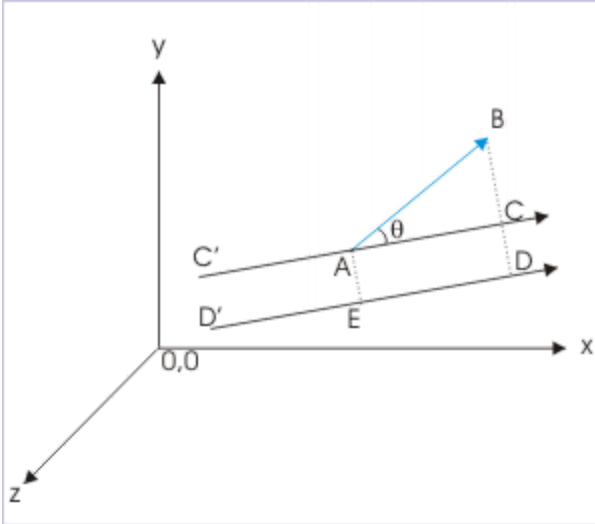
Components of a vector

A scalar component, also known as projection, of a vector **AB** in the positive direction of a straight line C'C is defined as :

$$AC = |\mathbf{AB}| \cos \theta = AB \cos \theta$$

Where θ is the angle that vector AB makes with the specified direction C'C as shown in the figure.

Analytical method



It should be understood that the component so determined is not rooted to the common point “A” or any specific straight line like $C'C$ in a given direction. Matter of fact, component of vector AB drawn on any straight lines like $D'D$ parallel to CC' are same. Note that the component i.e. projection of the vector is graphically obtained by drawing two perpendicular lines from the ends of the vector on the straight line in the specified direction.

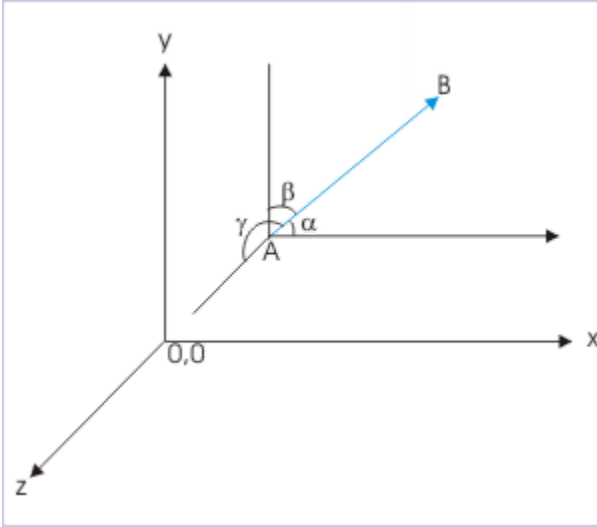
Equation:

$$ED = AC = AB \cos \theta$$

It is clear that scalar component can either be positive or negative depending on the value of angle that the vector makes with the referred direction. The angle lies between the range given by $0 \leq \theta \leq 180^\circ$. This interval means that we should consider the smaller angle between two vectors.

In accordance with the above definition, we resolve a given vector in three components in three mutually perpendicular directions of rectangular coordinate system. Note here that we measure angle with respect to parallel lines to the axes. By convention, we denote components by using the non-bold type face of the vector symbol with a suffix representing direction (x or y or z).

Components of a vector



$$AB_x = AB \cos \alpha$$

$$AB_y = AB \cos \beta$$

$$AB_z = AB \cos \gamma$$

Where α , β and γ are the angles that vector AB makes with the positive directions of x, y and z directions respectively.

As a vector can be resolved in a set of components, the reverse processing of components in a vector is also expected. A vector is composed from vector components in three directions along the axes of rectangular coordinate system by combining three component vectors. The vector component is the vector counterpart of the scalar component, which is obtained by multiplying the scalar component with the unit vector in axial direction. The vector components of a vector **a**, are $a_x \mathbf{i}$, $a_y \mathbf{j}$ and $a_z \mathbf{k}$.

A vector, **a**, is equal to the vector sum of component vectors in three mutually perpendicular directions of rectangular coordinate system.

Equation:

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$$

where **i**, **j** and **k** are unit vectors in the respective directions.

Equation:

$$\mathbf{a} = a \cos \alpha \mathbf{i} + a \cos \beta \mathbf{j} + a \cos \gamma \mathbf{k}$$

Magnitude of the sum of two vectors ($a_x \mathbf{i} + a_y \mathbf{j}$) in x and y direction, which are perpendicular to each other, is :

$$a' = \sqrt{(a_x^2 + a_y^2)} = (a_x^2 + a_y^2)^{\frac{1}{2}}$$

We observe here that resultant vector lies in the plane formed by the two component vectors being added. The resultant vector is, therefore, perpendicular to third component vector. Thus, the magnitude of the sum of vector ($a_x \mathbf{i} + a_y \mathbf{j}$) and third vector , $a_z \mathbf{k}$, which are perpendicular to each other, is :

Equation:

$$\Rightarrow a = \sqrt{\left[\left\{ (a_x^2 + a_y^2)^{\frac{1}{2}} \right\}^2 + a_z^2 \right]}$$

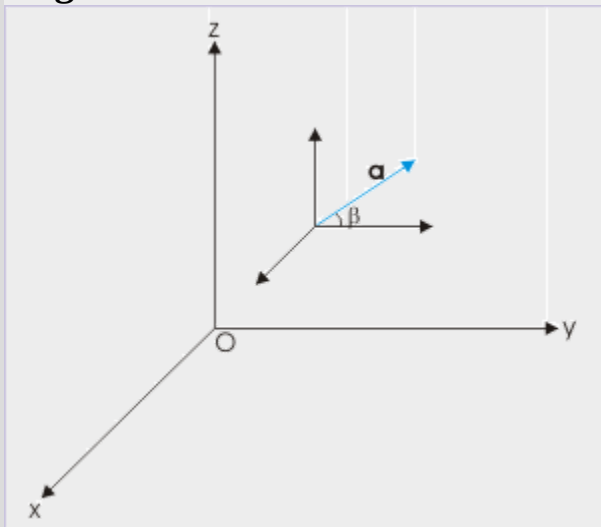
$$a = \sqrt{(a_x^2 + a_y^2 + a_z^2)}$$

Example:

Problem : Find the angle that vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ makes with y-axis.

Solution : We can answer this question with the help of expression for the cosine of angle that a vector makes with a given axis. We know that component along y-axis is :

Angle



Vector makes an angle with y-axis.

$$a_y = a \cos \beta$$
$$\Rightarrow \cos \beta = \frac{a_y}{a}$$

Here,

$$a_y = 1$$

and

$$a = \sqrt{2^2 + 1^2 + 1^2}$$

Hence,

$$\Rightarrow \cos \beta = \frac{a_y}{a} = \frac{1}{\sqrt{6}}$$

$$\Rightarrow \beta = \cos^{-1} \frac{1}{\sqrt{6}}$$

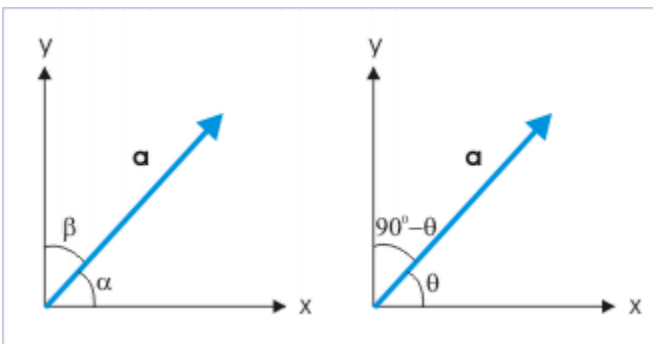
Planar components of a vector

The planar components of a vector lies in the plane of vector. Since there are two perpendicular axes involved with a plane, a vector is resolved in two components which lie in the same plane as that of vector. Clearly, a vector is composed of components in only two directions :

$$\mathbf{a} = a \cos \alpha \mathbf{i} + a \cos \beta \mathbf{j}$$

From the figure depicting a planar coordinate, it is clear that angle “ β ” is compliment of angle “ α ”. If $\alpha = \theta$, then

Planar vector



The direction of a planar vector with respect to rectangular axes can be described by a single angle.

$$\alpha = \theta \text{ and } \beta = 90^\circ - \theta$$

Putting in the expression for the vector,

Equation:

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$$

$$\mathbf{a} = a \cos \theta \mathbf{i} + a \sin \theta \mathbf{j}$$

$$\mathbf{a} = a \cos \theta \mathbf{i} + a \sin \theta \mathbf{j}$$

From graphical representation, the tangent of the angle that vector makes with x-axis is :

Equation:

$$\tan \alpha = \tan \theta = \frac{a \sin \theta}{a \cos \theta} = \frac{a_y}{a_x}$$

Similarly, the tangent of the angle that vector makes with y-axis is :

Equation:

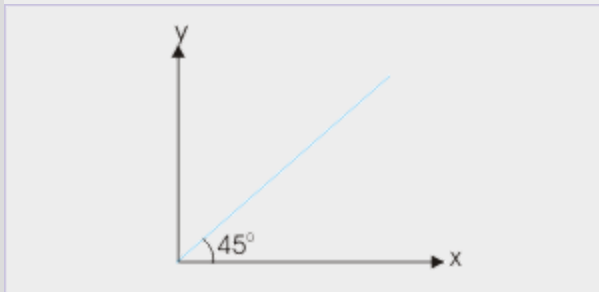
$$\tan \beta = \tan(90^\circ - \theta) = \cot \theta = \frac{a \cos \theta}{a \sin \theta} = \frac{a_x}{a_y}$$

Example:

Problem : Find the unit vector in the direction of a bisector of the angle between a pair of coordinate axes.

Solution : The unit vector along the direction of a bisector lies in the plane formed by two coordinates. The bisector makes an angle of 45° with either of the axes. Hence, required unit vector is :

Unit vector



Unit vector along the bisector.

$$\mathbf{n} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} = \cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}$$

$$\Rightarrow \mathbf{n} = \frac{1}{\sqrt{2}} \times (\mathbf{i} + \mathbf{j})$$

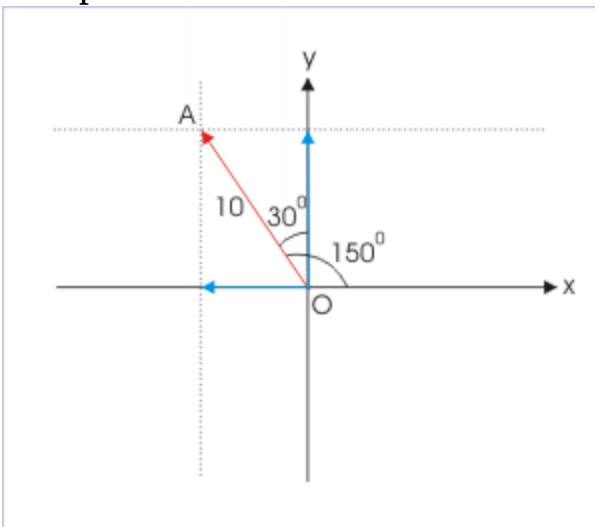
Note : We may check that the magnitude of the unit vector is indeed 1.

$$|\mathbf{n}| = n = \sqrt{\left\{ \left(\frac{1}{\sqrt{2}} \right)^2 + \left(\frac{1}{\sqrt{2}} \right)^2 \right\}} = \sqrt{1} = 1$$

Representation of a vector in component form

The angle involved in determination of components plays important role. There is a bit of ambiguity about angle being used. Actually, there are two ways to write a vector in component form. We shall illustrate these two methods with an illustration. Let us consider an example. Here, we consider a vector **OA** having magnitude of 10 units. The vector makes 150° and 30° angle with x and y axes as shown in the figure.

Component of a vector



Component of a vector

Using definition of components, the vector **OA** is represented as :

$$\mathbf{OA} = 10 \cos 150^\circ \mathbf{i} + 10 \cos 30^\circ \mathbf{j}$$

$$\Rightarrow \mathbf{OA} = 10X - \frac{1}{2}\mathbf{i} + 10X \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\Rightarrow \mathbf{OA} = -5\mathbf{i} + 5\sqrt{3}\mathbf{j}$$

Note that we use angles that vector makes with the axes to determine scalar components. We obtain corresponding component vectors by multiplying scalar components with respective unit vectors of the axes involved. We can, however, use another method in which we only consider acute angle – irrespective of directions of axes involved. Here, vector **OA** makes acute angle of 60° with x-axis. While representing vector, we put a negative sign if the component is opposite to the positive directions of axes. We can easily determine this by observing projection of vector on the axes. Following this :

$$\mathbf{OA} = -10 \cos 60^\circ \mathbf{i} + 10 \sin 60^\circ \mathbf{j}$$

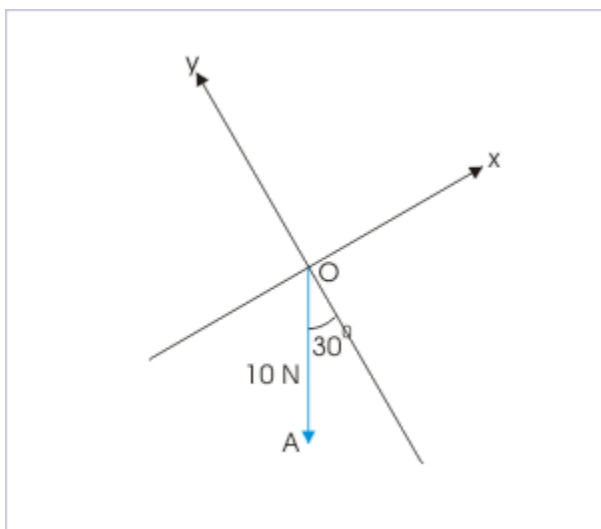
$$\Rightarrow \mathbf{OA} = -5\mathbf{i} + 5\sqrt{3}\mathbf{j}$$

Note that we put a negative sign before component along x-direction as projection of vector on x-axis is in opposite direction with respect to positive direction of x-axis. The y - projection, however, is in the positive direction of y-axis. As such, we do not need to put a negative sign before the component. Generally, people prefer second method as trigonometric functions are positive in first quadrant. We are not worried about the sign of trigonometric function at all.

Exercise:

Problem: Write force **OA** in component form.

Component of a vector



Component of a vector.

Solution:

$$F_x = 10 \sin 30^\circ = 10 \times \frac{1}{2} = 5 \quad N$$

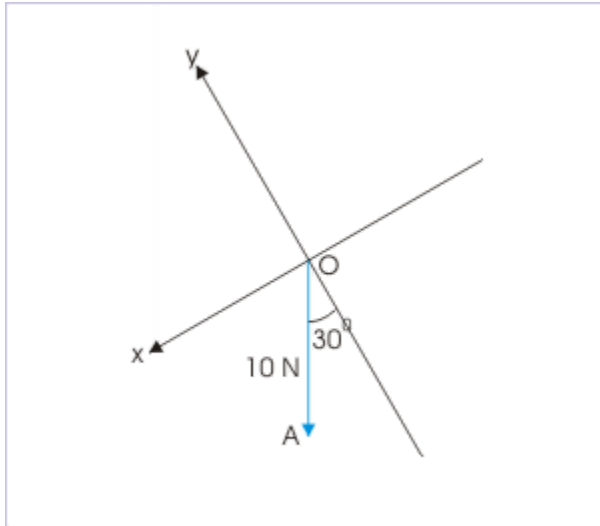
$$F_y = 10 \cos 30^\circ = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \quad N$$

$$\mathbf{F} = -F_x \mathbf{i} - F_y \mathbf{j} = -5\mathbf{i} - 5\sqrt{3}\mathbf{j} \quad N$$

Exercise:

Problem: Write force **OA** in component form.

Component of a vector



Component of a vector.

Solution:

$$F_x = 10 \sin 30^\circ = 10 \times \frac{1}{2} = 5 \quad N$$

$$F_y = 10 \cos 30^\circ = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \quad N$$

$$\mathbf{F} = F_x \mathbf{i} - F_y \mathbf{j} = 5\mathbf{i} - 5\sqrt{3}\mathbf{j} \quad N$$

Vector addition : Algebraic method

Graphical method is meticulous and tedious as it involves drawing of vectors on a scale and measurement of angles. More importantly, it does not allow algebraic operations that otherwise would give a simple solution. We can, however, extend algebraic techniques to vectors, **provided vectors are represented on a rectangular coordinate system**. The representation of a vector on coordinate system uses the concept of unit vector and component.

Now, the stage is set to design a frame work, which allows vector addition with algebraic methods. The frame work for vector addition draws on two important concepts. The first concept is that a vector can be equivalently expressed in terms of three component vectors :

$$\begin{aligned} \mathbf{a} &= a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \\ \mathbf{b} &= b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k} \end{aligned}$$

The component vector form has important significance. It ensures that component vectors to be added are restricted to three known directions only. This paradigm eliminates the possibility of unknown direction. The second concept is that vectors along a direction can be treated algebraically. If two vectors are along the same line, then resultant is given as :

When $\theta = 0^\circ$, $\cos\theta = \cos 0^\circ = 1$ and,

$$\Rightarrow R = \sqrt{P^2 + 2PQ + Q^2} = \sqrt{(P + Q)^2} = P + Q$$

When $\theta = 180^\circ$, $\cos\theta = \cos 180^\circ = -1$ and,

$$\Rightarrow R = \sqrt{P^2 - 2PQ + Q^2} = \sqrt{(P - Q)^2} = P - Q$$

Thus, we see that the magnitude of the resultant is equal to algebraic sum of the magnitudes of the two vectors.

Using these two concepts, the addition of vectors is affected as outlined here :

1: Represent the vectors (**a** and **b**) to be added in terms of components :

$$\begin{aligned}\mathbf{a} &= a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \\ \mathbf{b} &= b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}\end{aligned}$$

2: Group components in a given direction :

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} + b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k} \\ \Rightarrow \mathbf{a} + \mathbf{b} &= (a_x + b_x) \mathbf{i} + (a_y + b_y) \mathbf{j} + (a_z + b_z) \mathbf{k}\end{aligned}$$

3: Find the magnitude and direction of the sum, using analytical method

Equation:

$$\Rightarrow a = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2 + (a_z + b_z)^2}$$

Exercises

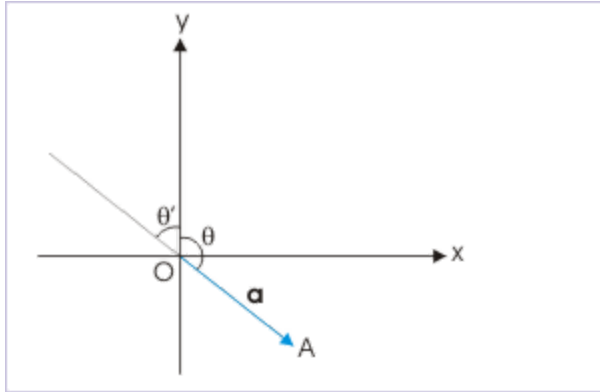
Exercise:

Problem: Find the angle that vector $\sqrt{3}\mathbf{i} - \mathbf{j}$ makes with y-axis.

Solution:

From graphical representation, the angle that vector makes with y-axis has following trigonometric function :

Angle



Vector makes an angle with y-axis.

$$\tan \theta = \frac{a_x}{a_y}$$

Now, we apply the formulae to find the angle, say θ , with y-axis,

$$\tan \theta = -\frac{\sqrt{3}}{1} = -\sqrt{3} = \tan 120^\circ$$

$$\Rightarrow \theta = 120^\circ$$

In case, we are only interested to know the magnitude of angle between vector and y-axis, then we can neglect the negative sign,

$$\tan \theta' = \sqrt{3} = \tan 60^\circ$$

$$\Rightarrow \theta' = 60^\circ$$

Exercise:**Problem:**

If a vector makes angles α, β and γ with x, y and z axes of a rectangular coordinate system, then prove that :

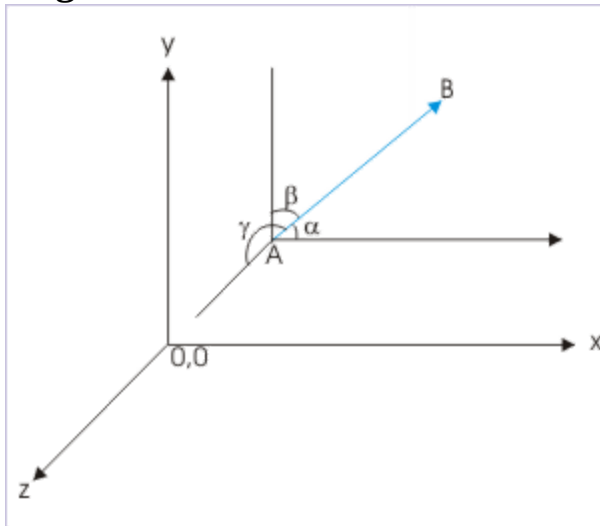
$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Solution:

The vector can be expressed in terms of its component as :

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$$

where $a_x = a \cos \alpha$; $a_y = a \cos \beta$; $a_z = a \cos \gamma$.

Angles

Vector makes different angles
with axes..

The magnitude of the vector is given by :

$$a = \sqrt{(a_x^2 + a_y^2 + a_z^2)}$$

Putting expressions of components in the equation,

$$\Rightarrow a = \sqrt{(a^2 \cos^2 \alpha + a^2 \cos^2 \beta + a^2 \cos^2 \gamma)}$$

$$\Rightarrow \sqrt{(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)} = 1$$

Squaring both sides,

$$\Rightarrow \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Exercise:

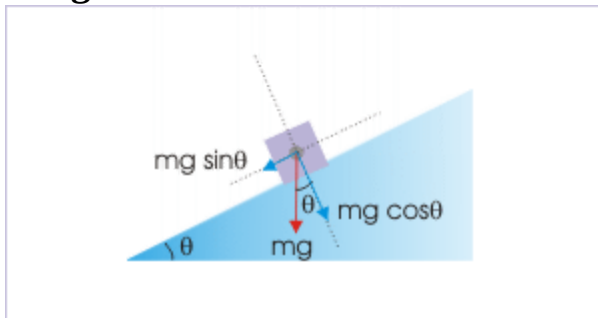
Problem:

Find the components of weight of a block along the incline and perpendicular to the incline.

Solution:

The component of the weight along the incline is :

Weight on an incline



Components of weight along the incline and perpendicular to the incline.

$$W_x = mg \sin \theta = 100 \times \sin 30^\circ = 100 \times \frac{1}{2} = 50 \text{ N}$$

and the component of weight perpendicular to incline is :

$$W_y = mg \cos \theta = 100 \times \cos 30^\circ = 100 \times \frac{\sqrt{3}}{2} = 50\sqrt{3} \text{ N}$$

Exercise:

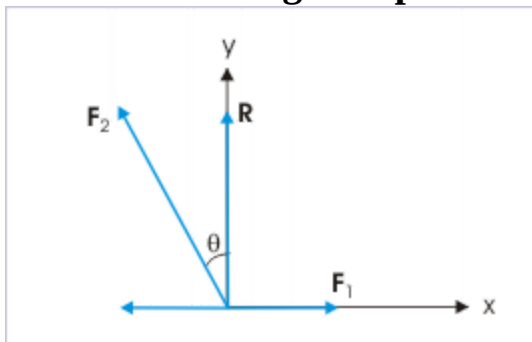
Problem:

The sum of magnitudes of two forces acting at a point is 16 N. If the resultant of the two forces is 8 N and it is normal to the smaller of the two forces, then find the forces.

Solution:

We depict the situation as shown in the figure. The resultant force R is shown normal to small force F_1 . In order that the sum of the forces is equal to R , the component of larger force along the x-direction should be equal to smaller force :

Two forces acting on a point



The resultant force is perpendicular to smaller of the two forces.

$$F_2 \sin \theta = F_1$$

Also, the component of the larger force along y-direction should be equal to the magnitude of resultant,

$$F_2 \cos \theta = R = 8$$

Squaring and adding two equations, we have :

$$F_2^2 = F_1^2 + 64$$

$$\Rightarrow F_2^2 - F_1^2 = 64$$

However, according to the question,

$$\Rightarrow F_1 + F_2 = 16$$

Substituting, we have :

$$\Rightarrow F_2 - F_1 = \frac{64}{16} = 4$$

Solving,

$$\Rightarrow F_1 = 6 \text{ N}$$

$$\Rightarrow F_2 = 10 \text{ N}$$

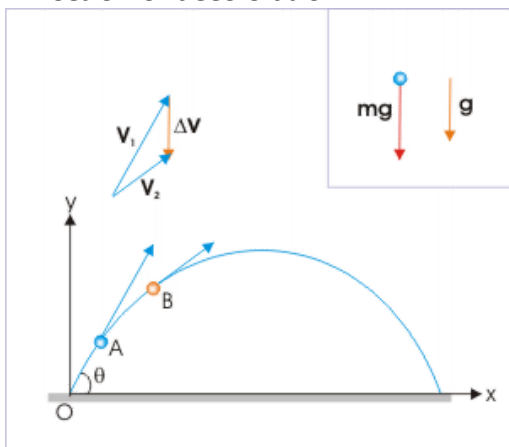
More illustrations on the subject are available in the module titled [Resolution of forces](#)

Scalar (dot) product

In physics, we require to multiply a vector with other scalar and vector quantities. The vector multiplication, however, is not an unique mathematical construct like scalar multiplication. The multiplication depends on the nature of quantities (vector or scalar) and on the physical process, necessitating scalar or vector multiplication.

The rules of vector multiplication have been formulated to encapsulate physical processes in their completeness. This is the core consideration. In order to explore this aspect, let us find out the direction of acceleration in the case of parabolic motion of a particle. There may be two ways to deal with the requirement. We may observe the directions of velocities at two points along the path and find out the direction of the change of velocities. Since we know that the direction of change of velocity is the direction of acceleration, we draw the vector diagram and find out the direction of acceleration. We can see that the direction of acceleration turns out to act in vertically downward direction.

Direction of acceleration



The conceptualization of physical laws in vector form, however, provides us with powerful means to arrive at the result in relatively simpler manner. If we look at the flight of particle in parabolic motion, then we observe that the motion of particle is under the force of gravity, which is acting vertically downward. There is no other force (neglecting air resistance). Now, from second law of motion, we know that :

$$\mathbf{F}_{\text{Resultant}} = m\mathbf{a}$$

This equation reveals that the direction of acceleration is same as that of the resultant force acting on the particle. Thus, acceleration of the particle in parabolic motion is acting vertically downward. We see that this second approach is more elegant of the two methods. We could arrive at the correct answer in a very concise manner, without getting into the details of the motion. It is possible, because Newton's second law in vector form states that net force on the body is product of acceleration vector with scalar mass. As multiplication of scalar with a vector does not change the direction of resultant vector, we conclude that direction of acceleration is same as that of net force acting on the projectile.

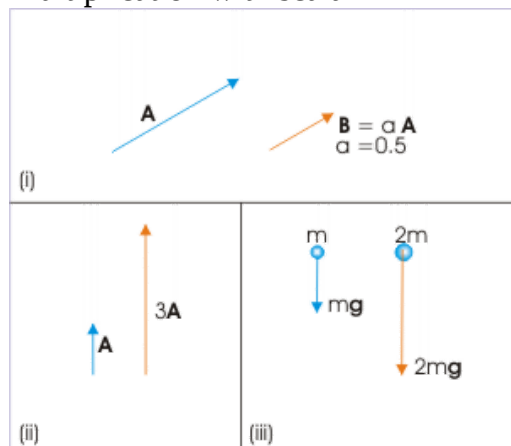
Multiplication with scalar

Multiplication of a vector, **A**, with another scalar quantity, a , results in another vector, **B**. The magnitude of the resulting vector is equal to the product of the magnitude of vector with the scalar quantity. The direction of the resulting vector, however, is same as that of the original vector (See Figures below).

Equation:

$$\mathbf{B} = a\mathbf{A}$$

Multiplication with scalar



We have already made use of this type of multiplication intuitively in expressing a vector in component form.

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

In this vector representation, each component vector is obtained by multiplying the scalar component with the unit vector. As the unit vector has the magnitude of 1 with a specific direction, the resulting component vector retains the magnitude of the scalar component, but acquires the direction of unit vector.

Equation:

$$\mathbf{A}_x = A_x\mathbf{i}$$

Products of vectors

Some physical quantities are themselves a scalar quantity, but are composed from the product of vector quantities. One such example is “work”. On the other hand, there are physical quantities like torque and magnetic force on a moving charge, which are themselves vectors and are also composed from vector quantities.

Thus, products of vectors are defined in two distinct manner – one resulting in a scalar quantity and the other resulting in a vector quantity. The product that results in scalar value is scalar product, also known as dot product as a "dot" (.) is the symbol of operator for this product. On the other hand, the product that results in vector value is vector product, also known as cross product as a "cross" ($\times$) is the symbol of operator for this product. We shall discuss scalar product only in this module. We shall cover vector product in a separate module.

Scalar product (dot product)

Scalar product of two vectors **a** and **b** is a scalar quantity defined as :

Equation:

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

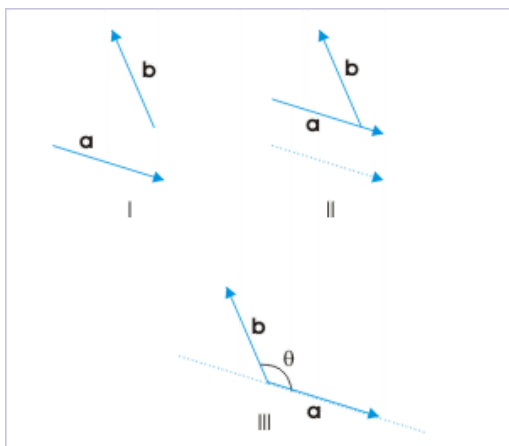
where “a” and “b” are the magnitudes of two vectors and “ θ ” is the angle between the direction of two vectors. It is important to note that vectors have two angles θ and $2\pi - \theta$. We can use either of them as cosine of both “ θ ” and “ $2\pi - \theta$ ” are same. However, it is suggested to use the smaller of the enclosed angles to be consistent with cross product in which it is required to use the smaller of the enclosed angles. This approach will maintain consistency with regard to enclosed angle in two types of vector multiplications.

The notation “**a.b**” is important and should be mentally noted to represent a scalar quantity – even though it involves bold faced vectors. It should be noted that the quantity on the right hand side of the equation is a scalar.

Angle between vectors

The angle between vectors is measured with precaution. The direction of vectors may sometimes be misleading. The basic consideration is that it is the angle between vectors at the common point of intersection. This intersection point, however, should be the common tail of vectors. If required, we may be required to shift the vector parallel to it or along its line of action to obtain common point at which tails of vectors meet.

Angle between vectors



Angle between vectors

See the steps shown in the figure. First, we need to shift one of two vectors say, **a** so that it touches the tail of vector **b**. Second, we move vector **a** along its line of action till tails of two vectors meet at the common point. Finally, we measure the angle θ such that $0 \leq \theta \leq \pi$.

Meaning of scalar product

We can read the definition of scalar product in either of the following manners :

Equation:

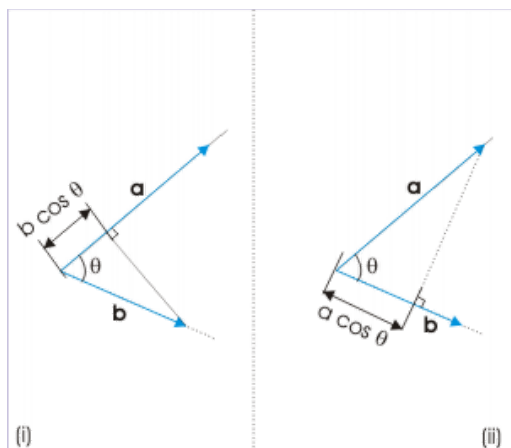
$$\mathbf{a} \cdot \mathbf{b} = a(b \cos \theta)$$

$$\mathbf{a} \cdot \mathbf{b} = b(a \cos \theta)$$

Recall that “ $b \cos \theta$ ” is the scalar component of vector **b** along the direction of vector **a** and “ $a \cos \theta$ ” is the scalar component of vector **a** along the direction of vector **b**. Thus, we may consider the scalar product of vectors **a** and **b** as the product of the magnitude of one vector and the scalar component of other vector along the first vector.

The figure below shows drawing of scalar components. The scalar component of vector in figure (i) is obtained by drawing perpendicular from the tip of the vector, **b**, on the direction of vector, **a**. Similarly, the scalar component of vector in figure (ii) is obtained by drawing perpendicular from the tip of the vector, **a**, on the direction of vector, **b**.

Scalar product



The two alternate ways of evaluating dot product of two vectors indicate that the product is commutative i.e. independent of the order of two vectors :

Equation:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

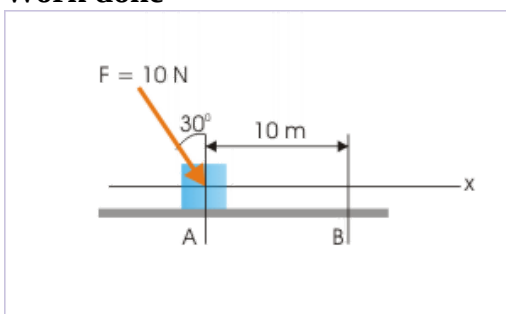
Exercise:

Problem:

A block of mass “m” moves from point A to B along a smooth plane surface under the action of force as shown in the figure. Find the work done if it is defined as :

$$W = \mathbf{F} \cdot \Delta \mathbf{x}$$

Work done

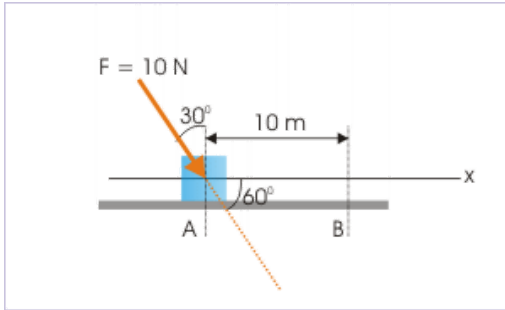


where $\mathbf{F}$ and $\Delta \mathbf{x}$ are force and displacement vectors.

Solution:

Expanding the expression of work, we have :

Work done



$$W = \mathbf{F} \cdot \Delta \mathbf{x} = F \Delta x \cos \theta$$

Here, $F = 10 \text{ N}$, $\Delta x = 10 \text{ m}$ and $\cos \theta = \cos 60^\circ = 0.5$.

$$\Rightarrow W = 10 \times 10 \times 0.5 = 50 \text{ J}$$

Values of scalar product

The value of dot product is maximum for the maximum value of $\cos \theta$. Now, the maximum value of cosine is $\cos 0^\circ = 1$. For this value, dot product simply evaluates to the product of the magnitudes of two vectors.

$$(\mathbf{a} \cdot \mathbf{b})_{\max} = ab$$

For $\theta = 180^\circ$, $\cos 180^\circ = -1$ and

$$\mathbf{a} \cdot \mathbf{b} = -ab$$

Thus, we see that dot product can evaluate to negative value as well. This is a significant result as many scalar quantities in physics are given negative value. The work done, for example, can be negative, when displacement is in the opposite direction to the component of force along that direction.

The scalar product evaluates to zero for $\theta = 90^\circ$ and 270° as cosine of these angles are zero. These results have important implication for unit vectors. The dot products of same unit vector evaluates to 1.

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

The dot products of combination of different unit vectors evaluate to zero.

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

Example:**Problem :** Find the angle between vectors $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{k}$.**Solution :** The cosine of the angle between two vectors is given in terms of dot product as :

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab}$$

Now,

$$\mathbf{a} \cdot \mathbf{b} = (2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - \mathbf{k})$$

Ignoring dot products of different unit vectors (they evaluate to zero), we have :

$$\mathbf{a} \cdot \mathbf{b} = 2\mathbf{i} \cdot \mathbf{i} + (-\mathbf{k}) \cdot (-\mathbf{k}) = 2 + 1 = 3$$

$$a = \sqrt{(2^2 + 1^2 + 1^2)} = \sqrt{6}$$

$$b = \sqrt{(1^2 + 1^2)} = \sqrt{2}$$

$$ab = \sqrt{6} \times \sqrt{2} = \sqrt{(12)} = 2\sqrt{3}$$

Putting in the expression of cosine, we have :

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} = \cos 30^\circ$$

$$\theta = 30^\circ$$

Scalar product in component form

Two vectors in component forms are written as :

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

In evaluating the product, we make use of the fact that multiplication of the same unit vectors is 1, while multiplication of different unit vectors is zero. The dot product evaluates to scalar terms as :

Equation:

$$\mathbf{a} \cdot \mathbf{b} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \cdot (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k})$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{b} = a_x\mathbf{i} \cdot b_x\mathbf{i} + a_y\mathbf{j} \cdot b_y\mathbf{j} + a_z\mathbf{k} \cdot b_z\mathbf{k}$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{b} = a_xb_x + a_yb_y + a_zb_z$$

Component as scalar (dot) product

A closer look at the expansion of dot product of two vectors reveals that the expression is very similar to the expression for a component of a vector. The expression of the dot product is :

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

On the other hand, the component of a vector in a given direction is :

$$a_x = a \cos \theta$$

Comparing two equations, we can define component of a vector in a direction given by unit vector " $\mathbf{n}$ " as :

Equation:

$$a_x = \mathbf{a} \cdot \mathbf{n} = a \cos \theta$$

This is a very general and useful relation to determine component of a vector in any direction. Only requirement is that we should know the unit vector in the direction in which component is to be determined.

Example:

Problem : Find the components of vector $2\mathbf{i} + 3\mathbf{j}$ along the direction $\mathbf{i} + \mathbf{j}$.

Solution : The component of a vector " $\mathbf{a}$ " in a direction, represented by unit vector " $\mathbf{n}$ " is given by dot product :

$$a_n = \mathbf{a} \cdot \mathbf{n}$$

Thus, it is clear that we need to find the unit vector in the direction of $\mathbf{i} + \mathbf{j}$. Now, the unit vector in the direction of the vector is :

$$n = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|}$$

Here,

$$|\mathbf{i} + \mathbf{j}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Hence,

$$n = \frac{1}{\sqrt{2}} \times (\mathbf{i} + \mathbf{j})$$

The component of vector $2\mathbf{i} + 3\mathbf{j}$ in the direction of “ $\mathbf{n}$ ” is :

$$\begin{aligned}a_n &= \mathbf{a} \cdot \mathbf{n} = (2\mathbf{i} + 3\mathbf{j}) \cdot \frac{1}{\sqrt{2}} \times (\mathbf{i} + \mathbf{j}) \\ \Rightarrow a_n &= \frac{1}{\sqrt{2}} \times (2\mathbf{i} + 3\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) \\ \Rightarrow a_n &= \frac{1}{\sqrt{2}} \times (2 \times 1 + 3 \times 1) \\ \Rightarrow a_n &= \frac{5}{\sqrt{2}}\end{aligned}$$

Attributes of scalar (dot) product

In this section, we summarize the properties of dot product as discussed above. Besides, some additional derived attributes are included for reference.

1: Dot product is commutative

This means that the dot product of vectors is not dependent on the sequence of vectors :

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

We must, however, be careful while writing sequence of dot product. For example, writing a sequence involving three vectors like $\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$ is incorrect. For, dot product of any two vectors is a scalar. As dot product is defined for two vectors (not one vector and one scalar), the resulting dot product of a scalar ($\mathbf{a} \cdot \mathbf{b}$) and that of third vector $\mathbf{c}$ has no meaning.

2: Distributive property of dot product :

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

3: The dot product of a vector with itself is equal to the square of the magnitude of the vector.

$$\mathbf{a} \cdot \mathbf{a} = a \times a \cos \theta = a^2 \cos 0^\circ = a^2$$

4: The magnitude of dot product of two vectors can be obtained in either of the following manner :

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta = a \times (b \cos \theta) = a \times \text{component of } \mathbf{b} \text{ along } \mathbf{a}$$

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta = (a \cos \theta) \times b = b \times \text{component of } \mathbf{a} \text{ along } \mathbf{b}$$

The dot product of two vectors is equal to the algebraic product of magnitude of one vector and component of second vector in the direction of first vector.

5: The cosine of the angle between two vectors can be obtained in terms of dot product as :

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

$$\Rightarrow \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab}$$

6: The condition of two perpendicular vectors in terms of dot product is given by :

$$\mathbf{a} \cdot \mathbf{b} = ab \cos 90^\circ = 0$$

7: Properties of dot product with respect to unit vectors along the axes of rectangular coordinate system are :

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

8: Dot product in component form is :

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

9: The dot product does not yield to cancellation. For example, if $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then we can not conclude that $\mathbf{b} = \mathbf{c}$. Rearranging, we have :

$$\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} = 0$$

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$$

This means that $\mathbf{a}$ and $(\mathbf{b} - \mathbf{c})$ are perpendicular to each other. In turn, this implies that $(\mathbf{b} - \mathbf{c})$ is not equal to zero (null vector). Hence, $\mathbf{b}$ is not equal to $\mathbf{c}$ as we would get after cancellation.

We can understand this difference with respect to cancellation more explicitly by working through the problem given here :

Example:

Problem : Verify vector equality $\mathbf{B} = \mathbf{C}$, if $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$.

Solution : The given equality of dot products is :

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$$

We should understand that dot product is not a simple algebraic product of two numbers (read magnitudes). The angle between two vectors plays a role in determining the magnitude of the dot product. Hence, it is entirely possible that vectors **B** and **C** are different yet their dot products with common vector **A** are equal. Let θ_1 and θ_2 be the angles for first and second pairs of dot products. Then,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$$

$$AB \cos \theta_1 = AC \cos \theta_2$$

If $\theta_1 = \theta_2$, then $B = C$. However, if $\theta_1 \neq \theta_2$, then $B \neq C$.

Law of cosine and dot product

Law of cosine relates sides of a triangle with one included angle. We can determine this relationship using property of a dot product. Let three vectors are represented by sides of the triangle such that closing side is the sum of other two vectors. Then applying triangle law of addition :

Cosine law

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Cosine law

$$\mathbf{c} = (\mathbf{a} + \mathbf{b})$$

We know that the dot product of a vector with itself is equal to the square of the magnitude of the vector. Hence,

$$c^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$$

$$c^2 = a^2 + 2ab \cos \theta + b^2$$

$$c^2 = a^2 + 2ab \cos(\pi - \varphi) + b^2$$

$$c^2 = a^2 - 2ab \cos \varphi + b^2$$

This is known as cosine law of triangle. Curiously, we may pay attention to first two equations above. As a matter of fact, second equation gives the square of the magnitude of resultant of two vectors **a** and **b**.

Differentiation and dot product

Differentiation of a vector expression yields a vector. Consider a vector expression given as :

$$\mathbf{a} = (x^2 + 2x + 3)\mathbf{i}$$

The derivative of the vector with respect to x is :

$$\mathbf{a}' = (2x + 2)\mathbf{i}$$

As the derivative is a vector, two vector expressions with dot product is differentiated in a manner so that dot product is retained in the final expression of derivative. For example,

$$\frac{d}{dx}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a}' \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}'$$

Exercises

Exercise:

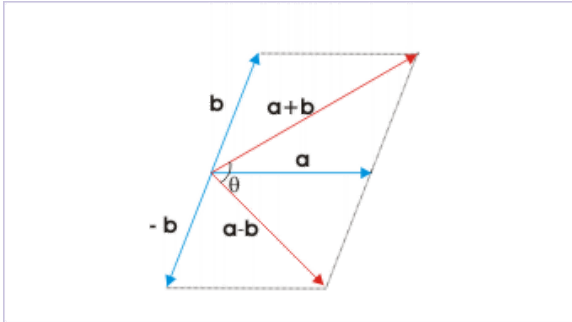
Problem:

Sum and difference of two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other. Find the relation between two vectors.

Solution:

The sum $\mathbf{a} + \mathbf{b}$ and difference $\mathbf{a} - \mathbf{b}$ are perpendicular to each other. Hence, their dot product should evaluate to zero.

Sum and difference of two vectors



Sum and difference of two vectors are perpendicular to each other.

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$$

Using distributive property,

$$\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = 0$$

Using commutative property, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, Hence,

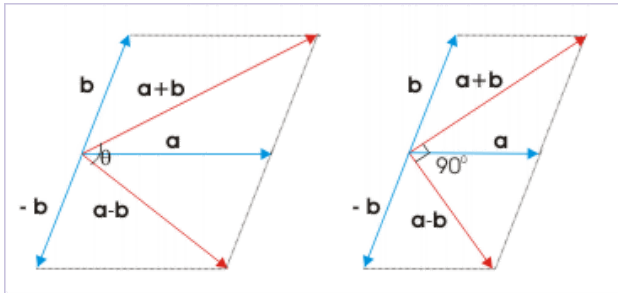
$$\mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = 0$$

$$a^2 - b^2 = 0$$

$$a = b$$

It means that magnitudes of two vectors are equal. See figure below for enclosed angle between vectors, when vectors are equal :

Sum and difference of two vectors



Sum and difference of two vectors are perpendicular to each other, when vectors are equal.

Exercise:

Problem: If $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, then find the angle between vectors $\mathbf{a}$ and $\mathbf{b}$.

Solution:

A question that involves modulus or magnitude of vector can be handled in specific manner to find information about the vector (s). The specific identity that is used in this circumstance is :

$$\mathbf{A} \cdot \mathbf{A} = A^2$$

We use this identity first with the sum of the vectors $(\mathbf{a} + \mathbf{b})$,

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a} + \mathbf{b}|^2$$

Using distributive property,

$$\Rightarrow \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = a^2 + b^2 + 2ab \cos \theta = |\mathbf{a} + \mathbf{b}|^2$$

$$\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = a^2 + b^2 + 2ab \cos \theta$$

Similarly, using the identity with difference of the vectors (a-b),

$$\Rightarrow |\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab \cos \theta$$

It is, however, given that :

$$\Rightarrow |\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$$

Squaring on either side of the equation,

$$\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$$

Putting the expressions,

$$\Rightarrow a^2 + b^2 + 2ab \cos \theta = a^2 + b^2 - 2ab \cos \theta$$

$$\Rightarrow 4ab \cos \theta = 0$$

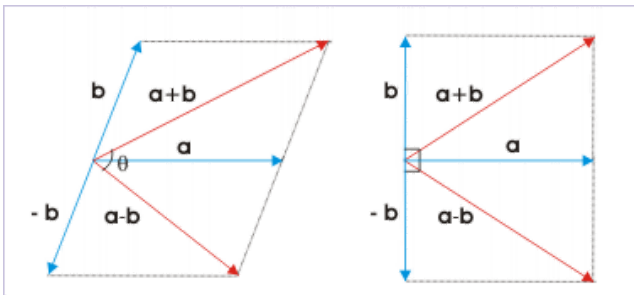
$$\Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = 90^\circ$$

Note : We can have a mental picture of the significance of this result. As given, the magnitude of sum of two vectors is equal to the magnitude of difference of two vectors. Now, we know that difference of vectors is similar to vector sum with one exception that one of the operand is rendered negative. Graphically, it means that one of the vectors is reversed.

Reversing one of the vectors changes the included angle between two vectors, but do not change the magnitudes of either vector. It is, therefore, only the included angle between the vectors that might change the magnitude of resultant. In order that magnitude of resultant does not change even after reversing direction of one of the vectors, it is required that the included angle between the vectors is not changed. This is only possible, when included angle between vectors is 90° . See figure.

Sum and difference of two vectors



Magnitudes of Sum and difference of two vectors are same when vectors at right angle to each other.

Exercise:

Problem:

If $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear unit vectors and $|\mathbf{a} + \mathbf{b}| = \sqrt{3}$, then find the value of expression :

$$(\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b})$$

Solution:

The given expression is scalar product of two vector sums. Using distributive property we can expand the expression, which will comprise of scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$.

$$(\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = 2\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot 2\mathbf{a} + (-\mathbf{b}) \cdot (-\mathbf{b}) = 2a^2 - \mathbf{a} \cdot \mathbf{b} - b^2$$

$$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = 2a^2 - b^2 - ab \cos \theta$$

We can evaluate this scalar product, if we know the angle between them as magnitudes of unit vectors are each 1. In order to find the angle between the vectors, we use the identity,

$$\mathbf{A} \cdot \mathbf{A} = A^2$$

Now,

$$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = a^2 + b^2 + 2ab \cos \theta = 1 + 1 + 2 \times 1 \times 1 \times \cos \theta$$

$$\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 2 + 2 \cos \theta$$

It is given that :

$$|\mathbf{a} + \mathbf{b}|^2 = (\sqrt{3})^2 = 3$$

Putting this value,

$$\Rightarrow 2 \cos \theta = |\mathbf{a} + \mathbf{b}|^2 - 2 = 3 - 2 = 1$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ$$

Using this value, we now proceed to find the value of given identity,

$$(\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = 2a^2 - b^2 - ab \cos \theta = 2 \times 1^2 - 1^2 - 1 \times 1 \times \cos 60^\circ$$

$$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = \frac{1}{2}$$

Exercise:

Problem:

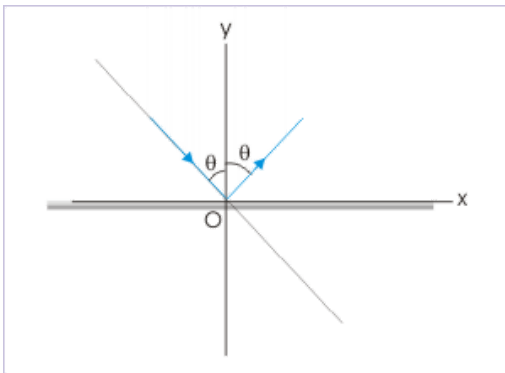
In an experiment of light reflection, if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the unit vectors in the direction of incident ray, reflected ray and normal to the reflecting surface, then prove that :

$$\Rightarrow \mathbf{b} = \mathbf{a} - 2(\mathbf{a} \cdot \mathbf{c})\mathbf{c}$$

Solution:

Let us consider vectors in a coordinate system in which “x” and “y” axes of the coordinate system are in the direction of reflecting surface and normal to the reflecting surface respectively as shown in the figure.

Reflection



Angle of incidence is equal to angle of reflection.

We express unit vectors with respect to the incident and reflected as :

$$\mathbf{a} = \sin \theta \mathbf{i} - \cos \theta \mathbf{j}$$

$$\mathbf{b} = \sin \theta \mathbf{i} + \cos \theta \mathbf{j}$$

Subtracting first equation from the second equation, we have :

$$\Rightarrow \mathbf{b} - \mathbf{a} = 2 \cos \theta \mathbf{j}$$

$$\Rightarrow \mathbf{b} = \mathbf{a} + 2 \cos \theta \mathbf{j}$$

Now, we evaluate dot product, involving unit vectors :

$$\mathbf{a} \cdot \mathbf{c} = 1 \times 1 \times \cos(180^\circ - \theta) = -\cos \theta$$

Substituting for $\cos \theta$, we have :

$$\Rightarrow \mathbf{b} = \mathbf{a} - 2(\mathbf{a} \cdot \mathbf{c})\mathbf{c}$$

Scalar product (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the scalar vector product. For this reason, questions are categorized in terms of the characterizing features of the subject matter :

- Angle between two vectors
- Condition of perpendicular vectors
- Component as scalar product
- Nature of scalar product
- Scalar product of a vector with itself
- Evaluation of dot product

Angle between two vectors

Example:

Problem : Find the angle between vectors $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{k}$.

Solution : The cosine of the angle between two vectors is given in terms of dot product as :

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab}$$

Now,

$$\mathbf{a} \cdot \mathbf{b} = (2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - \mathbf{k})$$

Ignoring dot products of different unit vectors (they evaluate to zero), we have :

$$\mathbf{a} \cdot \mathbf{b} = 2\mathbf{i} \cdot \mathbf{i} + (-\mathbf{k}) \cdot (-\mathbf{k}) = 2 + 1 = 3$$

$$a = \sqrt{(2^2 + 1^2 + 1^2)} = \sqrt{6}$$

$$b = \sqrt{(1^2 + 1^2)} = \sqrt{2}$$

$$ab = \sqrt{6} \times \sqrt{2} = \sqrt{(12)} = 2\sqrt{3}$$

Putting in the expression of cosine, we have :

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} = \cos 30^\circ$$

$$\theta = 30^\circ$$

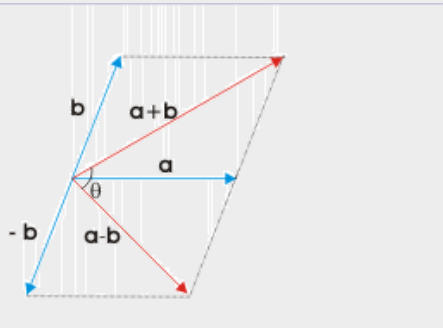
Condition of perpendicular vectors

Example:

Problem : Sum and difference of two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other. Find the relation between two vectors.

Solution : The sum $\mathbf{a} + \mathbf{b}$ and difference $\mathbf{a} - \mathbf{b}$ are perpendicular to each other. Hence, their dot product should evaluate to zero.

Sum and difference of two vectors



Sum and difference of two vectors are perpendicular to each other.

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$$

Using distributive property,

$$\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = 0$$

Using commutative property, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, Hence,

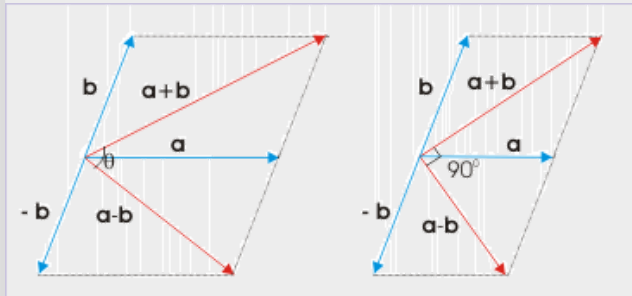
$$\mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = 0$$

$$a^2 - b^2 = 0$$

$$a = b$$

It means that magnitudes of two vectors are equal. See figure below for enclosed angle between vectors, when vectors are equal :

Sum and difference of two vectors



Sum and difference of two vectors are perpendicular to each other, when vectors are equal.

Component as scalar product

Example:

Problem : Find the components of vector $2\mathbf{i} + 3\mathbf{j}$ along the direction $\mathbf{i} + \mathbf{j}$.

Solution : The component of a vector “ $\mathbf{a}$ ” in a direction, represented by unit vector “ $\mathbf{n}$ ” is given by dot product :

$$a_n = \mathbf{a} \cdot \mathbf{n}$$

Thus, it is clear that we need to find the unit vector in the direction of $\mathbf{i} + \mathbf{j}$. Now, the unit vector in the direction of the vector is :

$$\mathbf{n} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|}$$

Here,

$$|\mathbf{i} + \mathbf{j}| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$$

Hence,

$$\mathbf{n} = \frac{1}{\sqrt{2}} \times (\mathbf{i} + \mathbf{j})$$

The component of vector $2\mathbf{i} + 3\mathbf{j}$ in the direction of “ $\mathbf{n}$ ” is :

$$a_n = \mathbf{a} \cdot \mathbf{n} = (2\mathbf{i} + 3\mathbf{j}) \cdot \frac{1}{\sqrt{2}} \times (\mathbf{i} + \mathbf{j})$$

$$\Rightarrow a_n = \frac{1}{\sqrt{2}} \times (2\mathbf{i} + 3\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})$$

$$\Rightarrow a_n = \frac{1}{\sqrt{2}} \times (2 \times 1 + 3 \times 1)$$

$$\Rightarrow a_n = \frac{5}{\sqrt{2}}$$

Nature of scalar product

Example:

Problem : Verify vector equality $\mathbf{B} = \mathbf{C}$, if $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$.

Solution : The given equality of dot products is :

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$$

The equality will result if $\mathbf{B} = \mathbf{C}$. We must, however, understand that dot product is not a simple algebraic product of two numbers (read magnitudes). The angle between two vectors plays a role in determining the magnitude of the dot product. Hence, it is entirely possible that vectors $\mathbf{B}$ and $\mathbf{C}$ are different yet their dot products with common vector $\mathbf{A}$ are equal.

We can attempt this question mathematically as well. Let θ_1 and θ_2 be the angles for first and second pairs of dot products. Then,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$$

$$AB \cos\theta_1 = AC \cos\theta_2$$

If $\theta_1 = \theta_2$, then $B = C$. However, if $\theta_1 \neq \theta_2$, then $B \neq C$.

Scalar product of a vector with itself

Example:

Problem : If $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, then find the angle between vectors $\mathbf{a}$ and $\mathbf{b}$.

Solution : A question that involves modulus or magnitude of vector can be handled in specific manner to find information about the vector (s). The specific identity that is used in this circumstance is :

$$\mathbf{A} \cdot \mathbf{A} = A^2$$

We use this identity first with the sum of the vectors $(\mathbf{a} + \mathbf{b})$,

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a} + \mathbf{b}|^2$$

Using distributive property,

$$\begin{aligned}\Rightarrow \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} &= a^2 + b^2 + 2ab \cos \theta = |\mathbf{a} + \mathbf{b}|^2 \\ \Rightarrow |\mathbf{a} + \mathbf{b}|^2 &= a^2 + b^2 + 2ab \cos \theta\end{aligned}$$

Similarly, using the identity with difference of the vectors $(\mathbf{a} - \mathbf{b})$,

$$\Rightarrow |\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab \cos \theta$$

It is, however, given that :

$$\Rightarrow |\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$$

Squaring on either side of the equation,

$$\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$$

Putting the expressions,

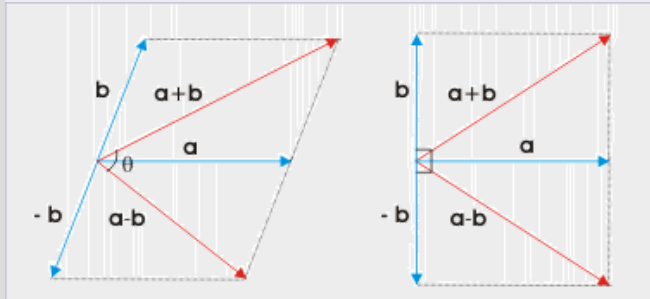
$$\begin{aligned}\Rightarrow a^2 + b^2 + 2ab \cos \theta &= a^2 + b^2 - 2ab \cos \theta \\ \Rightarrow 4ab \cos \theta &= 0 \\ \Rightarrow \cos \theta &= 0 \\ \Rightarrow \theta &= 90^\circ\end{aligned}$$

Note : We can have a mental picture of the significance of this result. As given, the magnitude of sum of two vectors is equal to the magnitude of difference of two vectors. Now, we know that difference of vectors is similar to vector sum with one exception

that one of the operand is rendered negative. Graphically, it means that one of the vectors is reversed.

Reversing one of the vectors changes the included angle between two vectors, but do not change the magnitudes of either vector. It is, therefore, only the included angle between the vectors that might change the magnitude of resultant. In order that magnitude of resultant does not change even after reversing direction of one of the vectors, it is required that the included angle between the vectors is not changed. This is only possible, when included angle between vectors is 90° . See figure.

Sum and difference of two vectors



Magnitudes of Sum and difference of two vectors are same when vectors at right angle to each other.

Example:

Problem : If $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear unit vectors and $|\mathbf{a} + \mathbf{b}| = \sqrt{3}$, then find the value of expression :

$$(\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b})$$

Solution : The given expression is scalar product of two vector sums. Using distributive property we can expand the expression, which will comprise of scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$.

$$(\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = 2\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot 2\mathbf{a} + (-\mathbf{b}) \cdot (-\mathbf{b}) = 2a^2 - \mathbf{a} \cdot \mathbf{b} - b^2$$

$$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = 2a^2 - b^2 - ab \cos \theta$$

We can evaluate this scalar product, if we know the angle between them as magnitudes of unit vectors are each 1. In order to find the angle between the vectors, we use the identity,

$$\mathbf{A} \cdot \mathbf{A} = A^2$$

Now,

$$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = a^2 + b^2 + 2ab \cos \theta = 1 + 1 + 2 \times 1 \times 1 \times \cos \theta$$

$$\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 2 + 2 \cos \theta$$

It is given that :

$$|\mathbf{a} + \mathbf{b}|^2 = (\sqrt{3})^2 = 3$$

Putting this value,

$$\Rightarrow 2 \cos \theta = |\mathbf{a} + \mathbf{b}|^2 - 2 = 3 - 2 = 1$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ$$

Using this value, we now proceed to find the value of given identity,

$$(\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = 2a^2 - b^2 - ab \cos \theta = 2 \times 1^2 - 1^2 - 1 \times 1 \times \cos 60^\circ$$

$$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = \frac{1}{2}$$

Evaluation of dot product

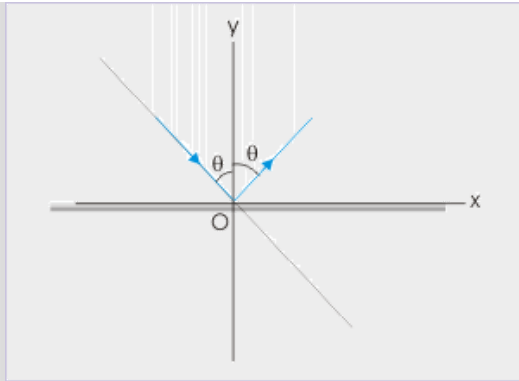
Example:

Problem : In an experiment of light reflection, if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the unit vectors in the direction of incident ray, reflected ray and normal to the reflecting surface, then prove that :

$$\Rightarrow \mathbf{b} = \mathbf{a} - 2(\mathbf{a} \cdot \mathbf{c})\mathbf{c}$$

Solution : Let us consider vectors in a coordinate system in which “x” and “y” axes of the coordinate system are in the direction of reflecting surface and normal to the reflecting surface respectively as shown in the figure.

Reflection



Angle of incidence is equal to angle of reflection.

We express unit vectors with respect to the incident and reflected as :

$$\mathbf{a} = \sin \theta \mathbf{i} - \cos \theta \mathbf{j}$$

$$\mathbf{b} = \sin \theta \mathbf{i} + \cos \theta \mathbf{j}$$

Subtracting first equation from the second equation, we have :

$$\Rightarrow \mathbf{b} - \mathbf{a} = 2 \cos \theta \mathbf{j}$$

$$\Rightarrow \mathbf{b} = \mathbf{a} + 2 \cos \theta \mathbf{j}$$

Now, we evaluate dot product, involving unit vectors :

$$\mathbf{a} \cdot \mathbf{c} = 1 \times 1 \times \cos(180^\circ - \theta) = -\cos \theta$$

Substituting for $\cos \theta$, we have :

$$\Rightarrow \mathbf{b} = \mathbf{a} - 2(\mathbf{a} \cdot \mathbf{c})\mathbf{c}$$

Vector (cross) product

Vector multiplication provides concise and accurate representation of natural laws, which involve vectors.

The cross product of two vectors **a** and **b** is a third vector. The magnitude of the vector product is given by the following expression :

Equation:

$$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = ab \sin \theta$$

where θ is the smaller of the angles between the two vectors. It is important to note that vectors have two angles θ and $2\pi - \theta$. We should use the smaller of the angles as sine of θ and $2\pi - \theta$ are different.

If "**n**" denotes unit vector in the direction of vector product, then

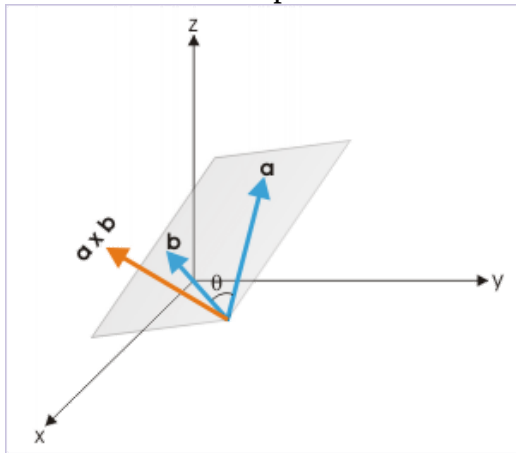
Equation:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = ab \sin \theta \mathbf{n}$$

Direction of vector product

The two vectors (**a** and **b**) define a unique plane. The vector product is perpendicular to this plane defined by the vectors as shown in the figure below. The most important aspect of the direction of cross product is that it is independent of the angle, θ , enclosed by the vectors. The enclosed angle, θ , only impacts the magnitude of the cross product and not its direction.

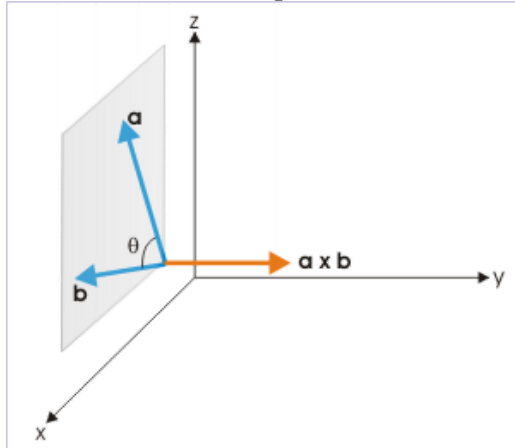
Direction of vector product



Incidentally, the requirement for determining direction suits extremely well with rectangular coordinate system. We know that rectangular coordinate system comprises of

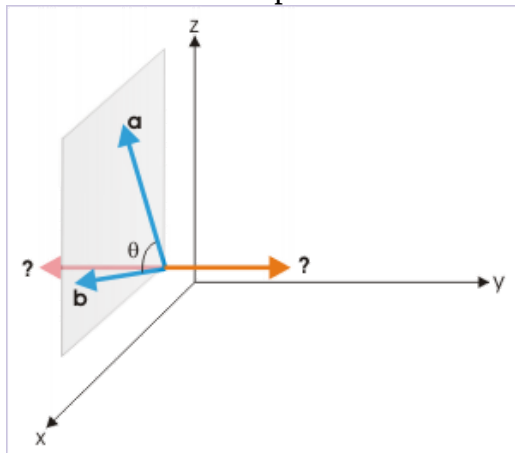
three planes, which are at right angles to each other. It is, therefore, easier if we orient our coordinate system in such a manner that vectors lie in one of the three planes defined by the rectangular coordinate system. The cross product is, then, oriented in the direction of axis, perpendicular to the plane of vectors.

Direction of vector product



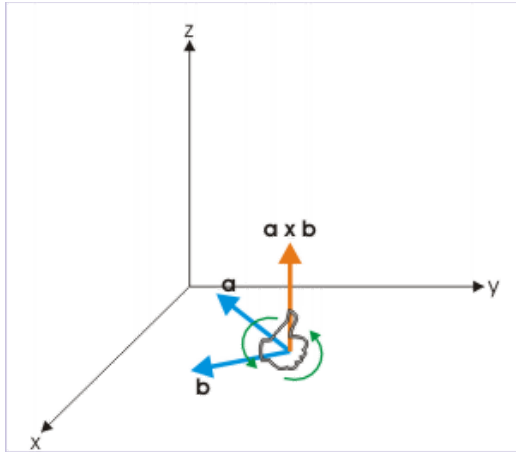
As a matter of fact, the direction of vector product is not yet actually determined. We can draw the vector product perpendicular to the plane on either of the two sides. For example, the product can be drawn either along the positive direction of y – axis or along the negative direction of y-axis (See Figure below).

Direction of vector product



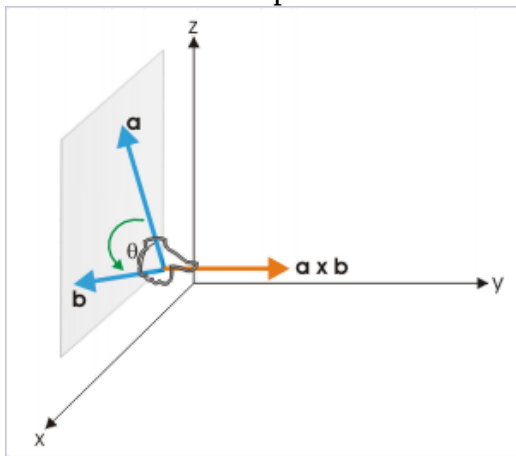
The direction of the vector product, including which side of the plane, is determined by right hand rule for vector products. According to this rule, we place right fist such that the curl of the fist follows as we proceed from the first vector, **a**, to the second vector, **b**. The stretched thumb, then gives the direction of vector product.

Direction of vector product



When we apply this rule to the case discussed earlier, we find that the vector product is in the positive y – direction as shown below :

Direction of vector product



Here, we notice that we move in the anti-clockwise direction as we move from vector, **a**, to vector, **b**, while looking at the plane formed by the vectors. This fact can also be used to determine the direction of the vector product. If the direction of movement is anticlockwise, then the vector product is directed towards us; otherwise the vector product is directed away on the other side of the plane.

It is important to note that the direction of cross product can be on a particular side of the plane, depending upon whether we take the product from **a** to **b** or from **b** to **a**. This implies :

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

Thus, vector product is not commutative like vector addition. It can be inferred from the discussion of direction that change of place of vectors in the sequence of cross product actually changes direction of the product such that :

Equation:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

Values of cross product

The value of vector product is maximum for the maximum value of $\sin\theta$. Now, the maximum value of sine is $\sin 90^\circ = 1$. For this value, the vector product evaluates to the product of the magnitude of two vectors multiplied. Thus maximum value of cross product is :

Equation:

$$(\mathbf{a} \times \mathbf{b})_{\max} = ab$$

The vector product evaluates to zero for $\theta = 0^\circ$ and 180° as sine of these angles are zero. These results have important implication for unit vectors. The cross product of same unit vector evaluates to 0.

Equation:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

The cross products of combination of different unit vectors evaluate as :

Equation:

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k}; \mathbf{j} \times \mathbf{k} = \mathbf{i}; \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k}; \mathbf{k} \times \mathbf{j} = -\mathbf{i}; \mathbf{i} \times \mathbf{k} = -\mathbf{j}\end{aligned}$$

There is a simple rule to determine the sign of the cross product. We write the unit vectors in sequence $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Now, we can form pair of vectors as we move from left to right like $\mathbf{i} \times \mathbf{j}$, $\mathbf{j} \times \mathbf{k}$ and right to left at the end like $\mathbf{k} \times \mathbf{i}$ in cyclic manner. The cross products of these pairs result in the remaining unit vector with positive sign. Cross products of other pairs result in the remaining unit vector with negative sign.

Cross product in component form

Two vectors in component forms are written as :

$$\begin{aligned}\mathbf{a} &= a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \\ \mathbf{b} &= b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}\end{aligned}$$

In evaluating the product, we make use of the fact that multiplication of the same unit vectors gives the value of 0, while multiplication of two different unit vectors result in

remaining vector with appropriate sign. Finally, the vector product evaluates to vector terms :

Equation:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \times (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) \\ \Rightarrow \mathbf{a} \times \mathbf{b} &= a_x \mathbf{i} \times b_y \mathbf{j} + a_x \mathbf{i} \times b_z \mathbf{k} + a_y \mathbf{j} \times b_x \mathbf{i} + a_y \mathbf{j} \times b_z \mathbf{k} + a_z \mathbf{k} \times b_x \mathbf{i} + a_z \mathbf{k} \times b_y \mathbf{j} \\ \Rightarrow \mathbf{a} \times \mathbf{b} &= a_x b_y \mathbf{k} - a_x b_z \mathbf{j} - a_y b_x \mathbf{k} + a_y b_z \mathbf{i} + a_z b_x \mathbf{j} - a_z b_y \mathbf{i} \\ \Rightarrow \mathbf{a} \times \mathbf{b} &= (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}\end{aligned}$$

Evidently, it is difficult to remember above expression. If we know to expand determinant, then we can write above expression in determinant form, which is easy to remember.

Equation:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Exercise:

Problem: If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = -3\mathbf{i} - 2\mathbf{j}$, find $\mathbf{A} \times \mathbf{B}$.

Solution:

$$\mathbf{a} \times \mathbf{b} = (2\mathbf{i} + 3\mathbf{j}) \times (-3\mathbf{i} - 2\mathbf{j})$$

Neglecting terms involving same unit vectors, we expand the multiplication algebraically as :

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (2\mathbf{i}) \times (-2\mathbf{j}) + (3\mathbf{j}) \times (-3\mathbf{i}) \\ \Rightarrow \mathbf{a} \times \mathbf{b} &= -4\mathbf{k} + 9\mathbf{k} = 5\mathbf{k}\end{aligned}$$

Exercise:

Problem: Consider the magnetic force given as :

$$\mathbf{F} = q (\mathbf{v} \times \mathbf{B})$$

Given $q = 10^{-6} \text{ C}$, $\mathbf{v} = (3\mathbf{i} + 4\mathbf{j}) \text{ m/s}$, $\mathbf{B} = 1\mathbf{i} \text{ Tesla}$. Find the magnetic force.

Solution:

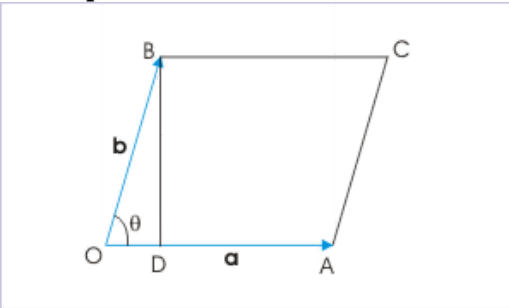
$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) = 10^{-6} \{ (3\mathbf{i} + 4\mathbf{j}) \times (-1\mathbf{i}) \}$$

$$\Rightarrow \mathbf{F} = 4 \times 10^{-6} \mathbf{k}$$

Geometric meaning vector product

In order to interpret the geometric meaning of the cross product, let us draw two vectors by the sides of a parallelogram as shown in the figure. Now, the magnitude of cross product is given by :

Cross product of two vectors



Two vectors are represented by two sides of a parallelogram.

$$|\mathbf{a} \times \mathbf{b}| = ab \sin \theta$$

We drop a perpendicular BD from B on the base line OA as shown in the figure. From ΔOAB ,

$$b \sin \theta = OB \sin \theta = BD$$

Substituting, we have :

$$|\mathbf{a} \times \mathbf{b}| = OA \times BD = \text{Base} \times \text{Height} = \text{Area of parallelogram}$$

It means that the magnitude of cross product is equal to the area of parallelogram formed by the two vectors. Thus,

Equation:

$$\text{Area of parallelogram} = |\mathbf{a} \times \mathbf{b}|$$

Since area of the triangle OAB is half of the area of the parallelogram, the area of the triangle formed by two vectors is :

Equation:

$$\text{Area of triangle} = \frac{1}{2} \times |\mathbf{a} \times \mathbf{b}|$$

Attributes of vector (cross) product

In this section, we summarize the properties of cross product :

1: Vector (cross) product is not commutative

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

A change of sequence of vectors results in the change of direction of the product (vector) :

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

The inequality resulting from change in the order of sequence, denotes “anti-commutative” nature of vector product as against scalar product, which is commutative.

Further, we can extend the sequence to more than two vectors in the case of cross product. This means that vector expressions like $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ is valid. Ofcourse, the order of vectors in sequence will impact the ultimate product.

2: Distributive property of cross product :

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

3: The magnitude of cross product of two vectors can be obtained in either of the following manner :

$$|\mathbf{a} \times \mathbf{b}| = ab \sin \theta$$

or,

$$|\mathbf{a} \times \mathbf{b}| = a \times (b \sin \theta)$$

$$\Rightarrow |\mathbf{a} \times \mathbf{b}| = a \times \text{component of } \mathbf{b} \text{ in the direction perpendicular to vector } \mathbf{a}$$

or,

$$|\mathbf{a} \times \mathbf{b}| = b \times (a \sin \theta)$$

$$\Rightarrow |\mathbf{b} \times \mathbf{a}| = a \times \text{component of } \mathbf{a} \text{ in the direction perpendicular to vector } \mathbf{b}$$

4: Vector product in component form is :

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

5: Unit vector in the direction of cross product

Let “ $\mathbf{n}$ ” be the unit vector in the direction of cross product. Then, cross product of two vectors is given by :

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \mathbf{n}$$

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a} \times \mathbf{b}| \mathbf{n}$$

$$\mathbf{n} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

6: The condition of two parallel vectors in terms of cross product is given by :

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \mathbf{n} = ab \sin 0^\circ \mathbf{n} = 0$$

If the vectors involved are expressed in component form, then we can write the above condition as :

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = 0$$

Equivalently, this condition can be also said in terms of the ratio of components of two vectors in mutually perpendicular directions :

$$\frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$$

7: Properties of cross product with respect to unit vectors along the axes of rectangular coordinate system are :

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}; \mathbf{j} \times \mathbf{k} = \mathbf{i}; \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}; \mathbf{k} \times \mathbf{j} = -\mathbf{i}; \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Vector product (application)

Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the vector product. For this reason, questions are categorized in terms of the characterizing features of the subject matter :

- Condition of parallel vectors
- Unit vector of cross product
- Nature of vector product
- Evaluation of vector product
- Area of parallelogram

Condition of parallel vectors

Example:

Problem : Determine whether vectors $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $3\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ are parallel to each other?

Solution : If the two vectors are parallel, then ratios of corresponding components of vectors in three coordinate directions are equal. Here,

$$\begin{aligned}\frac{a_x}{b_x} &= \frac{2}{3} \\ \frac{a_y}{b_y} &= \frac{1}{3} \\ \frac{a_z}{b_z} &= \frac{1}{3}\end{aligned}$$

The ratios are, therefore, not equal. Hence, given vectors are not parallel to each other.

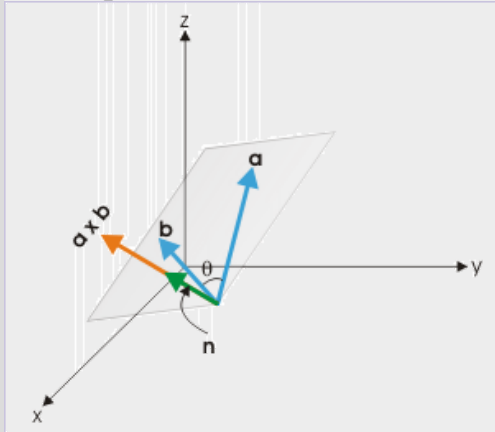
Unit vector of cross product

Example:

Problem : Find unit vector in the direction perpendicular to vectors $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

Solution : We know that cross product of two vectors is perpendicular to each of vectors. Thus, unit vector in the direction of cross product is perpendicular to the given vectors. Now, unit vector of cross product is given by :

Vector product



Unit vector in the direction of
vector product.

$$\mathbf{n} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

Here,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{vmatrix}$$

$$\Rightarrow \mathbf{a} \times \mathbf{b} = \{1 \times 3 - (-2 \times -1)\}\mathbf{i} + \{(2 \times -2) - 1 \times 3\}\mathbf{j} + \{(1 \times -1) - 1 \times 2\}\mathbf{k}$$

$$\Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{i} - 7\mathbf{j} - 3\mathbf{k}$$

$$\Rightarrow |\mathbf{a} \times \mathbf{b}| = \sqrt{1^2 + (-7)^2 + (-3)^2}$$

$$\Rightarrow \mathbf{n} = \frac{1}{\sqrt{59}} \times \mathbf{i} - 7\mathbf{j} - 3\mathbf{k}$$

Nature of vector product

Example:**Problem :** Verify vector equality $\mathbf{B} = \mathbf{C}$, if $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$.**Solution :** Let θ_1 and θ_2 be the angles for first and second pairs of cross products. Then,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \mathbf{A} \times \mathbf{C} \\ \Rightarrow AB \sin \theta_1 \mathbf{n}_1 &= AC \sin \theta_2 \mathbf{n}_2 \\ \Rightarrow B \sin \theta_1 \mathbf{n}_1 &= C \sin \theta_2 \mathbf{n}_2\end{aligned}$$

It is clear that $\mathbf{B}=\mathbf{C}$ is true only when $\sin \theta_1 \mathbf{n}_1 = \sin \theta_2 \mathbf{n}_2$. It is always possible that the angles involved or the directions of cross products are different. Thus, we can conclude that $\mathbf{B}$ need not be equal to $\mathbf{C}$.

Evaluation of vector product**Example:****Problem :** If $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a} \times \mathbf{b}|$ for unit vectors $\mathbf{a}$ and $\mathbf{b}$, then find the angle between unit vectors.**Solution :** According to question,

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a} \times \mathbf{b}| \\ \Rightarrow ab \cos \theta &= ab \sin \theta \\ \Rightarrow \tan \theta &= 1 \times 1 \times \tan 45^\circ \\ \Rightarrow \theta &= 45^\circ\end{aligned}$$

Example:**Problem :** Prove that :

$$|\mathbf{a} \cdot \mathbf{b}|^2 - |\mathbf{a} \times \mathbf{b}|^2 = a^2 \times b^2 \times \cos 2\theta$$

Solution : Expanding LHS, we have :

$$\begin{aligned}|\mathbf{a} \cdot \mathbf{b}|^2 - |\mathbf{a} \times \mathbf{b}|^2 &= (ab \cos \theta)^2 - (ab \sin \theta)^2 = a^2 b^2 \cos^2 \theta - \sin^2 \theta \\ \Rightarrow |\mathbf{a} \cdot \mathbf{b}|^2 - |\mathbf{a} \times \mathbf{b}|^2 &= a^2 b^2 \cos 2\theta\end{aligned}$$

Area of parallelogram

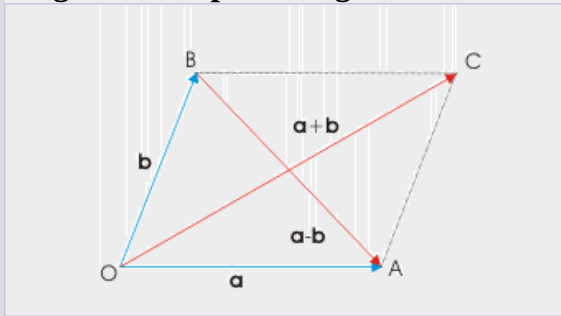
Example:

Problem : The diagonals of a parallelogram are represented by vectors $3\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}-\mathbf{j}-\mathbf{k}$. Find the area of parallelogram.

Solution : The area of parallelogram whose sides are formed by vectors $\mathbf{a}$ and $\mathbf{b}$, is given by :

$$\text{Area} = |\mathbf{a} \times \mathbf{b}|$$

However, we are given in question vectors representing diagonals – not the sides. But, we know that the diagonals are sum and difference of vectors representing sides of a parallelogram. It means that :

Diagonals of a parallelogram

The vectors along diagonals are sum and difference of two vectors representing the sides.

$$\mathbf{a} + \mathbf{b} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

and

$$\mathbf{a} - \mathbf{b} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

Now the vector product of vectors representing diagonals is :

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = \mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} + \mathbf{b} \times (-\mathbf{b}) = -\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a}$$

Using anti-commutative property of vector product,

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = -2\mathbf{a} \times \mathbf{b}$$

Thus,

$$\mathbf{a} \times \mathbf{b} = -\frac{1}{2} \times (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$$

$$\mathbf{a} \times \mathbf{b} = -\frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = -\frac{1}{2} \times \{ (1 \times -1 - 1 \times -1)\mathbf{i} + (1 \times 1 - 3 \times -1)\mathbf{j} + (3 \times -1 - 1 \times 1)\mathbf{k} \}$$

$$\mathbf{a} \times \mathbf{b} = -\frac{1}{2} \times (4\mathbf{j} - 4\mathbf{k})$$

$$\mathbf{a} \times \mathbf{b} = -2\mathbf{j} + 2\mathbf{k}$$

The volume of the parallelogram is :

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2} \text{ units}$$

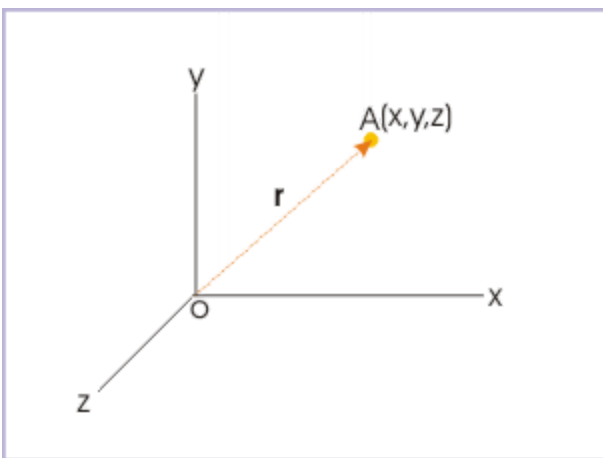
Position vector

Position vector is a convenient mathematical construct to encapsulate the twin ideas of magnitude (how far?) and direction (in which direction?) of the position, occupied by an object.

Position vector

Position vector is a vector that extends from the reference point to the position of the particle.

Position vector



Position vector is represented by a vector, joining origin to the position of point object

Generally, we take origin of the coordinate system as the reference point.

It is easy to realize that vector representation of position is appropriate, where directional properties of the motion are investigated. As a matter of fact, three important directional attributes of motion, namely displacement, velocity and acceleration are defined in terms of position vectors.

Consider the definitions : the “displacement” is equal to the change in position vector; the “velocity” is equal to the rate of change of position vector with respect to time; and “acceleration” is equal to the rate of change

of velocity with respect to time, which, in turn, is the rate of change of position vector. Thus, all directional attributes of motion is based on the processing of position vectors.

Position Vector in component form

One of the important characteristics of position vector is that it is rooted to the origin of the coordinate system. We shall find that most other vectors associated with physical quantities, having directional properties, are floating vectors and not rooted to a point of the coordinate system like position vector.

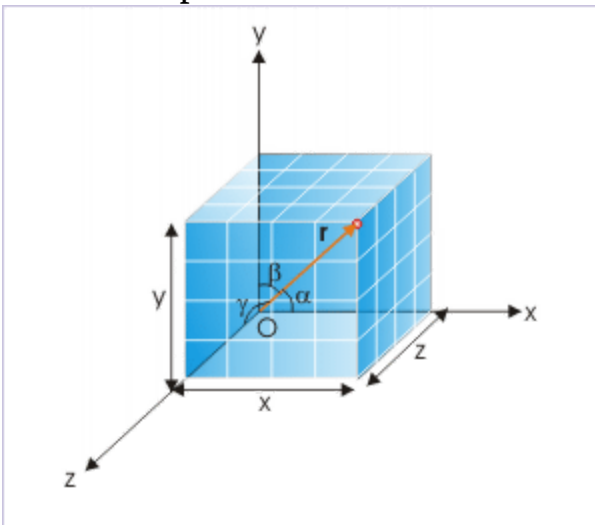
Recall that scalar components are graphically obtained by dropping two perpendiculars from the ends of the vector to the axes. In the case of position vector, one of the end is the origin itself. As position vector is rooted to the origin, the scalar components of position vectors in three mutually perpendicular directions of the coordinate system are equal to the coordinates themselves. The scalar components of position vector, $\mathbf{r}$, by definition in the designated directions of the rectangular axes are :

$$x = r \cos \alpha$$

$$y = r \cos \beta$$

$$z = r \cos \gamma$$

Scalar components of a vector



Scalar components are equal to
coordinates of the position

and position vector in terms of components (coordinates) is :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in x, y and z directions.

The magnitude of position vector is given by :

$$r = |\mathbf{r}| = \sqrt{(x^2 + y^2 + z^2)}$$

Example:

Position and distance

Problem : Position (in meters) of a moving particle as a function of time (in seconds) is given by :

$$\mathbf{r} = (3t^2 - 3)\mathbf{i} + (4 - 7t)\mathbf{j} + (-t^3)\mathbf{k}$$

Find the coordinates of the positions of the particle at the start of the motion and at time $t = 2$ s. Also, determine the linear distances of the positions of the particle from the origin of the coordinate system at these time instants.

Solution : The coordinates of the position are projection of position vector on three mutually perpendicular axes. Whereas linear distance of the position of the particle from the origin of the coordinate system is equal to the magnitude of the position vector. Now,

When $t = 0$ (start of the motion)

$$\mathbf{r} = (3 \times 0 - 3)\mathbf{i} + (4 - 7 \times 0)\mathbf{j} + (-0)\mathbf{k}$$

The coordinates are :

$$x = -3m \text{ and } y = 4m$$

and the linear distance from the origin is :

$$r = |\mathbf{r}| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5m$$

When $t = 2 \text{ s}$

$$\mathbf{r} = (3 \times 2^2 - 3)\mathbf{i} + (4 - 7 \times 2)\mathbf{j} + (-2^3)\mathbf{k} = 9\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$$

The coordinates are :

$$x = 9m, y = -10m \text{ and } z = -8m.$$

and the linear distance from the origin is :

$$r = |\mathbf{r}| = \sqrt{9^2 + (-10)^2 + (-8)^2} = \sqrt{245} = 15.65m$$

Motion types and position vector

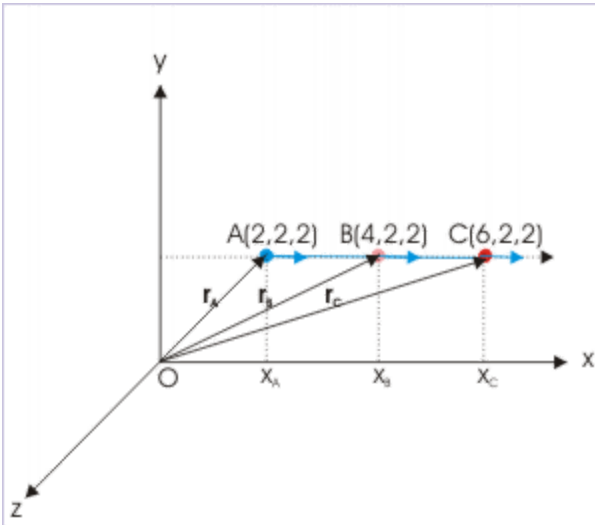
Position is a three dimensional concept, requiring three coordinate values to specify it. Motion of a particle, however, can take place in one (linear) and two (planar) dimensions as well.

In two dimensional motion, two of the three coordinates change with time. The remaining third coordinate is constant. By appropriately choosing the coordinate system, we can eliminate the need of specifying the third coordinate.

In one dimensional motion, only one of the three coordinates is changing with time. Other two coordinates are constant through out the motion. As such, it would suffice to describe positions of the particle with the values of changing coordinate and neglecting the remaining coordinates.

A motion along x –axis or parallel to x – axis is, thus, described by x - component of the position vector i.e. x – coordinate of the position as shown in the figure. It is only the x-coordinate that changes with time; other two coordinates remain same.

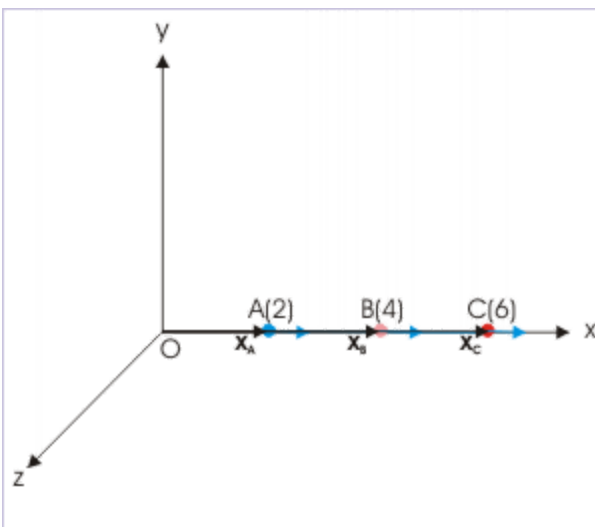
Motion in one dimesnion



The description of one dimensional motion is further simplified by shifting axis to the path of motion as shown below. In this case, other coordinates are individually equal to zero.

$$x = x; y = 0; z = 0$$

Motion in one dimesnion



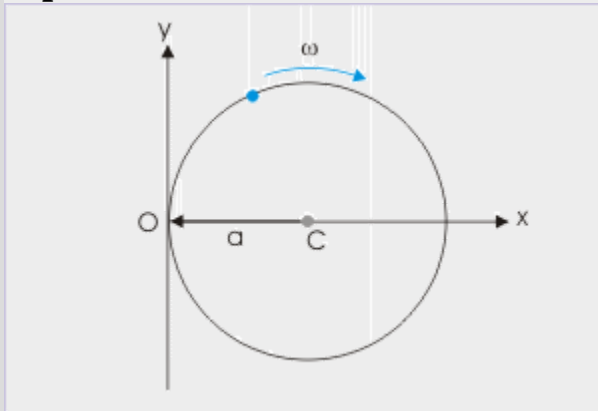
In this case, position vector itself is along x – axis and, therefore, its magnitude is equal to x – coordinate.

Examples

Example:

Problem : A particle is executing motion along a circle of radius “ a ” with a constant angular speed “ ω ” as shown in the figure. If the particle is at “O” at $t = 0$, then determine the position vector of the particle at an instant in xy - plane with “O” as the origin of the coordinate system.

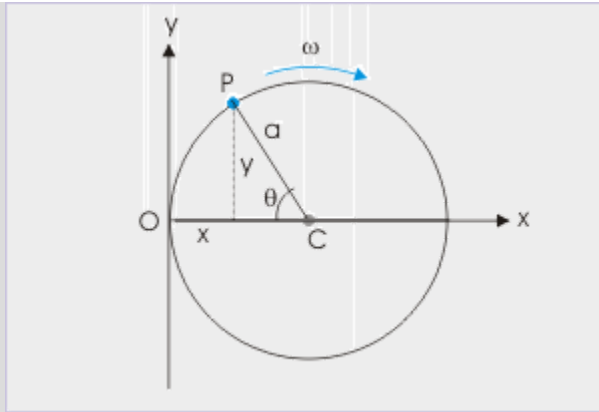
A particle in circular motion



The particle moves with a constant angular velocity.

Solution : Let the particle be at position “P” at a given time “ t ”. Then the position vector of the particle is :

A particle in circular motion



The particle moves with a constant angular velocity starting from “O” at $t = 0$.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

Note that "x" and "y" components of position vector is measured from the origin "O". From the figure,

$$y = a \sin \theta = a \sin \omega t$$

It is important to check that as particle moves in clockwise direction, y-coordinate increase in first quarter starting from origin, decreases in second quarter and so on. Similarly, x-coordinate is given by the expression :

$$x = a - a \cos \omega t = a(1 - \cos \omega t)$$

Thus, position vector of the particle in circular motion is :

$$\mathbf{r} = a(1 - \cos \omega t)\mathbf{i} + a \sin \omega t\mathbf{j}$$

Displacement

Motion involves two types of measurements : one which depends on the end points (displacement) and the other which depends on all points(distance) of motion.

Displacement is a measurement of change in position of the particle in motion. Its magnitude and direction are measured by the length and direction of the straight line joining initial and final positions of the particle. Obviously, the length of the straight line between the positions is the shortest distance between the points.

Displacement

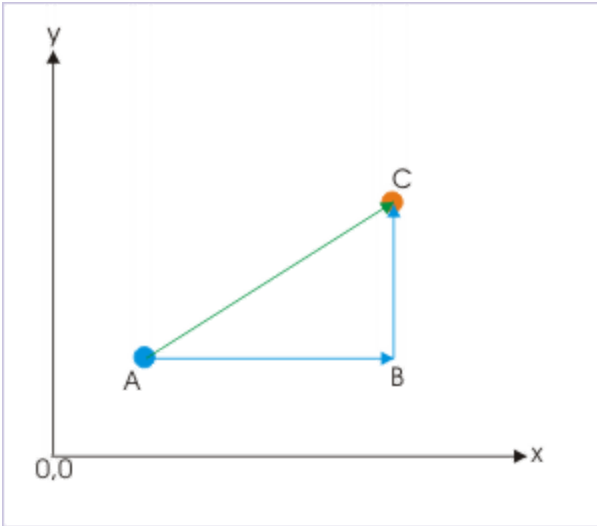
Displacement is the vector extending from initial to final positions of the particle in motion during an interval.

From physical view point, displacement conveys the meaning of shortest distance plus direction of the motion between two time instants or corresponding two positions. Initial and final positions of the point object are the only important consideration for measuring magnitude of displacement. Actual path between two positions has no consequence in so far as displacement is concerned.

The quantum of displacement is measured by the length of the straight line joining two ends of motion. If there is no change in the position at the end of a motion, the displacement is zero.

In order to illustrate the underlying concept of displacement, let us consider the motion of a particle from A to B to C. The displacement vector is represented by the vector AC and its magnitude by the length of AC.

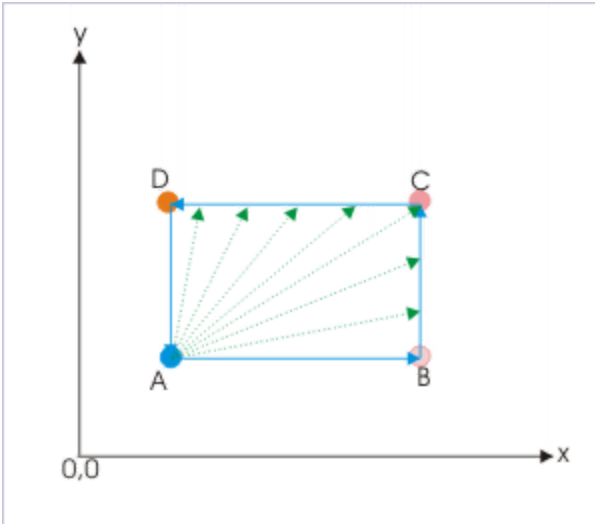
Displacement



Once motion has begun, magnitude of displacement may increase or decrease (at a slow, fast or constant rate) or may even be zero, if the object returns to its initial position. Since a body under motion can take any arbitrary path, it is always possible that the end point of the motion may come closer or may go farther away from the initial point. Thus, displacement, unlike distance, may decrease from a given level.

In order to understand the variations in displacement with the progress of motion, let us consider another example of the motion of a particle along the rectangular path from A to B to C to D to A. Magnitude of displacement, shown by dotted vectors, is increasing during motion from A to B to C. Whereas magnitude of displacement is decreasing as the particle moves from C to D to A, eventually being equal to zero, when the particle returns to A.

Displacement



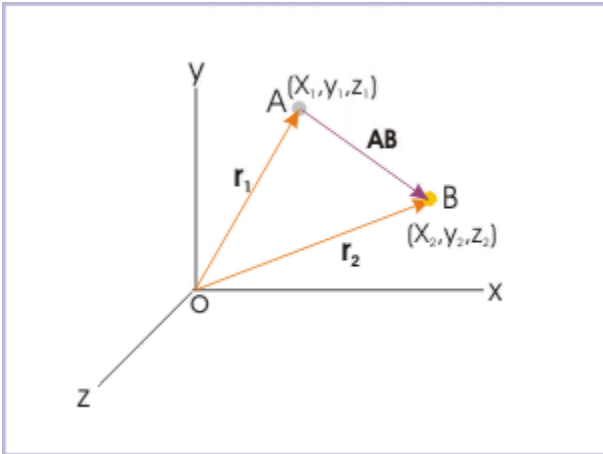
However, displacement is essentially a measurement of length combined with direction. As direction has no dimension, its dimensional formula is also $[L]$ like that of distance; and likewise, its SI measurement unit is 'meter'.

Displacement and Position vector

We have the liberty to describe displacement vector as an independent vector ($\mathbf{AB}$) or in terms of position vectors ($\mathbf{r}_1$ and $\mathbf{r}_2$). The choice depends on the problem in hand. The description, however, is equivalent.

Let us consider that a point object moves from point A (represented by position vector $\mathbf{r}_1$) to point B (represented by position vector $\mathbf{r}_2$) as shown in the figure. Now, using triangle law (moving from O to A to B to O), we have :

Displacement in terms of position vectors



Displacement is equal to the difference between final and initial position vectors

Equation:

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB}$$

$$\mathbf{r}_1 + \mathbf{AB} = \mathbf{r}_2$$

$$\Rightarrow \mathbf{AB} = \mathbf{r}_2 - \mathbf{r}_1 = \Delta \mathbf{r}$$

Thus, displacement is equal to the difference between final and initial position vectors. It is important to note that we obtain the difference between final and initial position vectors by drawing a third vector starting from the tip of the initial position vector and ending at the tip of the final position vector. This approach helps us to quickly draw the vector representing the difference of vectors and is a helpful procedural technique that can be used without any ambiguity. Equivalently, we can express displacement vector in terms of components of position vectors as :

Equation:

$$\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$

$$\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

$$\Rightarrow \mathbf{AB} = \Delta\mathbf{r} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

$$\Rightarrow \mathbf{AB} = \Delta\mathbf{r} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}$$

We must emphasize here that position vectors and displacement vector are different vector quantities. We need to investigate the relation between position vectors and displacement vector - a bit more closely. It is very important to mentally note that the difference of position vectors i.e. displacement, $\Delta\mathbf{r}$, has different directional property to that of the position vectors themselves ($\mathbf{r}_1$ and $\mathbf{r}_2$).

In the figure above, the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are directed along OA and OB respectively, while displacement vector, $\Delta\mathbf{r}$, is directed along AB. This means that the direction of displacement vector need not be same as that of either of the position vectors. Now, what would be the situation, when the motion begins from origin O instead of A? In that case, initial position vector is zero (null vector). Now, let the final position vector be denoted as $\mathbf{r}$. Then

$$\mathbf{r}_1 = 0$$

$$\mathbf{r}_2 = \mathbf{r}$$

and displacement is :

Equation:

$$\Rightarrow \mathbf{AB} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r} - 0 = \mathbf{r}$$

This is a special case, when final position vector itself is equal to the displacement. For this reason, when motion is studied from the origin of reference or origin of reference is chosen to coincide with initial position, then displacement and final position vectors are same and denoted by the symbol, " $\mathbf{r}$ ".

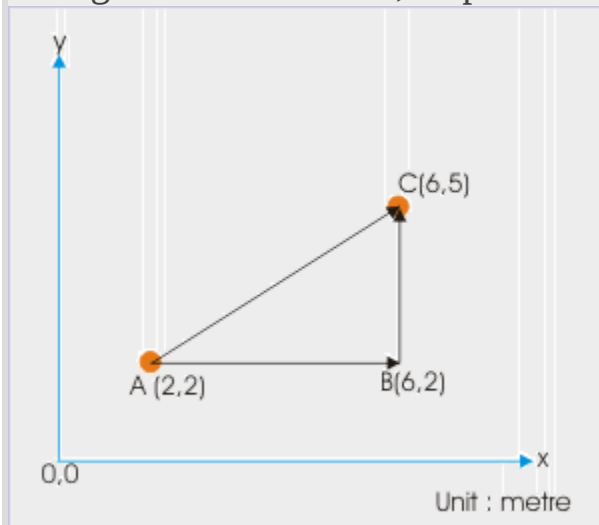
In general, however, it is the difference between final and initial position vectors, which is equal to the displacement and we refer displacement in terms of change in position vector and use the symbol $\Delta \mathbf{r}$ to represent displacement.

Magnitude of displacement is equal to the absolute value of the displacement vector. In physical sense, the magnitude of displacement is equal to the linear distance between initial and final positions along the straight line joining two positions i.e. the shortest distance between initial and final positions. This value may or may not be equal to the distance along the actual path of motion. In other words, magnitude of displacement represents the minimum value of distance between any two positions.

Example:

Displacement

Question : Consider a person walking from point A to B to C as shown in the figure. Find distance, displacement and magnitude of displacement.



Characteristics of motion : Two dimensional

Solution : The distance covered, s , during the motion from A to C is to the sum of the lengths AB and BC.

$$s = 4 + 3 = 7m$$

Displacement, **AC**, is given as :

$$\mathbf{AC} = \Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\mathbf{AC} = \Delta \mathbf{r} = (6 - 2)\mathbf{i} + (5 - 2)\mathbf{j} = 4\mathbf{i} + 3\mathbf{j}$$

The displacement vector makes an angle with the x – axis given by :

$$\theta = \tan^{-1}(3/4)$$

and the magnitude of the displacement is :

$$|\mathbf{AB}| = |\mathbf{r}_2 - \mathbf{r}_1|$$

$$\Rightarrow |\mathbf{AB}| = |\Delta \mathbf{r}| = |4\mathbf{i} + 3\mathbf{j}| = \sqrt{4^2 + 3^2} = 5 \text{ m}$$

The example above brings out nuances associated with terms used in describing motion. In particular, we see that distance and magnitude of displacement are not equal. This inequality arises due to the path of motion, which may be other than the shortest linear path between initial and final positions.

This means that distance and magnitude of displacement may not be equal. They are equal as a limiting case when particle moves in one direction without reversing direction; otherwise, distance is greater than the magnitude of displacement in most of the real time situation.

Equation:

$$s \geq |\Delta \mathbf{r}|$$

This inequality is important. It implies that displacement is not **distance plus direction** as may loosely be considered. As a matter of fact, displacement is **shortest distance plus direction** . For this reason, we need to avoid representing displacement by the symbol “s” as a vector counterpart of scalar distance, represented by “s”. In vector algebra, modulus of a vector, **A**, is represented by its non bold type face letter “A”. Going by this convention, if “s” and “s” represent displacement and distance respectively, then $s = |\mathbf{s}|$, which is incorrect.

When a body moves in a straight line maintaining its direction (unidirectional linear motion), then magnitude of displacement, $|\Delta \mathbf{r}|$ is equal to distance, “s”. Often, this situational equality gives the impression that two quantities are always equal, which is not so. For this reason, we would be careful to write magnitude of displacement by the modulus $|\Delta \mathbf{r}|$ or in terms of displacement vector like $|\mathbf{AB}|$ and not by “|s|”.

Example:

Displacement

Question : Position (in meters) of a moving particle as a function of time (in second) is given by :

$$\mathbf{r} = (3t^2 - 3)\mathbf{i} + (4 - 7t)\mathbf{j} + (-t^3)\mathbf{k}$$

Find the displacement in first 2 seconds.

Characteristics of motion : Three dimensional, variable velocity, variable speed

Solution : The position vector at $t = 0$ and 2 seconds are calculated to identify initial and final positions. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors at $t = 0$ and $t = 2$ s.

When $t = 0$ (start of the motion)

$$\mathbf{r}_1 = (3 \times 0 - 3)\mathbf{i} + (4 - 7 \times 0)\mathbf{j} + (-0)\mathbf{k} = -3\mathbf{i} + 4\mathbf{j}$$

When $t = 2$ s,

$$\mathbf{r}_2 = (3 \times 2^2 - 3)\mathbf{i} + (4 - 7 \times 2)\mathbf{j} + (-2^3)\mathbf{k} = 9\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$$

The displacement, $\Delta \mathbf{r}$, is given by :

$$\Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (9\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}) - (-3\mathbf{i} + 4\mathbf{j})$$

$$\Rightarrow \Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (9\mathbf{i} - 10\mathbf{j} - 8\mathbf{k} + 3\mathbf{i} - 4\mathbf{j}) = 12\mathbf{i} - 14\mathbf{j} - 8\mathbf{k}$$

Magnitude of displacement is given by :

$$\Rightarrow |\mathbf{r}| = \sqrt{\{12^2 + (-14)^2 + (-8)^2\}} = \sqrt{404} = 20.1m$$

Displacement and dimension of motion

We have so far discussed displacement as a general case in three dimensions. The treatment of displacement in one or two dimensions is relatively simplified. The expression for displacement in component form for these cases are given here :

1. Motion in two dimension : Let the motion takes place in the plane determined by x and y axes, then :

$$\Delta \mathbf{r} = \Delta x \mathbf{i} + \Delta y \mathbf{j}; \Delta z = 0$$

If the initial position of the particle coincides with the origin of reference system, then :

$$\Delta \mathbf{r} = \mathbf{r} = x \mathbf{i} + y \mathbf{j}; z = 0$$

2. Motion in one dimension : Let the motion takes place along the straight line parallel to x - axis, then :

$$\Delta \mathbf{r} = \Delta x \mathbf{i}; \Delta y = \Delta z = 0$$

If the initial position of the particle coincides with the origin of reference system, then :

$$\Delta \mathbf{r} = \mathbf{r} = x \mathbf{i}; y = z = 0$$

Displacement – time plot

Plotting displacement vector requires three axes. Displacement – time plot will, therefore, need a fourth axis for representing time. As such, displacement – time plot can not be represented on a three dimensional

Cartesian coordinate system. Even plotting two dimensional displacement with time is complicated.

One dimensional motion, having only two directions – along or opposite to the positive direction of axis, allows plotting displacement – time graph. One dimensional motion involves only one way of changing direction i.e. the particle under motion can reverse its direction of motion. Any other change of direction is not possible; otherwise the motion would not remain one dimensional motion.

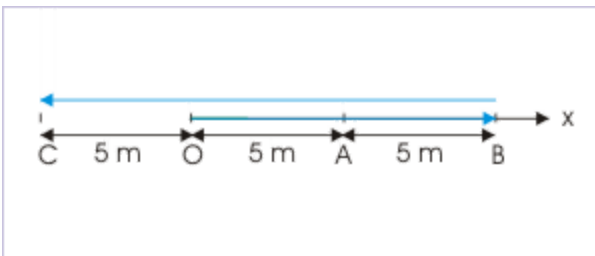
The simplification, in the case of one directional motion, allows us to do away with the need to use vector notation. Instead, the vectors are treated simply as scalars with one qualification that vectors in the direction of chosen reference is considered positive and vectors in the opposite direction to chosen reference is considered negative.

Representation of a displacement vector as a scalar quantity uses following construct :

1. Assign an axis along the motion
2. Assign the origin with the start of motion; It is, however, a matter of convenience and is not a requirement of the construct.
3. Consider displacement in the direction of axis as positive
4. Consider displacement in the opposite direction of axis as negative

To illustrate the construct, let us consider a motion of a ball which transverses from O to A to B to C to O along x-axis as shown in the figure.

Rectilinear motion



A particle moves along a straight line.

The magnitude of displacements (in meters) at various points of motion are :

$$x_1 = OA = 5$$

$$x_2 = OB = 10$$

$$x_3 = OC = -5$$

It is important to note from above data that when origin is chosen to coincide with initial position of the particle, then displacement and position vectors are equal.

The data for displacement as obtained above also reveals that by assigning proper sign to a scalar value, we can represent directional attribute of a vector quantity. In other words, plotting magnitude of displacement in one dimension with appropriate sign would completely represent the displacement vector in both magnitude and direction.

Interpreting change of position

There are different difference terms like Δr , $\Delta|\mathbf{r}|$, $|\Delta\mathbf{r}|$ and $\Delta\mathbf{r}$, which denote different aspects of change in position. It might appear trivial, but it is the understanding and ability to distinguish these quantities that will enable us to treat and describe motion appropriately in different dimensions. From the discussion so far, we have realized that the symbol " $\Delta\mathbf{r}$ " denotes displacement, which is equal to the difference in position vectors between "final" and "initial" positions. The meaning of the symbol " $|\Delta\mathbf{r}|$ " follows from the meaning of " $\Delta\mathbf{r}$ " as magnitude of displacement.

As a matter of fact, it is the symbol " Δr ", which creates certain "unexpected" confusion or ambiguity – if not handled appropriately. Going by the conventional understanding, we may be tempted to say that " Δr " represents magnitude of displacement and is equal to the absolute value of displacement i.e. $|\Delta\mathbf{r}|$. The point that we want to emphasize here is that it is not so.

Equation:

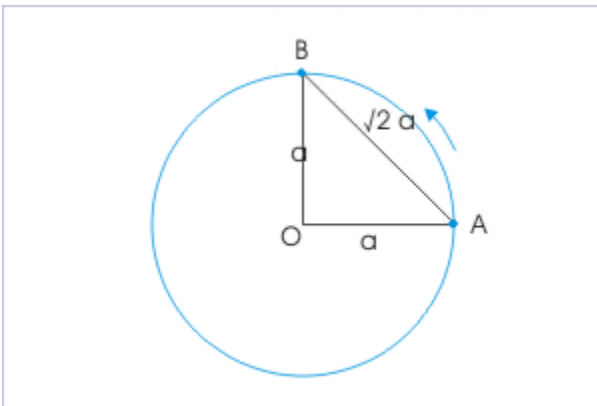
$$\Delta r \neq |\Delta \mathbf{r}|$$

Going by the plain meaning of the symbol, " Δ ", we can interpret two terms as :

- Δr = change in the magnitude of position vector, " r "
- $|\Delta \mathbf{r}|$ = magnitude of change in the position vector, " $\mathbf{r}$ "

Therefore, what we mean by the inequality emphasized earlier is that change in the magnitude of position vector, " r ", is not equal to the magnitude of change in the position vector. In order to appreciate the point, we can consider the case of two dimensional circular motion. Let us consider the motion from point "A" to "B" along a circle of radius " a ", as shown in the figure.

Circular motion



A particle transverses a quarter of circle from A to B.

Since radius of the circle remains same, the change in the magnitude of position vector, " r ", is zero during the motion. Hence,

$$\Delta r = 0$$

However, the magnitude of displacement during the motion is :

$$AB = \sqrt{(a^2 + a^2)} = \sqrt{2}a$$

Hence, the magnitude of change in the position vector is :

$$|\Delta \mathbf{r}| = AB = \sqrt{2}a$$

Clearly,

$$\Delta r \neq |\Delta \mathbf{r}|$$

Further, $|\mathbf{r}|$ represents the magnitude of position vector and is equal to “ r ” by conventional meaning. Hence,

$$\Delta r = \Delta |\mathbf{r}|$$

Therefore,

$$\Delta |\mathbf{r}| \neq |\Delta \mathbf{r}|$$

This completes the discussion on similarities and differences among the four symbols. But, the question remains why there are differences in the first place. The answer lies in the fact that position vector is a vector quantity with directional property. It means that it can change in either of the following manner :

1. change in magnitude
2. change in direction
3. change in both magnitude and direction

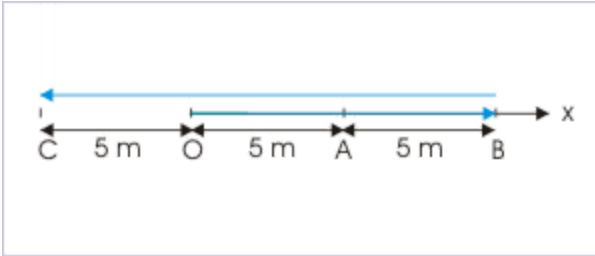
Thus we see that it is entirely possible, as in the case of circular motion, that change in position vector is attributed to the change in direction alone (not the magnitude). In that case, Δr and $|\Delta \mathbf{r}|$ are not same. We can see that such difference in meaning arises due to the consideration of direction. Will this difference persist even in one dimensional motion?

In one dimensional motion, representations of change in position vector and displacement are done with equivalent scalar system. Let us examine the meaning of equivalent scalar terms. Here, the change in the magnitude of position vector “ r ” is equivalent to the change in the magnitude of position

vector, represented by scalar equivalent " $|x|$ ". Also, the magnitude of change in the position vector " $\mathbf{r}$ " is equivalent to the magnitude of change in the position vector, " x " (note that signed scalar " x " denotes position vector).

Let us consider the case of a rectilinear motion (motion along a straight line) taking place from A to B to C. The magnitude of change in the position vector, " x ", considering "O" as the origin is :

Rectilinear motion



A particle moves along a straight line.

$$|\Delta x| = |x_2 - x_1| = |-5 - 5| = 10 \text{ m}$$

Now, the change in the magnitude of position vector, " $|x|$ ", is :

$$\Delta|x| = |x_2| - |x_1| = 5 - 5 = 0$$

Thus we see that difference in two terms exist even in one dimensional motion. This is expected also as one dimensional motion can involve reversal of direction as in the case considered above. Hence,

Equation:

$$\Delta|x| \neq |\Delta x|$$

Let us now consider unidirectional one dimensional motion like uniform motion in which velocity is constant and particle moves in only one direction. In this case, this difference disappears and

Equation:

$$\Delta|x| = |\Delta x|$$

The discussion here on this subtle difference is very important as this becomes an important consideration subsequently with velocity and acceleration as well, which are defined in terms of position vector.

Example

Example:

Problem : The displacement (x) of a particle is given by :

$$x = A \sin(\omega t + \theta)$$

At what time from the start of motion is the displacement maximum?

Solution : The displacement (x) depends on the value of sine function. It will be maximum for maximum value of $\sin(\omega t + \theta)$. The maximum value of sine function is 1. Hence,

$$\sin(\omega t + \theta) = 1 = \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \omega t + \theta = \pi/2$$

$$\Rightarrow t = \frac{\pi}{2\omega} - \frac{\theta}{\omega}$$

The motion is oscillating as expression for displacement is sinusoidal. The particle will attain maximum displacement at regular intervals.

Speed

Motion is the change of position with respect to time. Speed quantifies this change in position, but notably without direction. It tells us exactly : how rapidly this change is taking place with respect to time.

Motion

Speed is the rate of change of distance with respect to time and is expressed as distance covered in unit time.

Equation:

$$v = \frac{\Delta s}{\Delta t}$$

Equation:

$$\Rightarrow \Delta s = v\Delta t$$

Evaluation of ratio of distance and time for finite time interval is called “average” speed, where as evaluation of the ratio for infinitesimally small time interval, when $\Delta t \rightarrow 0$, is called instantaneous speed. In order to distinguish between average and instantaneous speed, we denote them with symbols v_a and v respectively.

Determination of speed allows us to compare motions of different objects. An aircraft, for example, travels much faster than a motor car. This is an established fact. But, we simply do not know how fast the aircraft is in comparison to the motor car. We need to measure speeds of each of them to state the difference in quantitative terms.

Speed is defined in terms of distance and time, both of which are scalar quantities. It follows that speed is a scalar quantity, having only magnitude and no sense of direction. When we say that a person is pacing at a speed of 3 km/hr, then we simply mean that the person covers 3 km in 1 hour. It is not known, however, where the person is actually heading and in which direction.

Dimension of speed is $\mathbf{LT_{-1}}$ and its SI unit is meter/second (m/s).

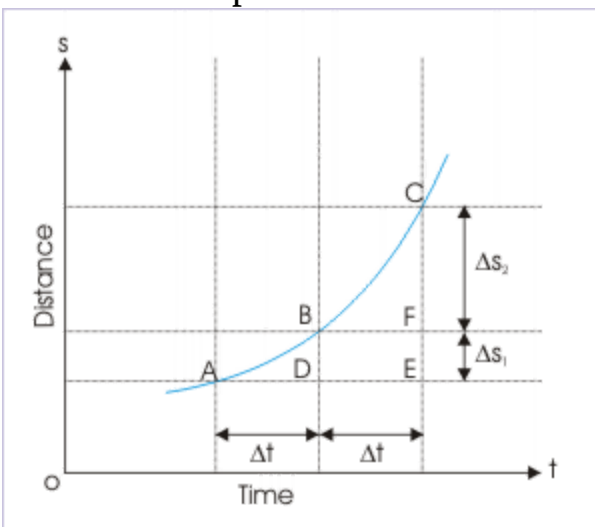
Some values of speed

- Light : 3×10^8 m/s
- Sound : 330 m/s
- Continental drift : 10^{-9} m/s

Distance .vs. time plots

Motion of an object over a period of time may vary. These variations are conveniently represented on a distance - time plot as shown in the figure.

Distance time plot



Distance is given by the vertical segment parallel to the axis representing distance.

The figure above displays distance covered in two equal time intervals. The vertical segment DB and FC parallel to the axis represents distances covered in the two equal time intervals. The distance covered in two equal time Δt intervals may not be equal as average speeds of the object in the two equal time intervals may be different.

$$s_1 = v_1 \Delta t = DB$$

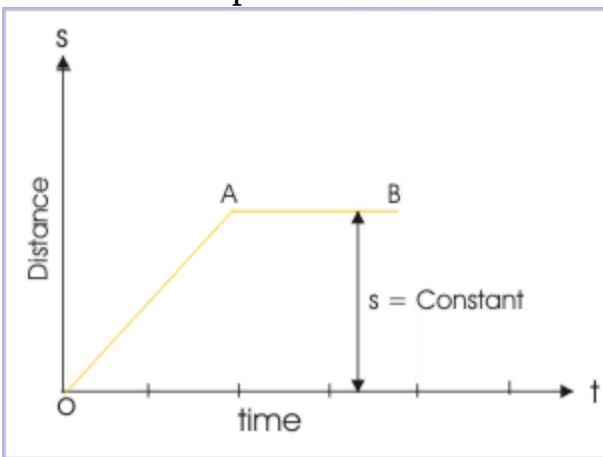
$$s_2 = v_2 \Delta t = FC$$

and

$$DB \neq FC$$

The distance - time plot characterizes the nature of distance. We see that the plot is always drawn in the first quadrant as distance can not be negative. Further, distance – time plot is ever increasing during the motion. It means that the plot can not decrease from any level at a given instant. When the object is at rest, the distance becomes constant and plot is a horizontal line parallel to time axis. Note that the portion of plot with constant speed does not add to the distance and the vertical segment representing distance remains constant during the motion.

Distance time plot



Static condition is represented
by a horizontal section on
distance - time plot.

Average speed

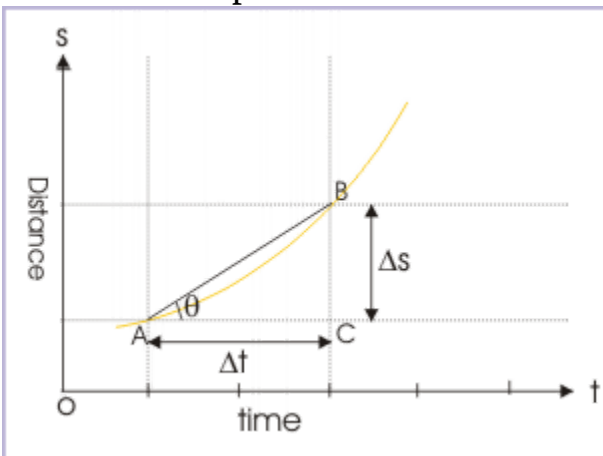
Average speed, as the name suggests, gives the overall view of the motion. It does not, however, give the details of motion. Let us take the example of the school bus. Ignoring the actual, let us consider that the average speed of the journey is 50 km/ hour. This piece of information about speed is very useful in planning the schedule, but the information is not complete as far as the motion is concerned. The school bus could have stopped at predetermined stoppages and crossings, besides traveling at different speeds for variety of reasons. Mathematically,

Equation:

$$v_a = \frac{\Delta s}{\Delta t}$$

On a distance time plot, average speed is equal to the slope of the straight line, joining the end points of the motion, makes with the time axis. Note that average speed is equal to the slope of the chord (AB) and **not** that of the tangent to the curve.

Distance time plot



Distance is equal to the tangent of the angle that the chord of motion makes with time axis.

$$v_a = \tan \theta = \frac{\Delta s}{\Delta t} = \frac{BC}{AC}$$

Example:**Average speed**

Problem : The object is moving with two different speeds v_1 and v_2 in two equal time intervals. Find the average speed.

Solution : The average speed is given by :

$$v_a = \frac{\Delta s}{\Delta t}$$

Let the duration of each time interval is t . Now, total distance is :

$$\Delta s = v_1 t + v_2 t$$

The total time is :

$$\Delta t = t + t = 2t$$

$$\Rightarrow v_a = \frac{v_1 t + v_2 t}{2t}$$

$$\Rightarrow v_a = \frac{v_1 + v_2}{2}$$

The average speed is equal to the arithmetic mean of two speeds.

Exercise:**Problem:**

The object is moving with two different speeds v_1 and v_1 in two equal intervals of distances. Find the average speed.

Solution:**Solution :**

The average speed is given by :

$$v = \frac{\Delta s}{\Delta t}$$

Let "s" be distance covered in each time interval t_1 and t_2 . Now,

$$\Delta s = s + s = 2s$$

$$\Delta t = t_1 + t_2 = \frac{s}{v_1} + \frac{s}{v_2}$$

$$\Rightarrow v_a = \frac{2v_1v_2}{v_1+v_2}$$

The average speed is equal to the harmonic mean of two velocities.

Instantaneous speed (v)

Instantaneous speed is also defined exactly like average speed i.e. it is equal to the ratio of total distance and time interval, but with one qualification that time interval is extremely (infinitesimally) small. This qualification of average speed has important bearing on the value and meaning of speed. The instantaneous speed is the speed at a particular instant of time and may have entirely different value than that of average speed. Mathematically,

Equation:

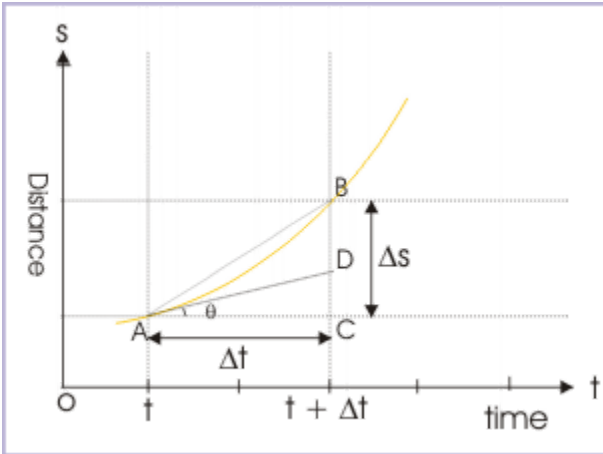
$$v = \lim_{\Delta s \rightarrow 0 \Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Where Δs is the distance traveled in time Δt .

As Δt tends to zero, the ratio defining speed becomes finite and equals to the first derivative of the distance. The speed at the moment 't' is called the instantaneous speed at time 't'.

On the distance - time plot, the speed is equal to the slope of the tangent to the curve at the time instant 't'. Let A and B points on the plot corresponds to the time t and $t + \Delta t$ during the motion. As Δt approaches zero, the chord AB becomes the tangent AC at A. The slope of the tangent equals ds/dt , which is equal to the instantaneous speed at 't'.

Speed time plot



Instantaneous speed is equal to the slope of the tangent at given instant.

$$v = \tan \theta = \frac{BC}{AC} = \frac{ds}{dt}$$

Speed - time plot

The distance covered in the small time period dt is given by :

$$ds = v dt$$

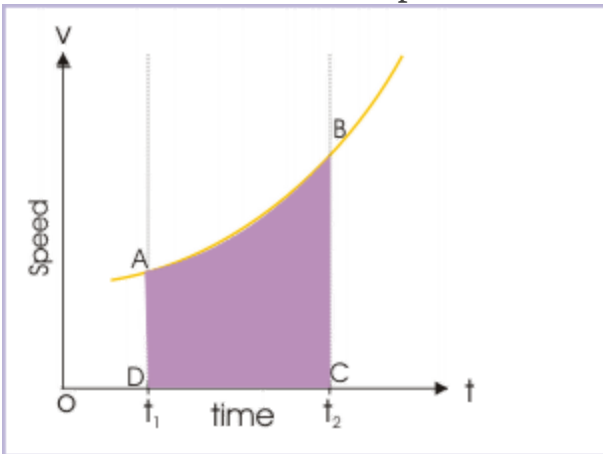
Integrating on both sides between time intervals t_1 and t_2 ,

Equation:

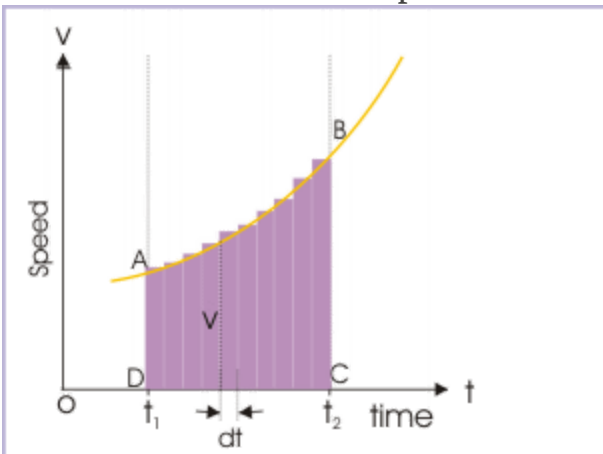
$$s = \int_{t_1}^{t_2} v dt$$

The right hand side of the integral represents an area on a plot drawn between two variables, speed (v) and time (t). The area is bounded by (i) v - t curve (ii) two time ordinates t_1 and t_2 and (iii) time (t) axis as shown by the shaded region on the plot.

Area under v-t plot



Area under v-t plot



Area under v-t plot gives the distance covered by the object in a given time interval.

Alternatively, we can consider the integral as the sum of areas of small strip of rectangular regions ($v \times dt$), each of which represents the distance covered (ds) in the small time interval (dt). As such, the area under speed - time plot gives the total distance covered in a given time interval.

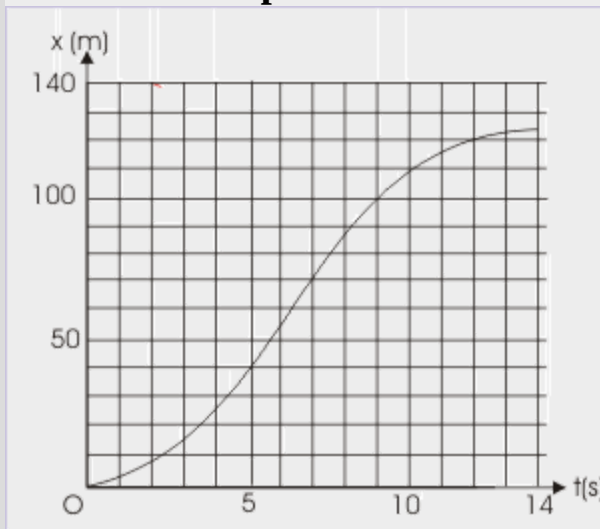
Position - time plot

The position – time plot is similar to distance – time plot in one dimensional motion. In this case, we can assess distance and speed from a position - time plot. Particularly, if the motion is unidirectional i.e. without any reversal of direction, then we can substitute distance by position variable. The example here illustrates this aspect of one-dimensional motion.

Example:

Problem The position – time plot of a particle's motion is shown below.

Position - time plot



Determine :

1. average speed in the first 10 seconds
2. Instantaneous speed at 5 seconds
3. Maximum speed
4. The time instant(s), when average speed measured from the beginning of motion equals instantaneous speed

Solution

1. Average speed in the first 10 seconds

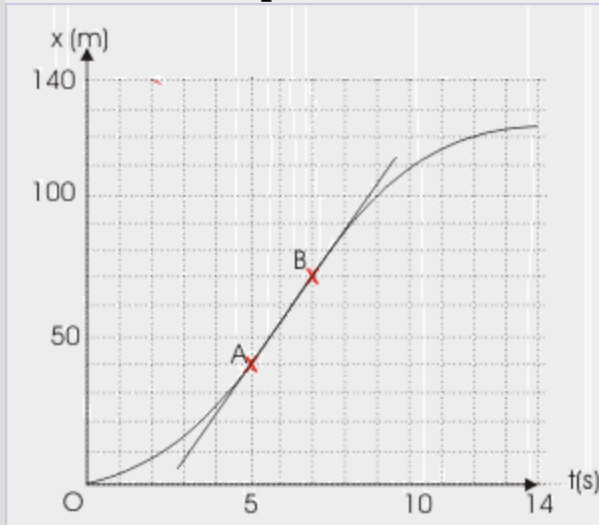
The particle covers a distance of 110 m. Thus, average speed in the first 10 seconds is :

$$v_a = \frac{110}{10} = 11 \text{ m/s}$$

2. Instantaneous speed at 5 second

We draw the tangent at $t = 5$ seconds as shown in the figure. The tangent coincides the curve between $t = 5$ s to 7 s. Now,

Position - time plot



$$v = \text{slope of the tangent} = \frac{30}{2} = 15 \text{ m/s}$$

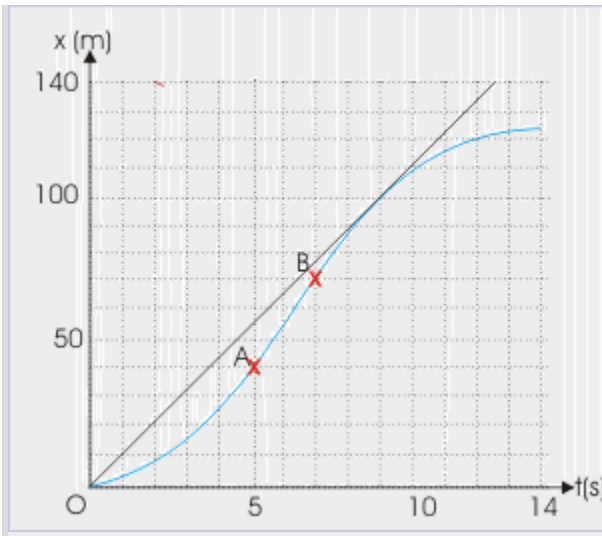
3. Maximum speed

We observe that the slope of the tangent to the curve first increase to become constant between A and B. The slope of the tangent after point B decreases to become almost flat at the end of the motion. It means the maximum speed corresponds to the constant slope between A and B, which is 15 m/s

4. Time instant(s) when average speed equals instantaneous speed

Average speed is equal to slope of chord between one point to another. On the other hand, speed is equal to slope of tangent at point on the plot. This means that slope of chord between two times is equal to tangent at the end point of motion. Now, as time is measured from the start of motion i.e. from origin, we draw a straight line from the origin, which is tangent to the curve. The only such tangent is shown in the figure below. Now, this straight line is the chord between origin and a point. This line is also tangent to the curve at that point. Thus, average speed equals instantaneous speeds at $t = 9$ s.

Position - time plot

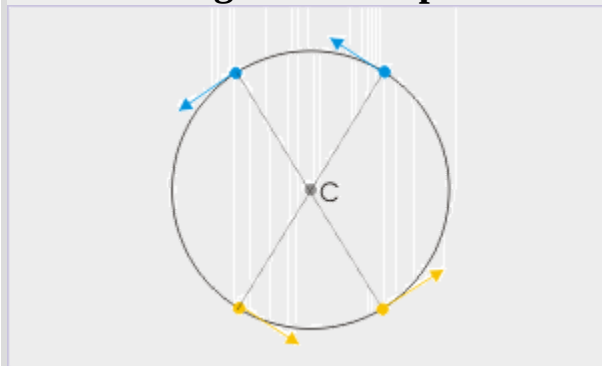


Example:

Problem : Two particles are moving with the same constant speed, but in opposite direction. Under what circumstance will the separation between two remains constant?

Solution : The condition of motion as stated in the question is possible, if particles are at diametrically opposite positions on a circular path. Two particles are always separated by the diameter of the circular path. See the figure below to evaluate the motion and separation between the particles.

Motion along a circular path



Two particles are always separated by the diameter of the circle transversed by the particles.

Velocity

Velocity is the measure of rapidity with which a particle covers shortest distance between initial and final positions, irrespective of the actual path. It also indicates the direction of motion as against speed, which is devoid of this information.

Velocity

Velocity is the rate of change of displacement with respect to time and is expressed as the ratio of displacement and time.

Equation:

$$\mathbf{v} = \frac{\text{Displacement}}{\Delta t}$$

$$\Rightarrow \text{Displacement} = \Delta \mathbf{v} t$$

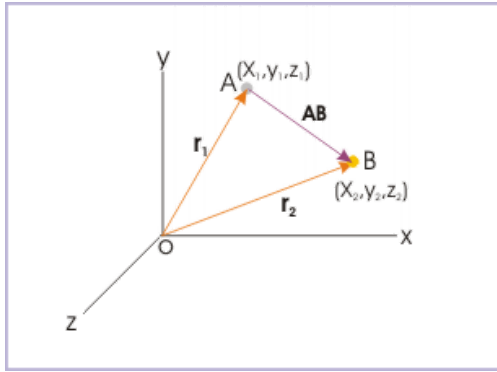
If the ratio of displacement and time is evaluated for finite time interval, we call the ratio “average” velocity, whereas if the ratio is evaluated for infinitesimally small time interval ($\Delta t \rightarrow 0$), then we call the ratio “instantaneous” velocity. Conventionally, we denote average and instantaneous velocities as $\mathbf{v}_a$ and $\mathbf{v}$ respectively to differentiate between the two concepts of velocity.

As against speed, which is defined in terms of distance, velocity is defined in terms of displacement. Velocity amounts to be equal to the multiplication of a scalar ($1/\Delta t$) with a vector (displacement). As scalar multiplication of a vector is another vector, velocity is a vector quantity, having both magnitude and direction. The direction of velocity is same as that of displacement and the magnitude of velocity is numerically equal to the absolute value of the velocity vector, denoted by the corresponding non bold face counterpart of the symbol.

Dimension of velocity is LT^{-1} and its SI unit is meter/second (m/s).

Position vector and velocity

The displacement is equal to the difference of position vectors between initial and final positions. As such, velocity can be conveniently expressed in terms of position vectors. Displacement in terms of position vector



Displacement is equal to the change in position vector

Let us consider that $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors corresponding to the object positions at time instants t_1 and t_2 . Then, displacement is given by :

Equation:

$$\mathbf{v} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} = \frac{\Delta \mathbf{r}}{\Delta t}$$

Velocity

Velocity is the rate of change of position vector with respect to time and is expressed as the ratio of change in position vector and time.

The expression of velocity in terms of position vectors is generally considered more intuitive and basic to the one expressed in terms of displacement. This follows from the fact that displacement vector itself is equal to the difference in position vectors between final and initial positions.

Average velocity

Average velocity is defined as the ratio of total displacement and time interval.

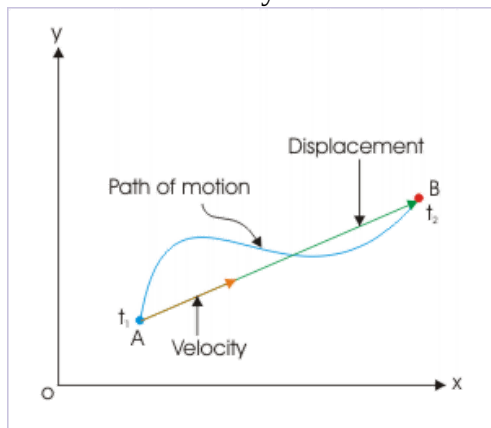
Equation:

$$\mathbf{v}_a = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$$

Average velocity gives the overall picture about the motion. The magnitude of the average velocity tells us the rapidity with which the object approaches final point along the straight line – not the rapidity along the actual path of motion. It is important to notice here that the magnitude of average velocity does not depend on the actual path as in the case of speed, but depends on the shortest path between two points represented by the straight line joining the

two ends. Further, the direction of average velocity is from the initial to final position along the straight line (See Figure).

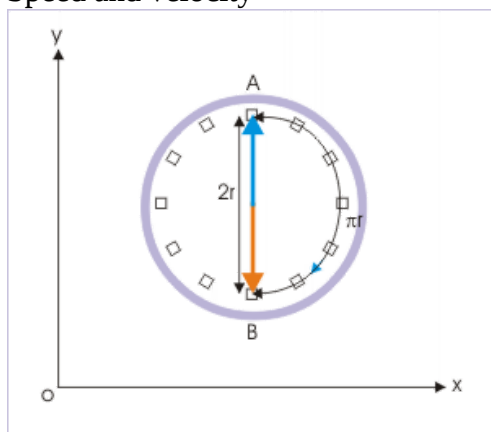
Direction of velocity



Average velocity may be different to instantaneous velocities in between the motion in either magnitude or direction or both. Consider the example of the tip of the second's hand of a wall clock. It moves along a circular path of $2\pi r$ in 60 seconds. The magnitude of average velocity is zero in this period (60 seconds) as the second's hand reaches the initial position. This is the overall picture. However, the tip of the second's hand has actually traveled the path of $2\pi r$, indicating that intermediate instantaneous velocities during the motion were not zero.

Also, the magnitude of average velocity may be entirely different than that of average speed. We know that distance is either greater than or equal to the magnitude of displacement. It follows then that average speed is either greater than or equal to the magnitude of average velocity. For the movement along semi-circle as shown in the figure below, the magnitude of velocity is $2r/30$ m/s, whereas average speed is $\pi r/30 = 3.14 r/30$ m/s. Clearly, the average speed is greater than the magnitude of average velocity.

Speed and velocity



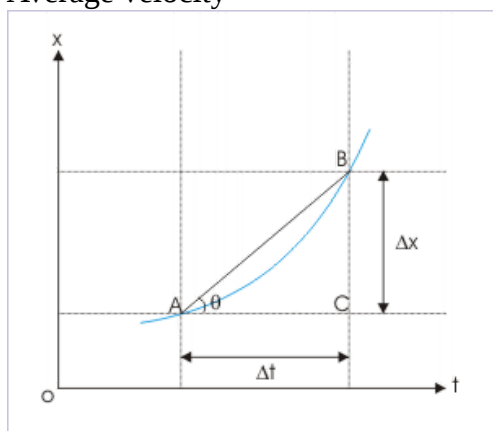
Position – time plot and average velocity

Position – time plot is a convenient technique to interpret velocity of a motion. The limitation here is that we can plot position – time graph only for one and two dimensional motions. As a matter of fact, it is only one dimensional (linear or rectilinear) motion, which renders itself for convenient drawing.

On the plot, positions are plotted with appropriate sign against time. A positive value of position indicates that particle is lying on the positive side of the origin, whereas negative value of position indicates that the particle is lying on the opposite side of the origin. It must, therefore, be realized that a position – time plot may extend to two quadrants of a two dimensional coordinate system as the value of x can be negative.

On a position – time plot, the vertical intercept parallel to position axis is the measure of displacement, whereas horizontal intercept is the measure of time interval (See Figure). On the other hand, the slope of the chord is equal to the ratio of two intercepts and hence equal to the magnitude of average velocity.

Average velocity



Equation:

$$|\mathbf{v}_a| = \frac{BC}{AC} = \frac{\Delta x}{\Delta t}$$

Example:

Average velocity

Problem : A particle completes a motion in two parts. It covers a straight distance of 10 m in 1 s in the first part along the positive x - direction and 20 m in 5 s in the second part along negative x - direction (See Figure). Find average speed and velocity.

Motion in straight line



Characteristics of motion : One dimensional

Solution : In order to find the average speed, we need to find distance and time. Here total time is $1 + 5 = 6$ s and total distance covered is $10 + 20 = 30$ m. Hence,

$$v_a = \frac{30}{6} = 5 \text{ m/s}$$

The displacement is equal to the linear distance between initial and final positions. The initial and final positions are at a linear distance = -10 m. The value is taken as negative as final position falls on the opposite side of the origin. Hence,

$$v_a = \frac{-10}{6} = -1.66 \text{ m/s}$$

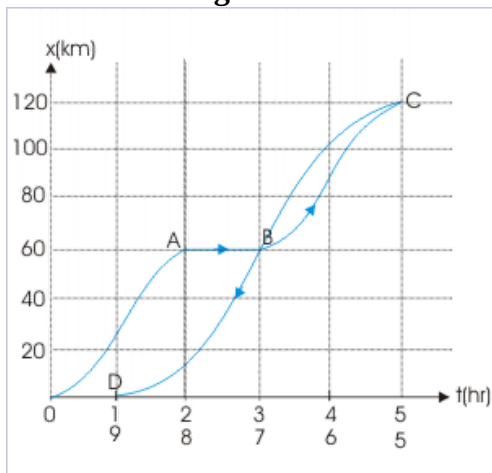
The negative value indicates that the average velocity is directed in the opposite direction to that of the positive reference direction. We have discussed earlier that one dimensional motion consists of only two direction and as such an one dimensional velocity can be equivalently represented by scalar value with appropriate sign scheme. Though, the symbol for average velocity is shown to be like a scalar symbol (not bold), but its value represents direction as well (the direction is opposite to reference direction).

Also significantly, we may note that average speed is not equal to the magnitude of average velocity.

Exercise:

Problem: Consider position – time plot as shown below showing a trip by a motor car.

Motion in straight line



Determine :

1. Total distance
2. Displacement
3. Average speed and average velocity for the round trip
4. Average speed and average velocity during motion from O to C
5. The parts of motion for which magnitudes of average velocity are equal in each direction.
6. Compare speeds in the portion OB and BD

Solution:

Characteristics of motion : One dimensional, variable speed

(i) Total distance in the round trip = $120 + 120 = 240$ Km

(ii) Displacement = $120 - 120 = 0$

(iii) Average speed for the round trip

$$v_{OD} = \frac{240}{9} = 26.67 \text{ km/hr} \quad \text{and average velocity for the round trip : } v_{OD} = \frac{0}{9} = 0$$

(iv) Average speed during motion from O to C

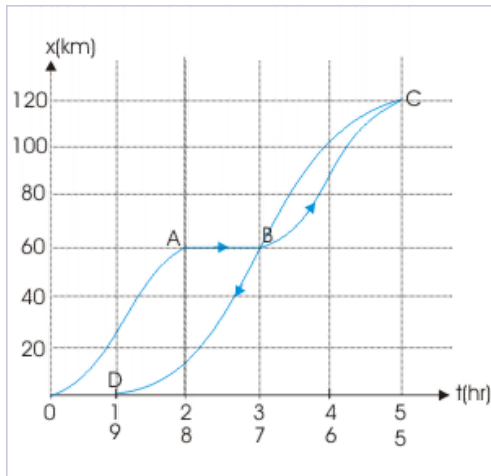
$$v_{OC} = \frac{120}{5} = 24 \text{ km/hr} \quad \text{and average velocity : } v_{OC} = \frac{120}{5} = 24 \text{ km/hr}$$

As magnitude of average velocity is positive, the direction of velocity is in the positive x – direction. It is important to note for motion in one dimension and in one direction (unidirectional), distance is equal to the magnitude of displacement and average speed is equal to the magnitude of average velocity. Such is the case for this portion of motion.

(v) The part of motion for which average velocity is equal in each direction.

By inspection of the plots, we see that time interval is same for motion from B to C and from C to B (on return). Also, the displacements in these two segments of motion are equal. Hence magnitudes of velocities in two segments are equal.

Motion in straight line



(vi) Compare speeds in the portion OB and BD.

By inspection of the plots, we see that the motor car travels equal distances of 60 m. We see that distances in each direction is covered in equal times i.e. 2 hrs. But, the car actually stops for 1 hour in the forward journey and as such average speed is smaller in this case.

$$v_{a,OB} = \frac{60}{3} = 20 \text{ km/hr} \quad \text{and} \quad v_{a,BD} = \frac{60}{2} = 30 \text{ km/hr}$$

Instantaneous velocity

velocity

Instantaneous velocity is equal to the rate of change of position vector i.e displacement with respect to time at a given time and is equal to the first differential of position vector.

Instantaneous velocity is defined exactly like speed. It is equal to the ratio of total displacement and time interval, but with one qualification that time interval is extremely (infinitesimally) small. Thus, instantaneous velocity can be termed as the average velocity at a particular instant of time when Δt tends to zero and may have entirely different value than that of average velocity. Mathematically,

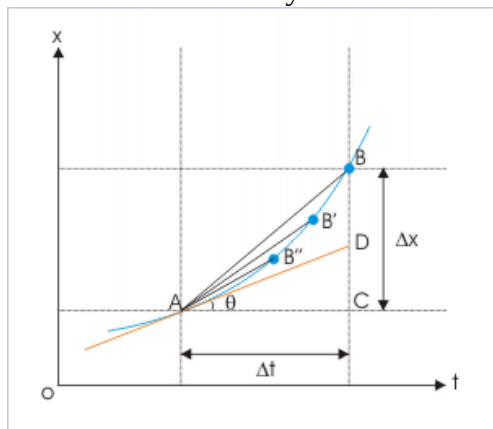
$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

As Δt tends to zero, the ratio defining velocity becomes finite and equals to the first derivative of the position vector. The velocity at the moment 't' is called the instantaneous velocity or simply velocity at time 't'.

Instantaneous velocity and position - time plot

Position - time plot provides for calculation of the magnitude of velocity, which is equal to speed. The discussion of position – time plot in the context of velocity, however, differs in one important respect that we can also estimate the direction of motion.

Instantaneous Velocity



In the figure above, as we proceed from point B to A through intermediate points B' and B'', the time interval becomes smaller and smaller and the chord becomes tangent to the curve at point A as $\Delta t \rightarrow 0$. The magnitude of instantaneous velocity (speed) at A is given by the slope of the curve.

$$|\mathbf{v}| = \frac{DC}{AC} = \frac{dx}{dt}$$

There is one important difference between average velocity and instantaneous velocity. The magnitude of average velocity $|\mathbf{v}_{avg}|$ and average speed v_{avg} may not be equal, but magnitude of instantaneous velocity $|\mathbf{v}|$ is always equal to instantaneous speed v .

We have discussed that magnitude of displacement and distance are different quantities. The magnitude of displacement is a measure of linear shortest distance, whereas distance is measure of actual path. As such, magnitude of average velocity $|\mathbf{v}_{avg}|$ and average speed v_{avg} are not be equal. However, if the motion is along a straight line and without any change in direction (i.e unidirectional), then distance and displacement are equal and so magnitude of average velocity and average speed are equal. In the case of instantaneous velocity, the time interval is infinitesimally small for which displacement and distance are infinitesimally small. In such situation, both displacement and distance are same. Hence, magnitude of instantaneous velocity $|\mathbf{v}|$ is always equal to instantaneous speed v .

Components of velocity

Velocity in a three dimensional space is defined as the ratio of displacement (change in position vector) and time. The object in motion undergoes a displacement, which has components in three mutually perpendicular directions in Cartesian coordinate system.

$$\Delta \mathbf{r} = \Delta x \mathbf{i} + \Delta y \mathbf{j} + \Delta z \mathbf{k}$$

It follows from the component form of displacement that a velocity in three dimensional coordinate space is the vector sum of component velocities in three mutually perpendicular directions. For a small time interval when $\Delta t \rightarrow 0$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

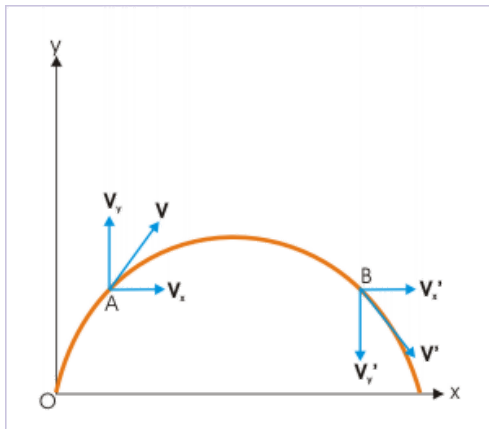
$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

For the sake of clarity, it must be understood that components of velocity is a conceptual construct for examining a physical situation. It is so because it is impossible for an object to have two velocities at a given time. If we have information about the variations of position along three mutually perpendicular directions, then we can find out component velocities along the axes leading to determination of resultant velocity. The resultant velocity is calculated using following relation :

$$v = |\mathbf{v}| = \sqrt{(v_x^2 + v_y^2 + v_z^2)}$$

The component of velocity is a powerful concept that makes it possible to treat a three or two dimensional motion as composition of component straight line motions. To illustrate the point, consider the case of two dimensional parabolic motions. Here, the velocity of the body is resolved in two mutually perpendicular directions; treating motion in each direction independently and then combining the component directional attributes by using rules of vector addition

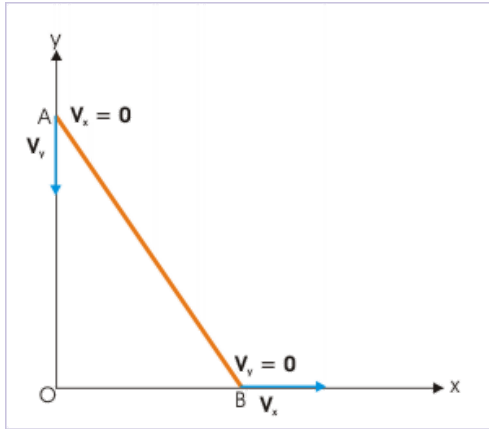
Parabolic motion



Motion is treated separately in two perpendicular directions

Similarly, the concept of component velocity is useful when motion is constrained. We may take the case of the motion of the edge of a pole as shown in the figure here. The motion of the ends of the pole is constrained in one direction, whereas other component of velocity is zero.

Constrained motion



The motion in space is determined by the component velocities in three mutually perpendicular directions. In two dimensional or planar motion, one of three components is zero. 10. The velocity of the object is determined by two relevant components of velocities in the plane. For example, motion in x and y direction yields :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$$

$$v = |\mathbf{v}| = \sqrt{(v_x^2 + v_y^2)}$$

Similarly, one dimensional motion (For example : x – direction) is described by one of the components of velocity.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i}$$

$$\mathbf{v} = v_x\mathbf{i}$$

$$v = |\mathbf{v}| = v_x$$

Few words of caution

Study of kinematics usually brings about closely related concepts, terms and symbols. It is always desirable to be precise and specific in using these terms and symbols. Following list of the terms along with their meaning are given here to work as reminder :

- 1: Position vector : $\mathbf{r}$: a vector specifying position and drawn from origin to the point occupied by point object
- 2: Distance : s : length of actual path : not treated as the magnitude of displacement
- 3: Displacement : $\mathbf{AB}$ or $\Delta\mathbf{r}$: a vector along the straight line joining end points A and B of the path : its magnitude, $|\mathbf{AB}|$ or $|\Delta\mathbf{r}|$ is not equal to distance, s .
- 4: Difference of position vector : $\Delta\mathbf{r}$: equal to displacement, $\mathbf{AB}$. Direction of $\Delta\mathbf{r}$ is not same as that of position vector ($\mathbf{r}$).

- 5: Magnitude of displacement : $|\mathbf{AB}|$ or $|\Delta \mathbf{r}|$: length of shortest path.
- 6: Average speed : v_a : ratio of distance and time interval : not treated as the magnitude of average velocity
- 7: Speed : v : first differential of distance with respect to time : equal to the magnitude of velocity, $|\mathbf{v}|$
- 8: Average velocity : $\mathbf{v}_a$: ratio of displacement and time interval : its magnitude, $|\mathbf{v}_a|$ is not equal to average speed, v_a .
- 9: Velocity : $\mathbf{v}$: first differential of displacement or position vector with respect to time

Summary

The paragraphs here are presented to highlight the similarities and differences between the two important concepts of speed and velocity with a view to summarize the discussion held so far.

1: Speed is measured without direction, whereas velocity is measured with direction. Speed and velocity both are calculated at a position or time instant. As such, both of them are independent of actual path. Most physical measurements, like speedometer of cars, determine instantaneous speed. Evidently, speed is the magnitude of velocity,

$$v = |\mathbf{v}|$$

2: Since, speed is a scalar quantity, it can be plotted on a single axis. For this reason, tangent to distance – time curve gives the speed at that point of the motion. As $ds = v \times dt$, the area under speed – time plot gives distance covered between two time instants.

3: On the other hand, velocity requires three axes to be represented on a plot. It means that a velocity – time plot would need 4 dimensions to be plotted, which is not possible on three dimensional Cartesian coordinate system. A two dimensional velocity and time plot is possible, but is highly complicated to be drawn.

4: One dimensional velocity can be treated as a scalar magnitude with appropriate sign to represent direction. It is, therefore, possible to draw one dimension velocity – time plot.

5: Average speed involves the length of path (distance), whereas average velocity involves shortest distance (displacement). As distance is either greater than or equal to the magnitude of displacement,

$$s \geq |\Delta \mathbf{r}| \text{ and } v_a \geq |\mathbf{v}_a|$$

Exercises

Exercise:

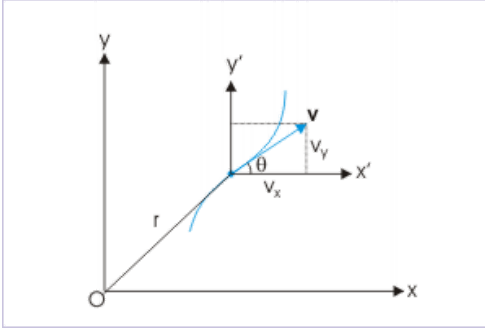
Problem: The position vector of a particle (in meters) is given as a function of time as :

$$\mathbf{r} = 2t\mathbf{i} + 2t^2\mathbf{j}$$

Determine the time rate of change of the angle " θ " made by the velocity vector with positive x-axis at time, $t = 2$ s.

Solution:

Solution : It is a two dimensional motion. The figure below shows how velocity vector makes an angle " θ " with x-axis of the coordinate system. In order to find the time rate of change of this angle " θ ", we need to express trigonometric ratio of the angle in terms of the components of velocity vector. From the figure :

Velocity of a particle in two dimensions

The velocity has two components.

$$\tan \theta = \frac{v_y}{v_x}$$

As given by the expression of position vector, its component in coordinate directions are :

$$x = 2t \quad \text{and} \quad y = 2t^2$$

We obtain expression of the components of velocity in two directions by differentiating "x" and "y" components of position vector with respect to time :

$$v_x = 2 \quad \text{and} \quad v_y = 4t$$

Putting in the trigonometric function, we have :

$$\tan \theta = \frac{v_y}{v_x} = \frac{4t}{2} = 2t$$

Since we are required to know the time rate of the angle, we differentiate the above trigonometric ratio with respect to time as,

$$\begin{aligned}\sec^2\theta \frac{d\theta}{dt} &= 2 \\ \Rightarrow (1 + \tan^2\theta) \frac{d\theta}{dt} &= 2 \\ \Rightarrow (1 + 4t^2) \frac{d\theta}{dt} &= 2 \\ \Rightarrow \frac{d\theta}{dt} &= \frac{2}{(1+4t^2)}\end{aligned}$$

At $t = 2$ s,

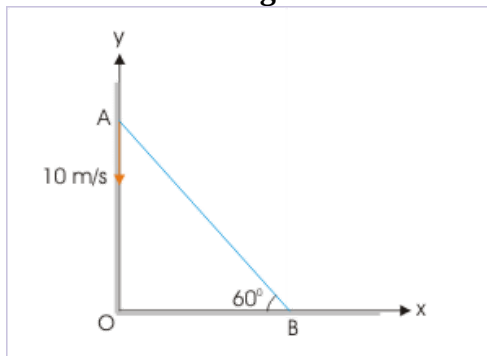
$$\Rightarrow \frac{d\theta}{dt} = \frac{2}{(1+4 \times 2^2)} = \frac{2}{17} \text{ rad /s}$$

Exercise:

Problem:

Two particles A and B are connected by a rigid rod AB. The rod slides along perpendicular rails as shown here. The velocity of A moving down is 10 m/s. What is the velocity of B when angle $\theta = 60^\circ$?

Motion of a leaning rod



One end of the rod is moving with a speed 10 m/s in vertically downward direction.

Solution:

Solution : The velocity of B is not an independent velocity. It is tied to the velocity of the particle “A” as two particles are connected through a rigid rod. The relationship between two velocities is governed by the inter-particles separation, which is equal to the length of rod.

The length of the rod, in turn, is linked to the positions of particles “A” and “B” . From figure,

$$x = \sqrt{(L^2 - y^2)}$$

Differentiating, with respect to time :

$$\Rightarrow v_x = \frac{dx}{dt} = -\frac{2y}{2\sqrt{(L^2 - y^2)}} \times \frac{dy}{dt} = -\frac{yv_y}{\sqrt{(L^2 - y^2)}} = -v_y \tan \theta$$

Considering magnitude only,

$$\Rightarrow v_x = v_y \tan \theta = 10 \tan 60^\circ = 10\sqrt{3} \quad \frac{m}{s}$$

Exercise:

Problem: The position vector of a particle is :

$$\mathbf{r} = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

where “a” is a constant. Show that velocity vector is perpendicular to position vector.

Solution:

Solution : In order to prove as required, we shall use the fact that scalar (dot) product of two perpendicular vectors is zero. Now, we need to find the expression of velocity to evaluate the dot product as intended. We can obtain the same by differentiating the expression of position vector with respect to time as :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}$$

To check whether velocity is perpendicular to the position vector, we evaluate the scalar product of $\mathbf{r}$ and $\mathbf{v}$, which should be equal to zero.

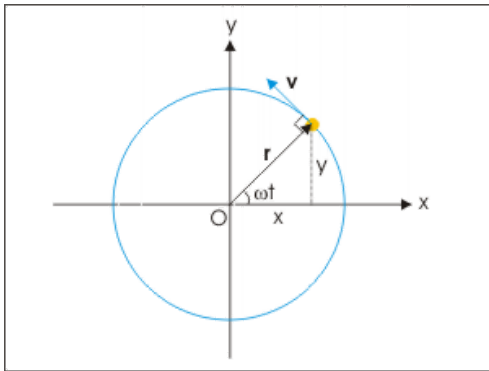
$$\mathbf{r} \cdot \mathbf{v} = 0$$

In this case,

$$\begin{aligned} \Rightarrow \mathbf{r} \cdot \mathbf{v} &= (a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}) \cdot (-a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}) \\ \Rightarrow -a^2 \omega \sin \omega t \cos \omega t + a^2 \omega \sin \omega t \cos \omega t &= 0 \end{aligned}$$

This means that the angle between position vector and velocity are at right angle to each other. Hence, velocity is perpendicular to position vector. It is pertinent to mention here that this result can also be inferred from the plot of motion. An inspection of position vector reveals that it represents uniform circular motion as shown in the figure here.

Circular motion



The particle describes a circular path.

The position vector is always directed radially, whereas velocity vector is always tangential to the circular path. These two vectors are, therefore, perpendicular to each other.

Exercise:

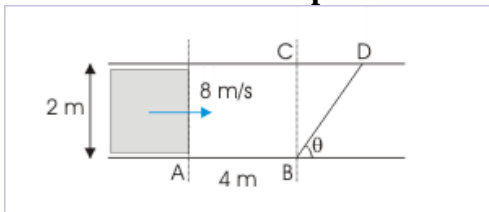
Problem:

Problem : A car of width 2 m is approaching a crossing at a velocity of 8 m/s. A pedestrian at a distance of 4 m wishes to cross the road safely. What should be the minimum speed of pedestrian so that he/she crosses the road safely?

Solution:

Solution : We draw the figure to illustrate the situation. Here, car travels the linear distance (AB + CD) along the direction in which it moves, by which time the pedestrian travels the linear distance BD. Let pedestrian travels at a speed “ v ” along BD, which makes an angle “ θ ” with the direction of car.

Motion of a car and a pedestrian



The pedestrian crosses the road at angle with direction of car.

We must understand here that there may be number of combination of angle and speed for which pedestrian will be able to safely cross before car reaches. However, we are required to find the minimum speed. This speed should, then, correspond to a particular value of θ .

We can also observe that pedestrian should move obliquely. In doing so he/she gains extra time to cross the road.

From triangle BCD,

$$\tan(90 - \theta) = \cot \theta = \frac{CD}{BC} = \frac{CD}{2}$$

$$\Rightarrow CD = 2 \cot \theta$$

Also,

$$\cos(90 - \theta) = \sin \theta = \frac{BC}{BD} = \frac{2}{BD}$$

$$\Rightarrow BD = \frac{2}{\sin \theta}$$

According to the condition given in the question, the time taken by car and pedestrian should be equal for the situation outlined above :

$$t = \frac{4 + 2 \cot \theta}{8} = \frac{\frac{2}{\sin \theta}}{v}$$

$$v = \frac{8}{2 \sin \theta + \cos \theta}$$

For minimum value of speed, $\frac{dv}{d\theta} = 0$,

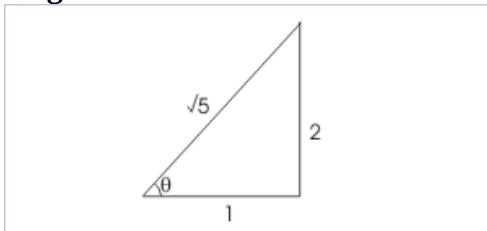
$$\Rightarrow \frac{dv}{d\theta} = \frac{-8 \times (2 \cos \theta - \sin \theta)}{(2 \sin \theta + \cos \theta)^2} = 0$$

$$\Rightarrow (2 \cos \theta - \sin \theta) = 0$$

$$\Rightarrow \tan \theta = 2$$

In order to evaluate the expression of velocity with trigonometric ratios, we take the help of right angle triangle as shown in the figure, which is consistent with the above result.

Trigonometric ratio



The tangent of angle is equal to 2.

From the triangle, defining angle “ θ ”, we have :

$$\sin \theta = 2\sqrt{5}$$

and

$$\cos \theta = \frac{1}{\sqrt{5}}$$

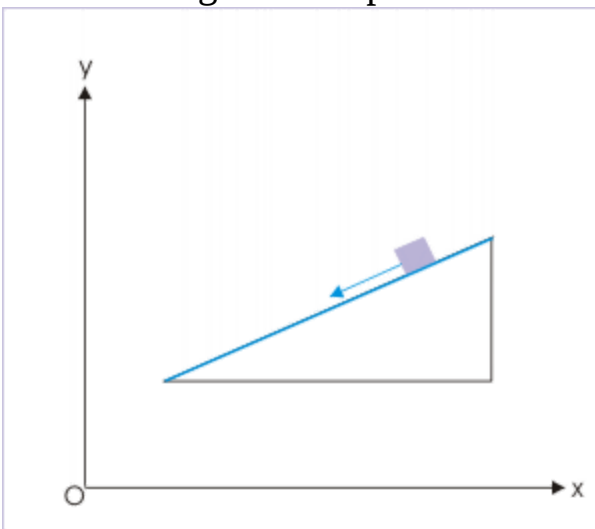
The minimum velocity is :

$$v = \frac{8}{2 \times 2\sqrt{5 + \frac{1}{\sqrt{5}}}} = \frac{8}{\sqrt{5}} = 3.57 \text{ m/s}$$

Rectilinear motion

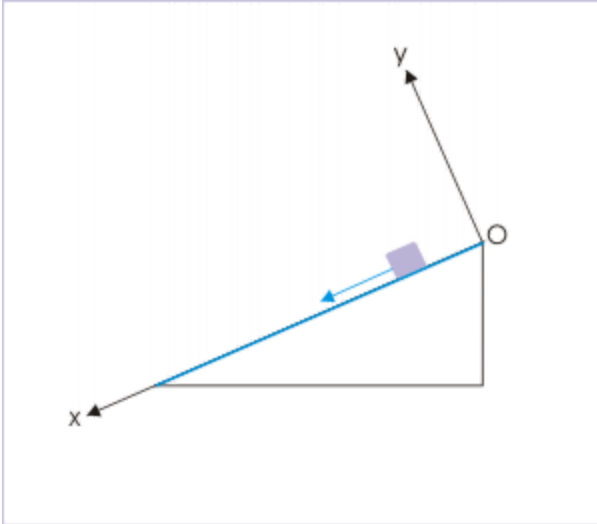
A motion along straight line is called rectilinear motion. In general, it need not be one – dimensional; it can take place in a two dimensional plane or in three dimensional space. But, it is always possible that rectilinear motion be treated as one dimensional motion, by suitably orienting axes of the coordinate system. This fact is illustrated here for motion along an inclined plane. The figure below depicts a rectilinear motion of the block as it slides down the incline. In this particular case, the description of motion in the coordinate system, as shown, involves two coordinates (x and y).

Motion along inclined plane



The reorientation of the coordinate system renders two dimensional description (requiring x and y values) of the motion to one dimensional (requiring only x value). A proper selection, most of the time, results in simplification of measurement associated with motion. In the case of the motion of the block, we may choose the orientation such that the progress of motion is along the positive x – direction as shown in the figure. A proper orientation of the coordinates results in positive values of quantities like displacement and velocity. It must be emphasized here that we have complete freedom in choosing the orientation of the coordinate system. The description of the rectilinear motion is independent of the orientation of axes.

Motion along inclined plane



In rectilinear motion, we are confined to the measurement of movement of body in only one direction. This simplifies expressions of quantities used to describe motion. In the following section, we discuss (also recollect from earlier discussion) the simplification resulting from motion in one dimension (say in x –direction).

Position vector in rectilinear motion

Position still requires three coordinates for specification. But, only one of them changes during the motion; remaining two coordinates remain constant. In practice, we choose one dimensional reference line to coincide with the path of the motion. It follows then that position of the particle under motion is equal to the value of x – coordinate (others being zero).

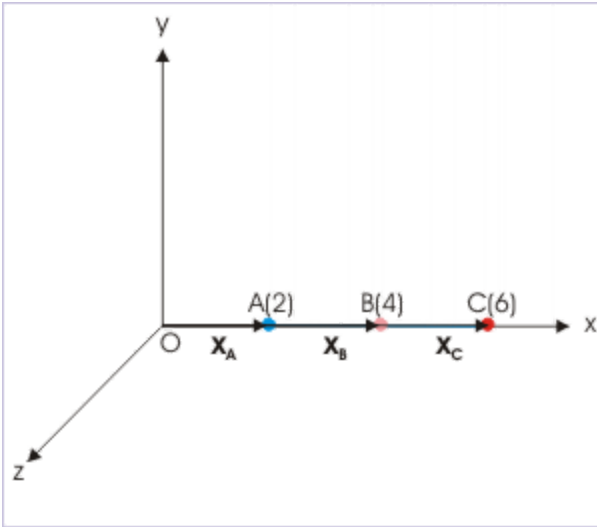
Corresponding position vector also remains a three dimensional quantity. However, if the path of motion coincides with the reference direction and origin of the reference coincides with origin, then position vector is simply equal to component vector in x – direction i.e

Equation:

$$\mathbf{r} = x\mathbf{i}$$

The position vectors corresponding to points A, B and C as shown in the figure are $2\mathbf{i}$, $4\mathbf{i}$ and $6\mathbf{i}$ units respectively.

Position vectors



As displacement is equal to change in position vector, the displacement for the indicated positions are given as :

$$\mathbf{AB} = (x_2 - x_1)\mathbf{i} = \Delta x\mathbf{i} = (4 - 2)\mathbf{i} = 2\mathbf{i}$$

$$\mathbf{BC} = (x_2 - x_1)\mathbf{i} = \Delta x\mathbf{i} = (6 - 4)\mathbf{i} = 2\mathbf{i}$$

$$\mathbf{AC} = (x_2 - x_1)\mathbf{i} = \Delta x\mathbf{i} = (6 - 2)\mathbf{i} = 4\mathbf{i}$$

Vector interpretation and equivalent system of scalars

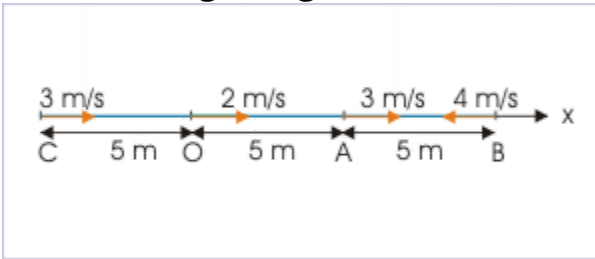
Rectilinear motion involves motion along straight line and thus is described usually in one dimension. Further, rectilinear motion involves only one way of changing direction i.e. the particle under motion can only reverse motion. The particle can move either in positive x-direction or in negative x-direction. There is no other possible direction valid in rectilinear motion.

This attribute of rectilinear motion allows us to do away with the need to use vector notation and vector algebra for quantities with directional attributes like position vector, displacement and velocity. Instead, the vectors are treated simply as scalars with one qualification that vectors in the direction of chosen reference is considered positive and vectors in the opposite direction to the chosen reference is considered negative.

The most important aspect of the sign convention is that a vector like velocity can be expressed by a scalar value say, 5 m/s. Though stated without any aid for specifying direction like using unit vector, the direction of the velocity is indicated, which is in the positive x-direction. If the velocity of motion is -5 m/s, then the velocity is in the direction opposite to the direction of reference.

To illustrate the construct, let us consider a motion of a ball which transverses from O to A to B to C to O along x-axis as shown in the figure.

Motion along straight line



The velocities at various points of motion in m/s (vector form) are :

$$\mathbf{v}_O = 2\mathbf{i}; \mathbf{v}_A = 3\mathbf{i}; \mathbf{v}_B = -4\mathbf{i}; \mathbf{v}_C = 3\mathbf{i}$$

Going by the scalar construct, we can altogether drop use of unit vector like "i" in describing all vector quantities used to describe motion in one dimension. The velocities at various points of motion in m/s (in equivalent scalar form) can be simply stated in scalar values for rectilinear motion as :

$$v_O = 2; v_A = 3; v_B = -4; v_C = 3$$

Similarly, we can represent position vector simply by the component in one direction, say x, in meters, as :

$$x_O = 0; x_A = 5; x_B = 10; x_C = -5$$

Also, the displacement vector (in meters) is represented with scalar symbol and value as :

$$OA = 5; OB = 10; OC = 5$$

Following the same convention, we can proceed to write defining equations of speed and velocity in rectilinear motion as :

Equation:

$$|v| = \left| \frac{dx}{dt} \right|$$

and

Equation:

$$v = \frac{dx}{dt}$$

Example:

Rectilinear motion

Problem : If the position of a particle along x – axis varies in time as :

$$x = 2t^2 - 3t + 1$$

Then :

1. What is the velocity at $t = 0$?
2. When does velocity become zero?
3. What is the velocity at the origin ?
4. Plot position – time plot. Discuss the plot to support the results obtained for the questions above.

Solution : We first need to find out an expression for velocity by differentiating the given function of position with respect to time as :

$$v = \frac{d}{dt} (2t^2 - 3t + 1) = 4t - 3$$

(i) The velocity at $t = 0$,

$$v = 4 \times 0 - 3 = -3\text{m/s}$$

(ii) When velocity becomes zero :

For $v = 0$,

$$4t - 3 = 0$$

$$\Rightarrow t = \frac{3}{4} = 0.75 \text{ s.}$$

(iii) The velocity at the origin :

At origin, $x = 0$,

$$x = 2t^2 - 3t + 1 = 0$$

$$\Rightarrow 2t^2 - 2t - t + 1 = 0$$

$$\Rightarrow 2t(t - 1) - (t - 1) = 0$$

$$\Rightarrow t = 0.5 \text{ s, } 1 \text{ s.}$$

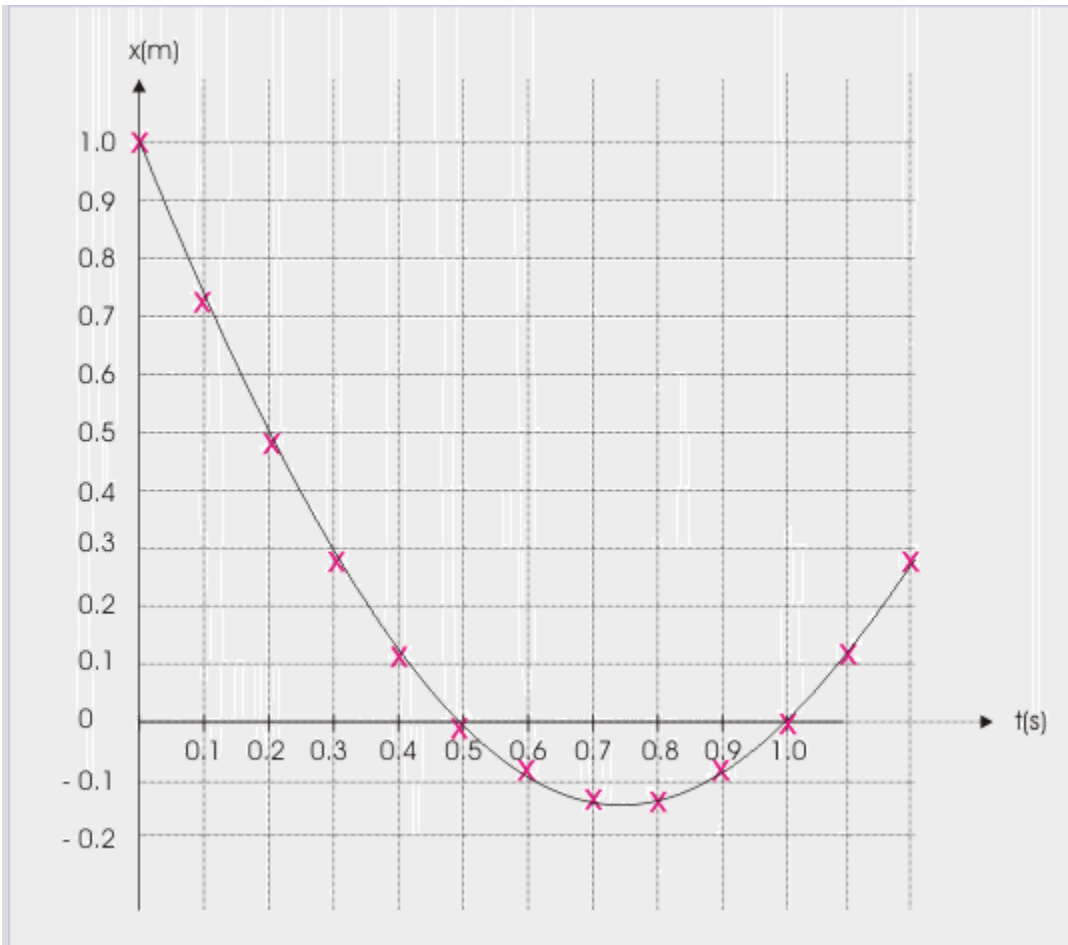
This means that particle is twice at the origin at $t = 0.5 \text{ s}$ and $t = 1 \text{ s}$. Now,

$$v(t = 0.5 \text{ s}) = 4t - 3 = 4 \times 0.5 - 3 = -1 \text{ m/s.}$$

Negative sign indicates that velocity is directed in the negative x – direction.

$$v(t = 1 \text{ s}) = 4t - 3 = 4 \times 1 - 3 = 1 \text{ m/s.}$$

Position – time plot



We observe that slope of the curve from $t = 0$ s to $t < 0.75$ s is negative, zero for $t = 0.75$ and positive for $t > 0.75$ s. The velocity at $t = 0$, thus, is negative. We can realize here that the slope of the tangent to the curve at $t = 0.75$ is zero. Hence, velocity is zero at $t = 0.75$ s.

The particle arrives at $x = 0$ for $t = 0.5$ s and $t = 1$ s. The velocity at first arrival is negative as the position falls on the part of the curve having negative slope, whereas the velocity at second arrival is positive as the position falls on the part of the curve having positive slope.

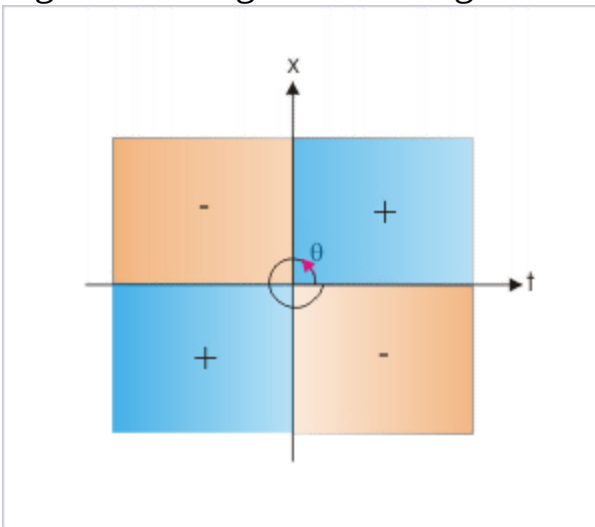
Position - time plot

We use different plots to describe rectilinear motion. Position-time plot is one of them. Position of the point object in motion is drawn against time. Evidently, it is a two dimensional plot. The position is plotted with appropriate sign as described earlier.

Nature of slope

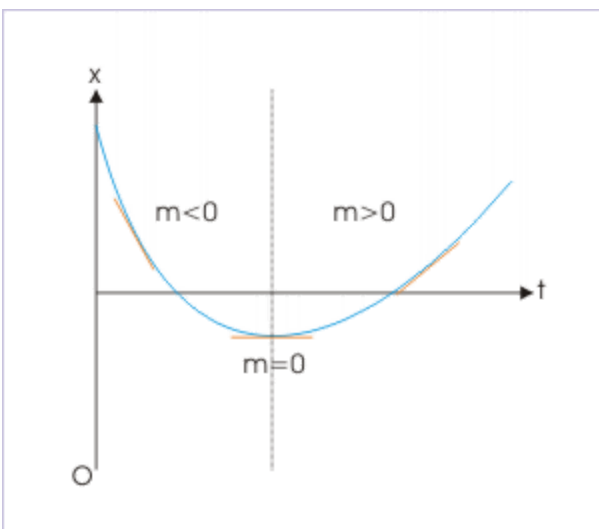
One of the important tool used to understand nature of such plots (as drawn above) is the slope of the tangent drawn on the plot. In particular, we need to qualitatively ascertain whether the slope is positive or negative. In this section, we seek to find out the ways to determine the nature of slope. Mathematically, the slope of a straight line is numerically equal to trigonometric tangent of the angle that the line makes with x – axis. It follows, therefore, that the slope of the straight line may be positive or negative depending on the angle. It is seen as shown in the figure below, the tangent of the angle in first and third quarters is positive, whereas it is negative in the remaining second and fourth quarter. This assessment of the slope of the position - time plot helps us to identify whether velocity is positive or negative?

Sign of the tangent of the angle



We may, however, use yet another simpler and effective technique to judge the nature of the slope. This employs physical interpretation of the plot. We know that the tangent of the angle is equal to the ratio of x (position) and t (time). In order to judge the nature of slope, we progress with the time and determine whether “ x ” increases or decreases. The increase in “ x ” corresponds to positive slope and a decrease, on the other hand, corresponds to negative slope. This assessment helps us to quickly identify whether velocity is positive or negative?

Slope of the curve



Direction of motion

The visual representation of the curve might suggest that the tangent to the position – time plot gives the direction of velocity. It is not true. It is contradictory to the assumption of the one dimensional motion. Motion is either in positive or negative x – direction and not in any other direction as would be suggested by the direction of tangent at various points. As a matter of fact, the curve of the position – time plot is not the representation of the path of motion. The path of the motion is simply a straight line. This distinction should always be kept in mind.

In reality, the nature of slope indicates the sense of direction, which can assume either of the two possible directions. A positive slope of the curve denotes motion along the positive direction of the referred axis, whereas negative slope indicates reversal of the direction of motion.

In the position – time plot as shown in the example at the beginning of the module ([See Figure](#)), the slope of the curve from $t = 0$ s to $t = 0.75$ s is negative, whereas slope becomes positive for $t > 0.75$ s. Clearly, an inversion of slope indicates reversal of direction. The particle, in the instant case, changes direction once at $t = 0.75$ s during the motion.

Variation in the velocity

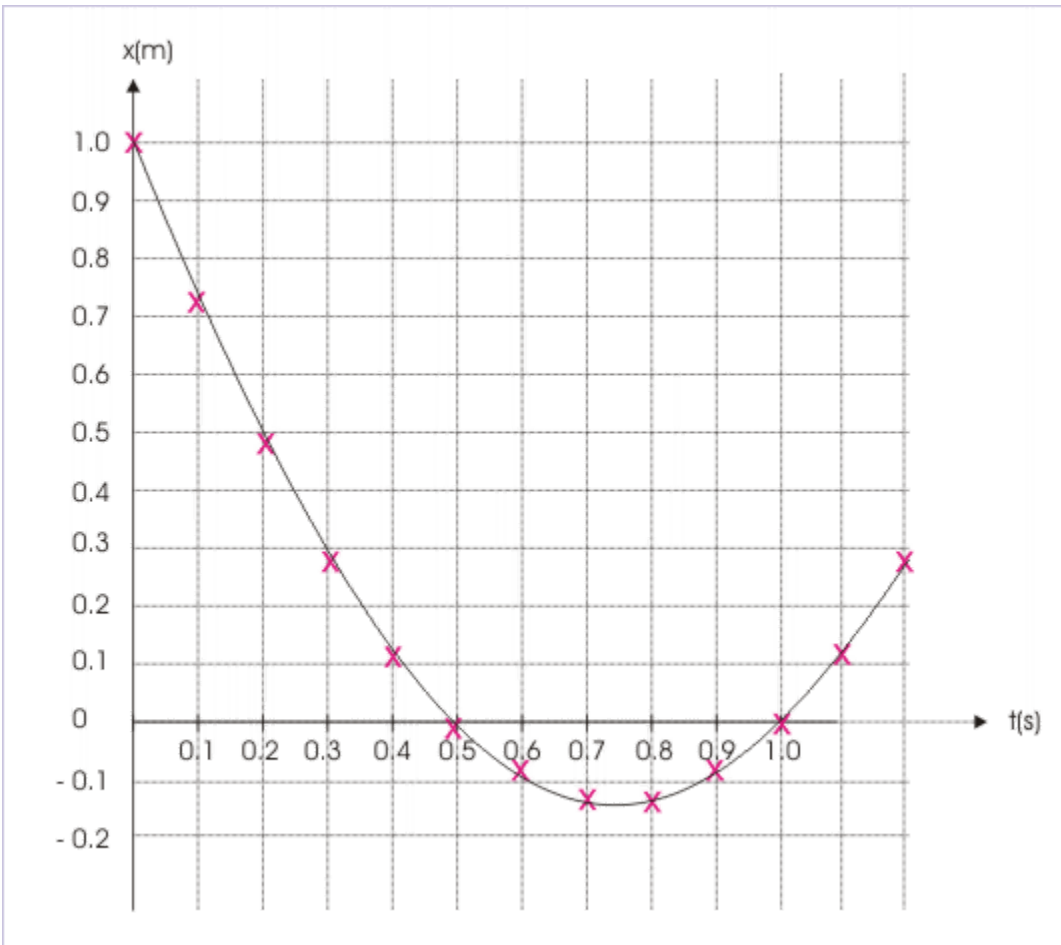
In addition to the sense of direction, the position - time plot allows us to determine the magnitude of velocity i.e. speed, which is equal to the magnitude of the slope. Here we shall see that the position – time plot is not only helpful in determining magnitude and direction of the velocity, but also in determining whether speed is increasing or decreasing or a constant.

Let us consider the plot generated in the example at the beginning of this module. The data set of the plot is as given here :

t (s)	x(m)	$\Delta x(m)$	-----	0.0	1.0	0.1
0.72	-0.28	0.2	0.48	-0.24	0.3	0.28
-0.16	0.5	0.00	-0.12	0.6	-0.08	-0.08
-0.12	0.8	-0.12	0.00	0.9	-0.08	0.04
1.1	0.12	0.12	1.2	0.28	0.16	-----

In the beginning of the motion starting from $t = 0$, we see that particle covers distance in decreasing magnitude in the negative x - direction. The magnitude of difference, Δx , in equal time interval decreases with the progress of time. Accordingly, the curve becomes flatter. This is reflected by the fact that the slope of the tangent becomes gentler till it becomes horizontal at $t = 0.75$ s. Beyond $t = 0.75$ s, the velocity is directed in the positive x – direction. We can see that particle covers more and more distances as the time progresses. It means that the velocity of the particle increases with time and the curve gets steeper with the passage of time.

Position – time plot



In general, we can conclude that a gentle slope indicates smaller velocity and a steeper slope indicates a larger velocity.

Velocity - time plot

In general, the velocity is a three dimensional vector quantity. A velocity – time would, therefore, require additional dimension. Hence, it is not possible to draw velocity – time plot on a three dimensional coordinate system. Two dimensional velocity - time plot is possible, but its drawing is complex.

One dimensional motion, having only two directions – along positive or negative direction of axis, allows plotting velocity – time graph. The velocity is treated simply as scalar speed with one qualification that velocity

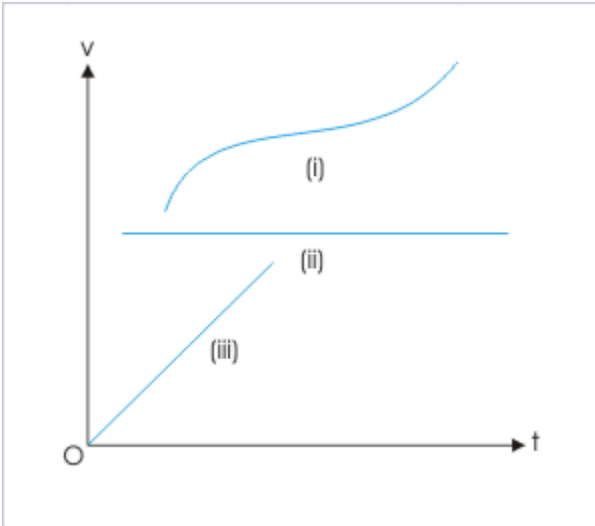
in the direction of chosen reference is considered positive and velocity in the opposite direction to chosen reference is considered negative.

Note:In "speed- time" and "velocity – time" plots use the symbol “ v ” to represent both speed and velocity. This is likely to create some confusion as these quantities are essentially different. We use the scalar symbol “ v ” to represent velocity as a special case for rectilinear motion, because scalar value of velocity with appropriate sign gives the direction of motion as well. Therefore, the scalar representation of velocity is consistent with the requirement of representing both magnitude and direction. Though current writings on the subject allows duplication of symbol, we must, however, be aware of the difference between two types of plot. The speed (v) is always positive in speed – time plot and drawn in the first quadrant of the coordinate system. On the other hand, velocity (v) may be positive or negative in velocity – time plot and drawn in first and fourth quadrants of the two dimensional coordinate system.

The nature of velocity – time plot

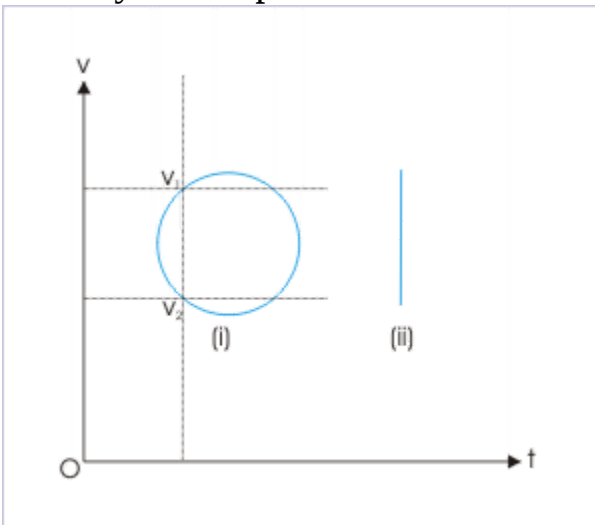
Velocity – time plot for rectilinear motion is a curve (Figure i). The nature of the curve is determined by the nature of motion. If the particle moves with constant velocity, then the plot is a straight line parallel to the time axis (Figure ii). On the other hand, if the velocity changes with respect to time at uniform rate, then the plot is a straight line (Figure iii).

Velocity – time plot



The representation of the variation of velocity with time, however, needs to be consistent with physical interpretation of motion. For example, we can not think of velocity - time plot, which is a vertical line parallel to the axis of velocity (Figure ii). Such plot is inconsistent as this would mean infinite numbers of values, against the reality of one velocity at a given instant. Similarly, the velocity-time plot should not be intersected by a vertical line twice as it would mean that the particle has more than one velocity at a given time (Figure i).

Velocity – time plot



Area under velocity – time plot

The area under the velocity – time plot is equal to displacement. The displacement in the small time period “dt” is given by :

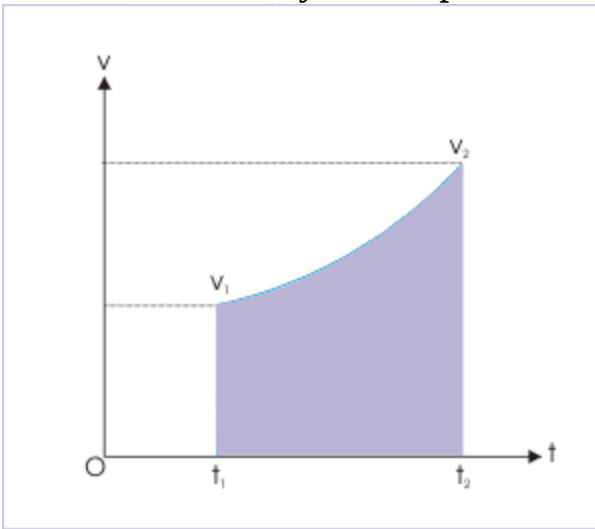
$$dx = v dt$$

Integrating on both sides between time intervals t_1 and t_2 ,

$$\Delta x = \int_{t_1}^{t_2} v dt$$

The right hand side integral graphically represents an area on a plot drawn between two variables : velocity (v) and time (t). The area is bounded by (i) v-t curve (ii) two time ordinates t_1 and t_2 and (iii) time (t) axis as shown by the shaded region on the plot. Thus, the area under v-t plot bounded by the ordinates give the magnitude of displacement (Δx) in the given time interval.

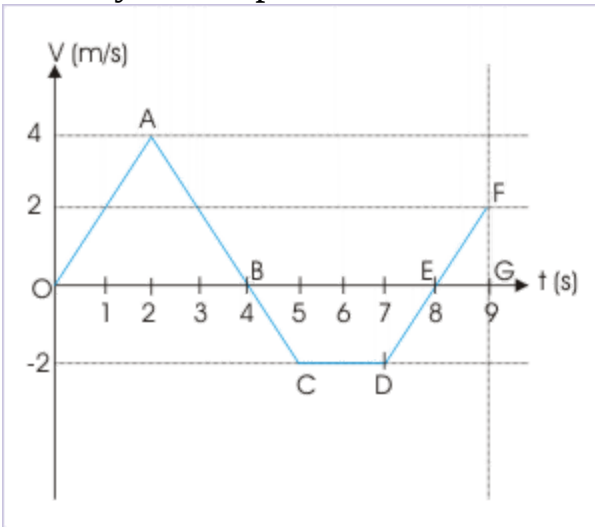
Area under velocity – time plot



When v-t curve consists of negative values of velocity, then the curve extends into fourth quadrant i.e. below time axis. In such cases, it is sometimes easier to evaluate area above and below time axis separately. The area above time axis represents positive displacement, whereas area under time axis represents negative displacement. Finally, areas are added with proper sign to obtain the net displacement during the motion.

To illustrate the working of the process for determining displacement, let us consider the rectilinear motion of a particle represented by the plot shown.

Velocity – time plot



Here,

$$\text{Area of triangle OAB} = \frac{1}{2} \times 4 \times 4 = 8 \text{ m}$$

$$\text{Area of trapezium BCDE} = -\frac{1}{2} \times 2 \times (2 + 4) = -6 \text{ m}$$

$$\text{Area of triangle EFG} = \frac{1}{2} \times 1 \times 2 = 1 \text{ m}$$

$$\Rightarrow \text{Net area} = 8 - 6 + 1 = 3 \text{ m}$$

A switch from positive to negative value of velocity and vice-versa is associated with change of direction of motion. It means that every intersection of the $v - t$ curve with time axis represents a reversal of direction. Note that it is not the change of slope (from positive to negative and vice versa) like on the position time that indicates a change in the direction of motion; but the intersection of time axis, which indicates change of direction of velocity. In the motion described in figure above, particle undergoes reversal of direction at two occasions at B and E.

It is also clear from the above example that displacement is given by the net area (considering appropriate positive and negative sign), while distance covered during the motion in the time interval is given by the cumulative area without considering the sign. In the above example, distance covered is :

$s = \text{Area of triangle OAB} + \text{Area of trapezium BCDE} + \text{Area of triangle EFG}$

$$\Rightarrow s = 8 + 6 + 1 = 15 \text{ m}$$

Exercise:

Problem:

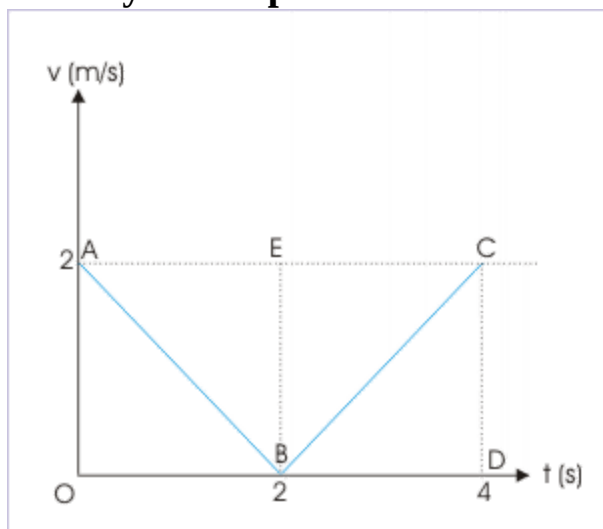
A person walks with a velocity given by $|t - 2|$ along a straight line. Find the distance and displacement for the motion in the first 4 seconds. What is the average velocity in this period?

Solution:

Here, the velocity is equal to the modulus of a function in time. It means that velocity is always positive. An inspection of the function reveals that velocity linearly decreases for the first 2 second from 2 m/s to zero. It, then, increases from zero to 2 m/s in the next 2 seconds. In order to obtain distance and displacement, we draw the Velocity – time plot as shown.

The area under the plot gives displacement. In this case, however, there is no negative displacement involved. As such, distance and displacement are equal.

Velocity – time plot



$$\text{Displacement} = \Delta OAB + \Delta BCD$$

$$\text{Displacement} = \frac{1}{2} \times OB \times OA + \frac{1}{2} \times BD \times CD$$

$$\text{Displacement} = \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 2 \times 2 = 4 \text{ m}$$

and the average velocity is given by :

$$v_{\text{avg}} = \frac{\text{Displacement}}{\Delta t} = \frac{4}{4} = 1 \text{ m/s}$$

Uniform motion

Uniform motion is a subset of rectilinear motion. It is the most simplified class of motion. In this case, the body under motion moves with constant velocity. It means that the body moves along a straight line without any change of magnitude and direction as velocity is constant. Also, a constant velocity implies that velocity is constant all through out the motion. The velocity at every instant during motion is, therefore, same.

It follows then that instantaneous and averages values of speed and velocity are all equal to a constant value for uniform motion :

$$v_a = |\mathbf{v}_a| = v = |\mathbf{v}| = \text{Constant}$$

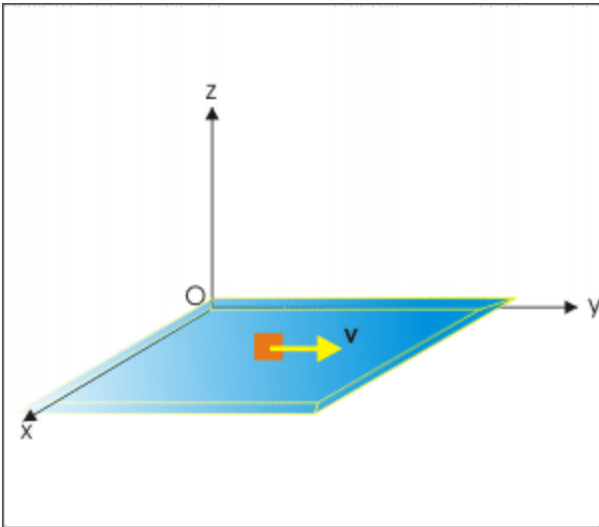
The motion of uniform linear motion has special significance, as this motion exactly echoes the principle enshrined in the first law of motion. The law states that all bodies in the absence of external force maintain their speed and direction. It follows, therefore, that the study of uniform motion is actually the description of motion, when no external force is in play.

The absence of external force is hypothetical in our experience as bodies are always subject to external force(s). The force of gravitation is sort of omnipresent force that can not be overlooked - atleast on earth.

Nevertheless, the concept of uniform motion has great theoretical significance as it gives us the reference for the accelerated or the non-uniform real motion.

On the earth, a horizontal motion of a block on a smooth plane approximates uniform motion as shown in the figure. The force of gravity acts and normal reaction force at the contact between surfaces act in vertically, but opposite directions. The two forces balances each other. As a result, there is no net force in the vertical direction. As the surface is smooth, we can also neglect horizontal force due to friction, which could have opposed the motion in horizontal direction. This situation is just an approximation for we can not think of a flawless smooth surface in the first place; and also there would be intermolecular attraction between the block and surface at the contact. In brief, we can not achieve zero friction - eventhough the surfaces in contact are perfectly smooth. However, the approximation like this is helpful for it provides us a situation, which is equivalent to the motion of an object without any external force(s).

Uniform motion

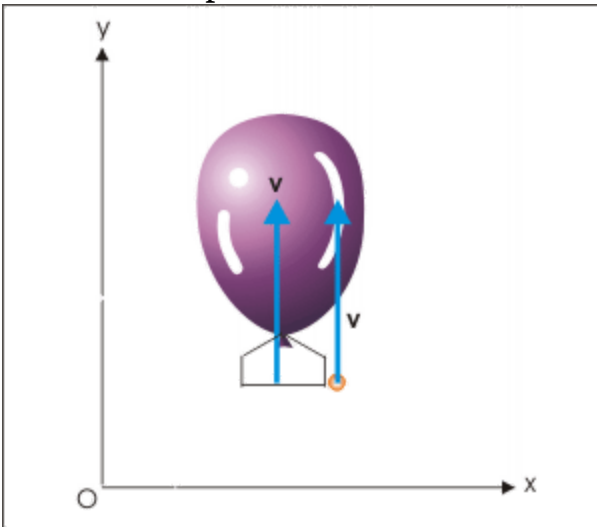


We can also imagine a volumetric space, where massive bodies like planets and stars are not nearby. The motion of an object in that space, therefore, would be free from any external force and the motion would be in accordance with the laws of motion. The study of astronauts walking in the space and doing repairs to the spaceship approximates the situation of the absence of external force. The astronaut in the absence of the force of gravitation and friction (as there is no atmosphere) moves along with the velocity of the spaceship.

Motion of separated bodies

The fact that the astronaut moves with the velocity of spaceship is an important statement about the state of motion of the separated bodies. A separated body acquires the velocity of the containing body. A pebble released from a moving train or dropped from a rising balloon is an example of the motion of separated body. The pebble acquires the velocity of the train or the balloon as the case may be.

Motion of separated bodies



It means that we must assign a velocity to the released body, which is equal in magnitude and direction to that of the body from which the released body has separated. The phenomena of imparting velocity to the separated body is a peculiarity with regard to velocity. We shall learn that acceleration (an attribute of non-uniform motion) does not behave in the same fashion. For example, if the train is accelerating at the time, the pebble is dropped, then the pebble would not acquire the acceleration of the containing body.

The reason that pebble does not acquire the acceleration of the train is very simple. We know that acceleration results from application of external force. In this case, the train is accelerated as the engine of the train pulls the compartment (i.e. applies force on the compartment). The pebble, being part of the compartment, is also accelerated till it is held in the hand of the passenger. However, as the pebble is dropped, the connection of the pebble with the rest of the system or with the engine is broken. No force is applied on the pebble in the horizontal direction. As such, pebble after being dropped has no acceleration in the horizontal direction.

In short, we can conclude that a separated body acquires velocity, but not the acceleration.

Rectilinear motion (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of rectilinear motion. The questions are categorized in terms of the characterizing features of the subject matter :

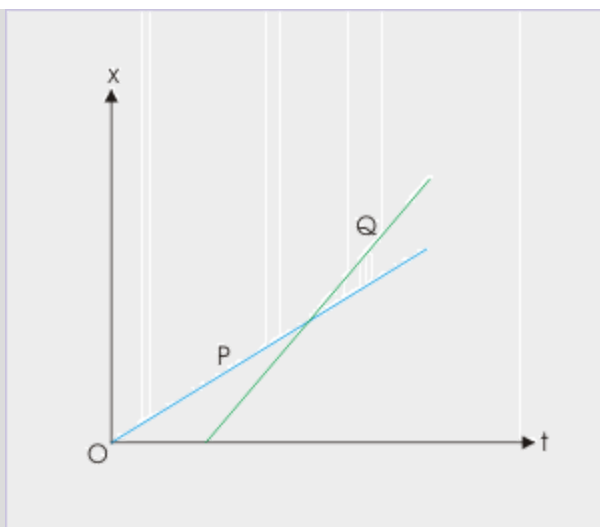
- Interpretation of position - time plot
- Interpretation of displacement - time plot
- Displacement
- Average velocity

Position vector

Example:

Problem : Two boys (P and Q) walk to their respective school from their homes in the morning on a particular day. Their motions are plotted on a position – time graph for the day as shown. If all schools and homes are situated by the side of a straight road, then answer the followings :

Motion of two boys

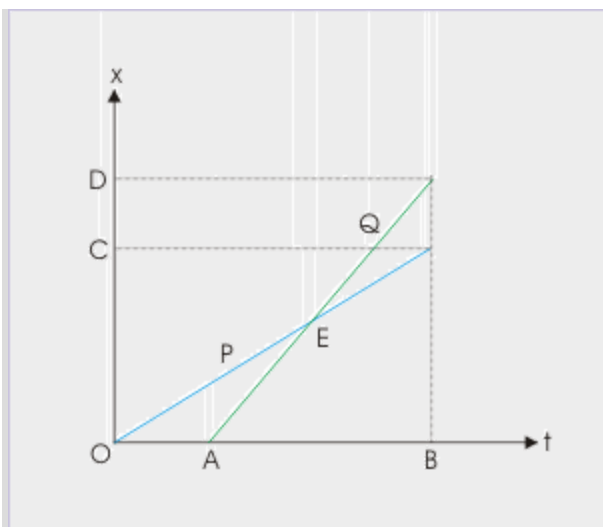


Position – time plot of a
rectilinear motion.

1. Which of the two resides closer to the school?
2. Which of the two starts earlier for the school?
3. Which of the two walks faster?
4. Which of the two reaches the school earlier?
5. Which of the two overtakes other during walk?

Solution : In order to answer questions, we complete the drawing with vertical and horizontal lines as shown here. Now,

Motion of two boys



Position – time plot of a rectilinear motion.

1: Which of the two resides closer to the school?

We can answer this question by knowing the displacements. The total displacements here are OC by P and OD by Q. From figure, $OC < OD$. Hence, P resides closer to the school.

2: Which of the two starts earlier for the school?

The start times are point O for P and A for Q on the time axis. Hence, P starts earlier for school.

3: Which of the two walks faster?

The speed is given by the slope of the plot. Slope of the motion of P is smaller than that of Q. Hence, Q walks faster.

4: Which of the two reaches the school earlier?

Both boys reach school at the time given by point B on time axis. Hence, they reach school at the same time.

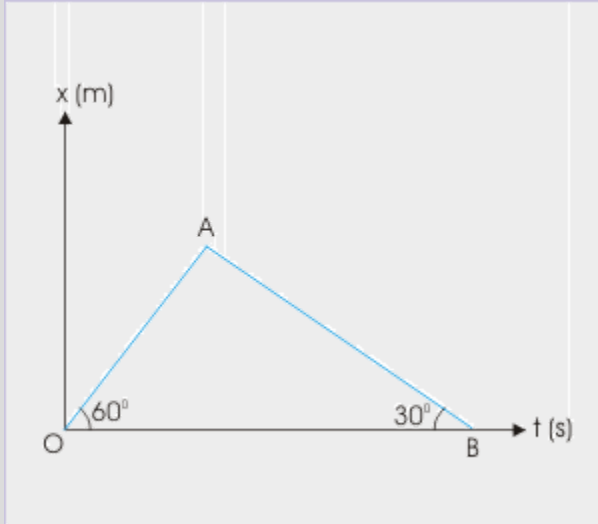
5: Which of the two overtakes other during walk?

The plots intersect at a point E. They are at the same location at this time instant. Since, speed of B is greater, he overtakes A.

Interpretation of displacement - time plot

Example:

Problem : A displacement – time plot in one dimension is as shown. Find the ratio of velocities represented by two straight lines.

Displacement-time plot

There two segments of different velocities.

Solution : The slope of displacement – time plot is equal to velocity. Let v_1 and v_2 be the velocities in two segments, then magnitudes of velocities in two segments are :

$$|v_1| = \tan 60^\circ = \sqrt{3}$$

$$|v_2| = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

We note that velocity in the first segment is positive, whereas velocity in the second segment is negative. Hence, the required ratio of two velocities is :

$$\frac{v_1}{v_2} = -\frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} = -3$$

Displacement

Example:

Problem : The displacement “x” of a particle moving in one dimension is related to time “t” as :

$$t = \sqrt{x} + 3$$

where “t” is in seconds and “x” is in meters. Find the displacement of the particle when its velocity is zero.

Solution : We need to find “x” when velocity is zero. In order to find this, we require to have an expression for velocity. This, in turn, requires an expression of displacement in terms of time. The given expression of time, therefore, is required to be re-arranged :

$$\sqrt{x} = t - 3$$

Squaring both sides, we have :

$$\Rightarrow x = (t - 3)^2 = t^2 - 6t + 9$$

Now, we obtain the required expression of velocity in one dimension by differentiating the above relation with respect to time,

$$v = \frac{dx}{dt} = 2t - 6$$

According to question,

$$v = 2t - 6 = 0$$

$$t = 3 \text{ s}$$

Putting this value of time in the expression of displacement, we have :

$$x = 3^2 - 6 \times 3 + 9 = 0$$

Average velocity

Example:

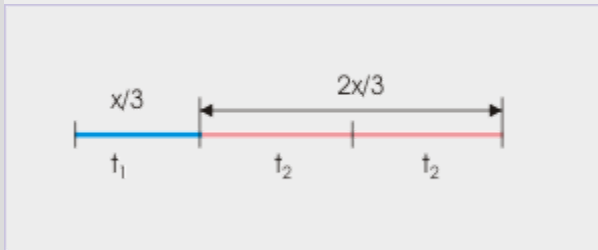
Problem : A particle moving in a straight line covers 1/3rd of the distance with a velocity 4 m/s. The remaining part of the linear distance is covered with velocities 2 m/s and 6 m/s for equal times. What is the average velocity during the total motion?

Solution : The average velocity is ratio of displacement and time. In one dimensional unidirectional motion, distance and displacement are same. As such, average velocity is ratio of total distance and time.

Let the total distance be “x”. Now, we need to find total time in order to find the average velocity. Let “ t_1 ”, “ t_2 ” and “ t_2 ” be the time periods of the motion in three parts of the motion as given in the question.

Considering the first part of the motion,

Unidirectional motion



The particle covers segments of the motion with different average velocities.

$$t_1 = \frac{x_1}{v_1} = \frac{\frac{x}{3}}{4} = \frac{x}{12}$$

For the second part of the motion, the particle covers the remaining distance at two different velocities for equal time, “ t_2 ”,

$$x - \frac{x}{3} = v_2 \times t_2 + v_3 \times t_2$$

$$\Rightarrow \frac{2x}{3} = 2t_2 + 6t_2 = 8t_2$$

$$\Rightarrow t_2 = \frac{2x}{24} = \frac{x}{12}$$

Thus, average velocity is :

$$v_a = \frac{x}{t} = \frac{x}{t_1 + 2t_2} = \frac{x}{\frac{x}{12} + \frac{x}{6}} = 4 \text{ m/s}$$

Understanding motion

The discussion of different attributes of motion in previous modules has led us to the study of motion from the point of view of a general consideration to a simplified consideration such as uniform or rectilinear motion. The time is now ripe to recapitulate and highlight important results - particularly where distinctions are to be made.

For convenience, we shall refer general motion as the one that involves non-linear, two/ three dimensional motion. The simplified motion, on the other hand, shall refer motion that involves one dimensional, rectilinear and uniform motion.

Consideration of scalar quantities like distance and speed are same for “general” as well as “simplified” cases. We need to score similarities or differences for vector quantities to complete our understanding up to this point. It is relevant here to point out that most of these aspects have already been dealt in detail in previous modules. As such, we shall limit our discussion on main points/ results and shall generally not use figures and details.

Similarities and differences

Similarity / Difference 1 : In general, the magnitude of displacement is not equal to distance.

Equation:

$$|\Delta \mathbf{r}| \leq s$$

For rectilinear motion (one dimensional case) also, displacement is not equal to distance as motion may involve reversal of direction along a line.

Equation:

$$|\Delta x| \leq s$$

For uniform motion (unidirectional motion),

Equation:

$$|\Delta x| = s$$

Similarity / Difference 2 : The change in the magnitude of position vector is not equal to the magnitude of change in position vector except for uniform motion i.e motion with constant velocity.

For two/three dimensional motion,

Equation:

$$\Delta r \neq |\Delta \mathbf{r}|$$

For one dimensional motion,

Equation:

$$\Delta x \neq |\Delta x|$$

For uniform motion (unidirectional),

Equation:

$$\Delta x = |\Delta x|$$

Similarity / Difference 3 : In all cases, we can draw a distance – time or speed – time plot. The area under speed – time plot equals distance (s).

Equation:

$$s = \int v dt$$

Similarity / Difference 4 : There is an ordered sequence of differentiation with respect to time that gives motional attributes of higher order. For example first differentiation of position vector or displacement yields velocity. We shall come to know subsequently that differentiation of

velocity, in turn, with respect to time yields acceleration. Differentiation, therefore, is a tool to get values for higher order attributes.

These differentiations are defining relations for the attributes of motion and hence applicable in all cases irrespective of the dimensions of motion or nature of velocity (constant or variable).

For two or three dimensional motion,

Equation:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

For one dimensional motion,

Equation:

$$v = \frac{dx}{dt}$$

Similarity / Difference 5 : Just like differentiation, there is an ordered sequence of integration that gives motional attributes of lower attributes. Since these integrations are based on basic/ defining differential equations, the integration is applicable in all cases irrespective of the dimensions of motion or nature of velocity (constant or variable).

For two or three dimensional motion,

Equation:

$$\Delta \mathbf{r} = \int \mathbf{v} dt$$

For one dimensional motion,

Equation:

$$\Delta x = \int v dt$$

Similarity / Difference 6 : We can not draw position – time, displacement – time or velocity – time plots for three dimensional motion. We can draw these plots for two dimensional motion, but the same would be complex and as such we would avoid drawing them.

We can, however, draw the same for one – dimensional motion by treating the vector attributes (position vector, displacement and velocity) as scalar with appropriate sign. Such drawing would be in the first and fourth quarters of two – dimensional plots. We should clearly understand that if we are drawing these plots, then the motion is either one or two dimensional. In general, we draw attribute .vs. time plots mostly for one – dimensional motion. We should also understand that graphical method is an additional tool for analysis in one dimensional motion.

The slope of the curve on these plots enables us to calculate the magnitude of higher attributes. The slope of position – time and displacement – time plot gives the magnitude of velocity; whereas the slope of velocity – time plot gives the magnitude of acceleration (This will be dealt in separate module).

Significantly, the tangent to the slopes on a "time" plot does not represent direction of motion. It is important to understand that the though the nature of slope (positive or negative) gives the direction of motion with respect to reference direction, but the tangent in itself does not indicate direction of motion. We must distinguish these “time” plots with simple position plots. The curve on the simple position plot is actual representation of the path of motion. Hence, tangent to the curve on position plot (plot on a x,y,z coordinate system) gives the direction of motion.

Similarity / Difference 7 : Needless to say that what is valid for one dimensional motion is also valid for the component motion in the case of two or three dimensional motion. This is actually a powerful technique to even treat a complex two or three dimensional motion, using one dimensional techniques. This aspect will be demonstrated on topics such as projectile and circular motion.

Similarity / Difference 8 : The area under velocity – time plot (for one dimensional motion) is equal to displacement.

Equation:

$$x = \int v dt$$

As area represents a vector (displacement), we treat area as scalar with appropriate sign for one dimensional motion. The positive area above the time axis gives the positive displacement, whereas the negative area below time axis gives negative displacement. The algebraic sum with appropriate sign results in net displacement. The algebraic sum without sign results in net distance.

Important thing to realize is that this analysis tool is not available for analysis of three dimensional motion as we can not draw the plot in the first place.

Similarity / Difference 9 : There is a difficulty in giving differentiating symbols to speed and velocity in one dimensional motion as velocity is treated as scalar. Both are represented as simple letter “v”. Recall that a non-bold faced letter “v” represents speed in two/three dimensional case. An equivalent representation of speed, in general, is $|v|$, but is seldom used in practice. As we do not use vector for one dimensional motion, there is a conflicting representation of the same symbol, “v”. We are left with no other solution as to be elaborate and specific so that we are able to convey the meaning either directly or by context. Some conventions, in this regard, may be helpful :

- If “v” is speed, then it can not be negative.
- If “v” is velocity, then it can be either positive or negative.
- If possible follow the convention as under :

$$v = \text{velocity}$$

$$|v| = \text{speed}$$

Similarity / Difference 10 : In the case of uniform motion (unidirectional motion), there is no distinction between scalar and vector attributes at all. The distance .vs. displacement and speed .vs. velocity differences have no relevance. The paired quantities are treated equal and same. Motion is one

dimensional and unidirectional; there being no question of negative value for attributes with direction.

Similarity / Difference 11 : Since velocity is a vector quantity being the time rate of change of position vector (displacement), there can be change in velocity due to the change in position vector (displacement) in any of the following three ways :

1. change in magnitude
2. change in direction
3. change in both magnitude and direction

This realization brings about important subtle differences in defining terms of velocity and their symbolic representation. In general motion, velocity is read as the "time rate of change of position vector" :

Equation:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

The speed i.e. the magnitude of velocity is read as the "absolute value (magnitude) of the time rate of change of position vector" :

Equation:

$$v = \left| \frac{d\mathbf{r}}{dt} \right|$$

But the important thing to realize is that “time rate of change in the magnitude of position vector” is not same as “magnitude of the time rate of change of position vector”. As such the time rate of change of the magnitude of position vector is not equal to speed. This fact can be stated mathematically in different ways :

Equation:

$$\begin{aligned} \frac{dr}{dt} &= \frac{d|\mathbf{r}|}{dt} \neq v \\ \frac{dr}{dt} &= \frac{d|\mathbf{r}|}{dt} \neq \left| \frac{d\mathbf{r}}{dt} \right| \end{aligned}$$

We shall work out an example of a motion in two dimensions (circular motion) subsequently in this module to illustrate this difference.

However, this difference disappears in the case of one dimensional motion. It is so because we use scalar quantity to represent vector attribute like position vector and velocity. Physically, we can interpret that there is no difference as there is no change of direction in one dimensional motion. It may be argued that there is a change in direction even in one dimensional motion in the form of reversal of motion, but then we should realize that we are interpreting instantaneous terms only – not the average terms which may be affected by reversal of motion. Here, except at the point of reversal of direction, the speed is :

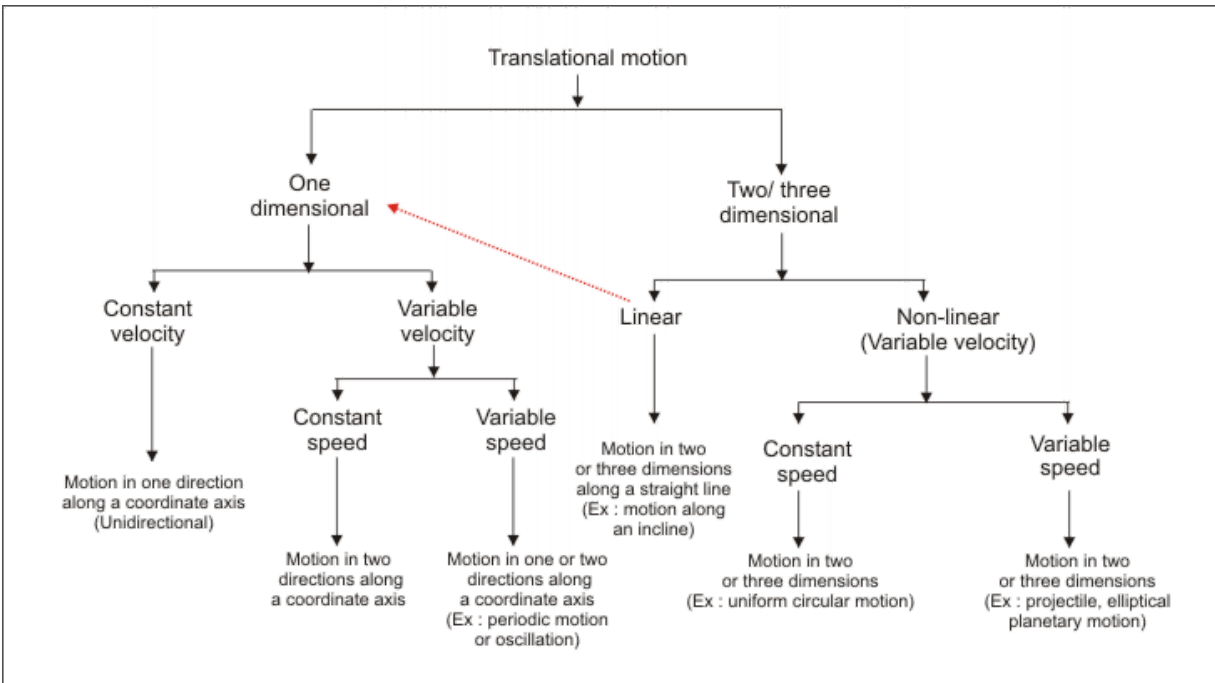
Equation:

$$v = \left| \frac{dx}{dt} \right| = \frac{d|x|}{dt}$$

Similarity / Difference 12 : Understanding of the class of motion is important from the point of view of analysis of motion (solving problem). The classification lets us clearly know which tools are available for analysis and which are not? Basically, our success or failure in understanding motion largely depends on our ability to identify motion according to a certain scheme of classification and then apply appropriate tool (formula/ defining equations etc) to analyze or solve the problem. It is, therefore, always advisable to write down the characteristics of motion for analyzing a situation involving motion in the correct context.

A simple classification of translational motion types, based on the study up to this point is suggested as given in the figure below. This classification is based on two considerations (i) dimensions of motion and (ii) nature of velocity.

Classification of motion



Classification based on (i) dimensions and (ii) velocity.

Example:

Problem : The position vector of a particle in motion is :

$$\mathbf{r} = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

where “a” is a constant. Find the time rate of change in the magnitude of position vector.

Solution : We need to know the magnitude of position vector to find its time rate of change. The magnitude of position vector is :

$$r = |\mathbf{r}| = \sqrt{(a \cos \omega t)^2 + (a \sin \omega t)^2}$$

$$\Rightarrow r = a \sqrt{\cos^2 \omega t + \sin^2 \omega t}$$

$$\Rightarrow r = a$$

But “a” is a constant. Hence, the time rate of change of the magnitude of position vector is zero :

$$\frac{dr}{dt} = 0$$

This result is an important result. This highlights that time rate of change of the magnitude of position vector is not equal to magnitude of time rate of change of the position vector (speed).

The velocity of the particle is obtained by differentiating the position vector with respect to time as :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}$$

The speed, which is magnitude of velocity, is :

$$\begin{aligned} v &= \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(-a\omega \sin \omega t)^2 + (a\omega \cos \omega t)^2} \\ \Rightarrow v &= a\omega \sqrt{(\sin^2 \omega t + \cos^2 \omega t)} \\ \Rightarrow v &= a\omega \end{aligned}$$

Clearly, speed of the particle is not zero. This illustrates that even if there is no change in the magnitude of position vector, the particle can have instantaneous velocity owing to the change in the direction.

As a matter of fact, the motion given by the position vector in the question actually represents uniform circular motion, where particle is always at constant distance (position) from the center, but has velocity of constant speed and varying directions. We can verify this by finding the equation of path for the particle in motion, which is nothing but a relation between coordinates. An inspection of the expression for “x” and “y” coordinates suggest that following trigonometric identity would give the desired equation of path,

$$\sin^2 \theta + \cos^2 \theta = 1$$

For the given case,

$$\mathbf{r} = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

$$\Rightarrow x = a \cos \omega t$$

$$\Rightarrow y = a \sin \omega t$$

Rearranging, we have :

$$\cos \omega t = \frac{x}{a}$$

and

$$\sin \omega t = \frac{y}{a}$$

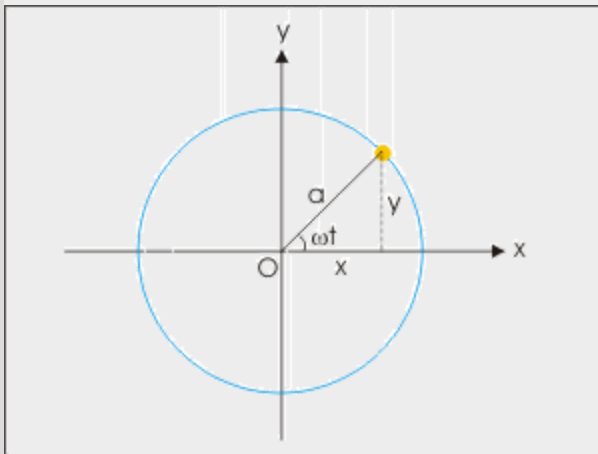
Now, using trigonometric identity :

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$\Rightarrow x^2 + y^2 = a^2$$

This is an equation of a circle of radius “a”.

Uniform circular motion



The particle moves along a circular path with a constant speed.

In the nutshell, for motion in general (for two/three dimensions),

$$\frac{dr}{dt} = \frac{d|\mathbf{r}|}{dt} \neq v$$

$$\frac{dr}{dt} = \frac{d|\mathbf{r}|}{dt} \neq \left| \frac{d\mathbf{r}}{dt} \right|$$

It is only in one dimensional motion that this distinction disappears as there is no change of direction as far as instantaneous velocity is concerned.

Velocity (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the concept of velocity. For this reason, questions are categorized in terms of the characterizing features of the subject matter :

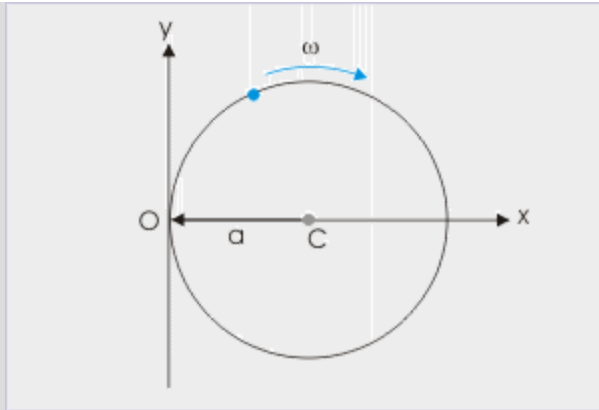
- Position vector
- Displacement
- Constrained motion
- Nature of velocity
- Comparing velocities

Position vector

Example:

Problem : A particle is executing motion along a circle of radius “a” with a constant angular speed “ ω ” as shown in the figure. If the particle is at “O” at $t = 0$, then determine the position vector of the particle at an instant in xy - plane with "O" as the origin of the coordinate system.

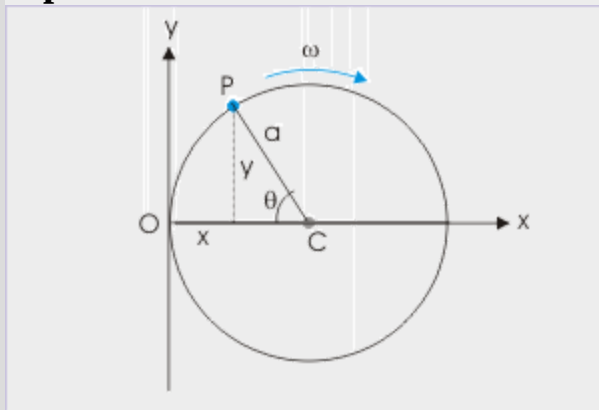
A particle in circular motion



The particle moves with a constant angular velocity.

Solution : Let the particle be at position “P” at a given time “t”. Then the position vector of the particle is :

A particle in circular motion



The particle moves with a constant angular velocity starting from “O” at $t = 0$.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

Note that "x" and "y" components of position vector is measured from the origin "O". From the figure,

$$y = a \sin \theta = a \sin \omega t$$

and

$$x = a - a \cos \omega t = a(1 - \cos \omega t)$$

Thus, position vector of the particle in circular motion is :

$$\mathbf{r} = a(1 - \cos \omega t)\mathbf{i} + a \sin \omega t\mathbf{j}$$

Example:

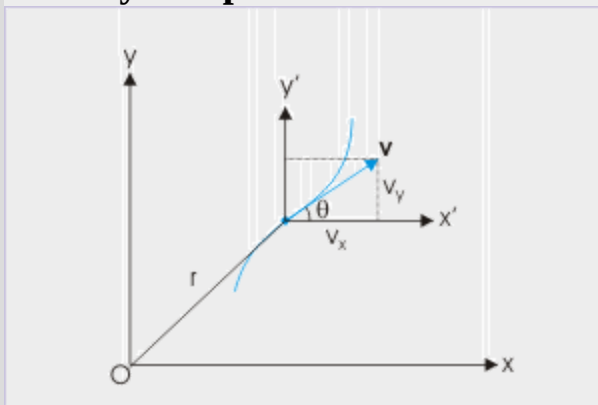
Problem : The position vector of a particle (in meters) is given as a function of time as :

$$\mathbf{r} = 2t\mathbf{i} + 2t^2\mathbf{j}$$

Determine the time rate of change of the angle “ θ ” made by the velocity vector with positive x-axis at time, $t = 2$ s.

Solution : It is a two dimensional motion. The figure below shows how velocity vector makes an angle “ θ ” with x-axis of the coordinate system. In order to find the time rate of change of this angle “ θ ”, we need to express trigonometric ratio of the angle in terms of the components of velocity vector. From the figure :

Velocity of a particle in two dimensions



The velocity has two

components.

$$\tan \theta = \frac{v_y}{v_x}$$

As given by the expression of position vector, its component in coordinate directions are :

$$x = 2t \text{ and } y = 2t^2$$

We obtain expression of the components of velocity in two directions by differentiating "x" and "y" components of position vector with respect to time :

$$v_x = 2 \text{ and } v_y = 4t$$

Putting in the trigonometric function, we have :

$$\tan \theta = \frac{v_y}{v_x} = \frac{4t}{2} = 2t$$

Since we are required to know the time rate of the angle, we differentiate the above trigonometric ratio with respect to time as,

$$\sec^2 \theta \frac{d\theta}{dt} = 2$$

$$\Rightarrow \left(1 + \tan^2 \theta\right) \frac{d\theta}{dt} = 2$$

$$\Rightarrow \left(1 + 4t^2\right) \frac{d\theta}{dt} = 2$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{2}{(1+4t^2)}$$

At $t = 2$ s,

$$\Rightarrow \frac{d\theta}{dt} = \frac{2}{(1+4 \times 2^2)} = \frac{2}{17} \text{ rad / s}$$

Displacement

Example:

Problem : The displacement (x) of a particle is given by :

$$x = A \sin(\omega t + \theta)$$

At what time is the displacement maximum?

Solution : The displacement (x) depends on the value of sine function. It will be maximum for maximum value of $\sin(\omega t + \theta)$. The maximum value of sine function is 1. Hence,

$$\sin(\omega t + \theta) = 1 = \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \omega t + \theta = \pi/2$$

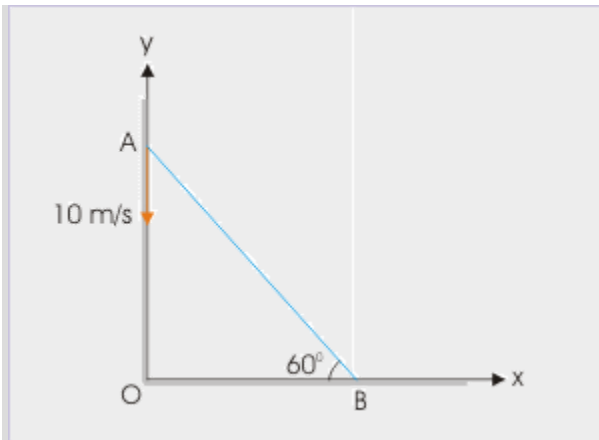
$$\Rightarrow t = \frac{\pi}{2\omega} - \frac{\theta}{\omega}$$

Constrained motion

Example:

Problem : Two particles A and B are connected by a rigid rod AB. The rod slides along perpendicular rails as shown here. The velocity of A moving down is 10 m/s. What is the velocity of B when angle $\theta = 60^\circ$?

Motion of a leaning rod



One end of the rod is moving with a speed 10 m/s in vertically downward direction.

Solution : The velocity of B is not an independent velocity. It is tied to the velocity of the particle “A” as two particles are connected through a rigid rod. The relationship between two velocities is governed by the inter-particles separation, which is equal to the length of rod. The length of the rod, in turn, is linked to the positions of particles “A” and “B” . From figure,

$$x = \sqrt{(L^2 - y^2)}$$

Differentiating, with respect to time :

$$\Rightarrow v_x = \frac{dx}{dt} = -\frac{2y}{2\sqrt{(L^2 - y^2)}} \times \frac{dy}{dt} = -\frac{yv_y}{\sqrt{(L^2 - y^2)}} = -v_y \tan \theta$$

Considering magnitude only,

$$\Rightarrow v_x = v_y \tan \theta = 10 \tan 60^\circ = 10\sqrt{3} \quad \frac{m}{s}$$

Nature of velocity

Example:

Problem : The position vector of a particle is :

$$\mathbf{r} = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

where “a” is a constant. Show that velocity vector is perpendicular to position vector.

Solution : In order to prove as required, we shall use the fact that scalar (dot) product of two perpendicular vectors is zero. Now, we need to find the expression of velocity to evaluate the dot product as intended. We can obtain the same by differentiating the expression of position vector with respect to time as :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}$$

To check whether velocity is perpendicular to the position vector, we evaluate the scalar product of $\mathbf{r}$ and $\mathbf{v}$, which should be equal to zero.

$$\mathbf{r} \cdot \mathbf{v} = 0$$

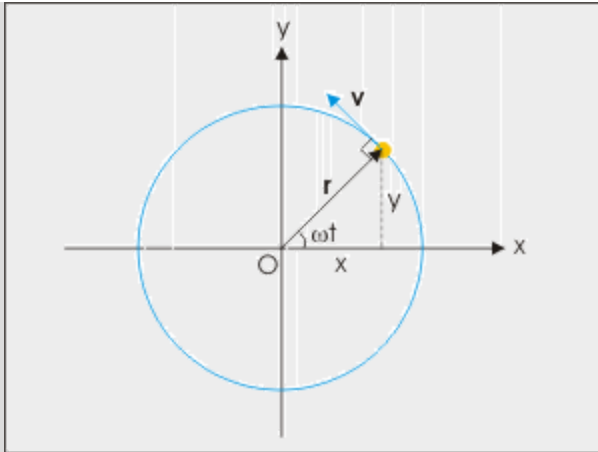
In this case,

$$\Rightarrow \mathbf{r} \cdot \mathbf{v} = (a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}) \cdot (-a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j})$$

$$\Rightarrow -a^2\omega \sin \omega t \cos \omega t + a^2\omega \sin \omega t \cos \omega t = 0$$

This means that the angle between position vector and velocity are at right angle to each other. Hence, velocity is perpendicular to position vector. It is pertinent to mention here that this result can also be inferred from the plot of motion. An inspection of position vector reveals that it represents uniform circular motion as shown in the figure here.

Circular motion



The particle describes a circular path.

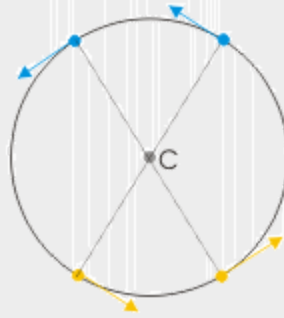
The position vector is always directed radially, whereas velocity vector is always tangential to the circular path. These two vectors are, therefore, perpendicular to each other.

Example:

Problem : Two particles are moving with the same constant speed, but in opposite direction. Under what circumstance will the separation between two remains constant?

Solution : The condition of motion as stated in the question is possible, if particles are at diametrically opposite positions on a circular path. Two particles are always separated by the diameter of the circular path. See the figure below to evaluate the motion and separation between the particles.

Motion along a circular path



Two particles are always separated by the diameter of the circle transversed by the particles.

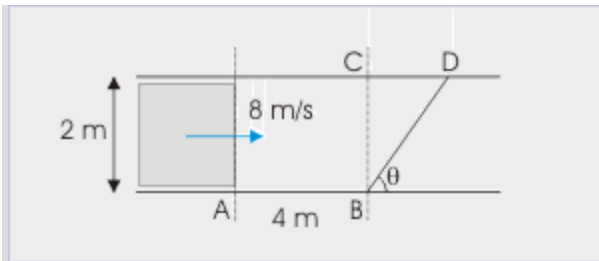
Comparing velocities

Example:

Problem : A car of width 2 m is approaching a crossing at a velocity of 8 m/s. A pedestrian at a distance of 4 m wishes to cross the road safely. What should be the minimum speed of pedestrian so that he/she crosses the road safely?

Solution : We draw the figure to illustrate the situation. Here, car travels the linear distance $(AB + CD)$ along the direction in which it moves, by which time the pedestrian travels the linear distance BD . Let pedestrian travels at a speed " v " along BD , which makes an angle " θ " with the direction of car.

Motion of a car and a pedestrian



The pedestrian crosses the road at angle with direction of car.

We must understand here that there may be number of combination of angle and speed for which pedestrian will be able to safely cross before car reaches. However, we are required to find the minimum speed. This speed should, then, correspond to a particular value of θ .

We can also observe that pedestrian should move obliquely. In doing so he/she gains extra time to cross the road.

From triangle BCD,

$$\tan(90 - \theta) = \cot \theta = \frac{CD}{BC} = \frac{CD}{2}$$

$$\Rightarrow CD = 2 \cot \theta$$

Also,

$$\cos(90 - \theta) = \sin \theta = \frac{BC}{BD} = \frac{2}{BD}$$

$$\Rightarrow BD = \frac{2}{\sin \theta}$$

According to the condition given in the question, the time taken by car and pedestrian should be equal for the situation outlined above :

$$t = \frac{4 + 2 \cot \theta}{8} = \frac{\frac{2}{\sin \theta}}{v}$$

$$v = \frac{8}{2 \sin \theta + \cos \theta}$$

For minimum value of speed, $\frac{dv}{d\theta} = 0$,

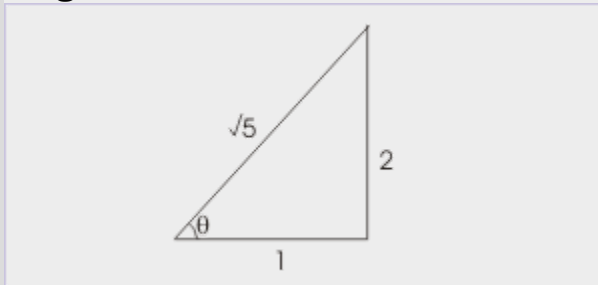
$$\Rightarrow \frac{dv}{d\theta} = \frac{-8 \times (2 \cos \theta - \sin \theta)}{(2 \sin \theta + \cos \theta)^2} = 0$$

$$\Rightarrow (2 \cos \theta - \sin \theta) = 0$$

$$\Rightarrow \tan \theta = 2$$

In order to evaluate the expression of velocity with trigonometric ratios, we take the help of right angle triangle as shown in the figure, which is consistent with the above result.

Trigonometric ratio



The tangent of angle is equal to 2.

From the triangle, defining angle “ θ ”, we have :

$$\sin \theta = \frac{2}{\sqrt{5}}$$

and

$$\cos \theta = \frac{1}{\sqrt{5}}$$

The minimum velocity is :

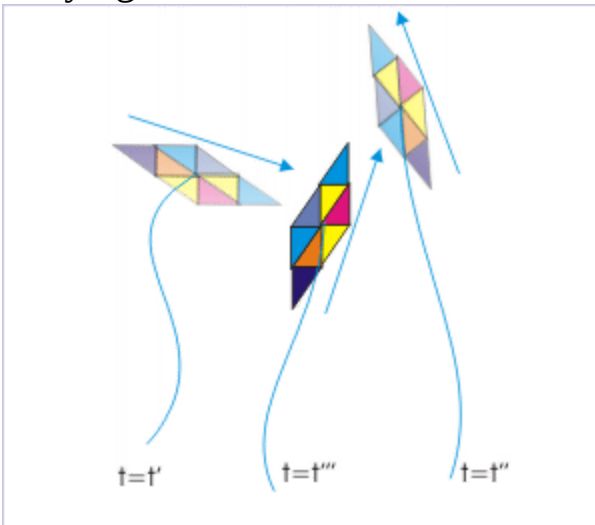
$$v = \frac{8}{2 \times 2\sqrt{5 + \frac{1}{\sqrt{5}}}} = \frac{8}{\sqrt{5}} = 3.57 \text{ m/s}$$

Acceleration

All bodies have intrinsic property to maintain its velocity. This is a fundamental nature of matter. However, a change in the velocity results when a net external force is applied. In that situation, [velocity](#) is not constant and is a function of time.

In our daily life, we are often subjected to the change in velocity. The incidence of the change in velocity is so common that we subconsciously treat constant velocity more as a theoretical consideration. We drive car with varying velocity, while negotiating traffic and curves. We take a ride on the train, which starts from rest and comes to rest. We use lift to ascend and descend floors at varying speeds. All these daily life routines involve change in the velocity. The spontaneous natural phenomena are also largely subjected to force and change in velocity. A flying kite changes its velocity in response to wind force as shown in the figure below.

A flying kite



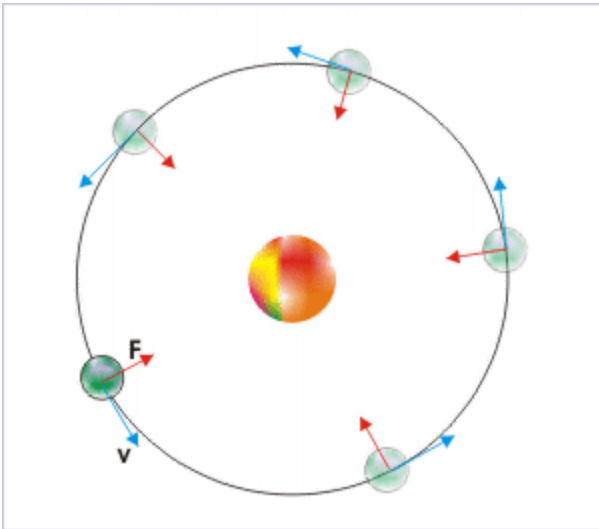
The kite changes its velocity in response to wind force.

The presence of external [force](#) is a common feature of our life and not an exception. Our existence on earth, as a matter of fact, is under the moderation of force due to gravity and friction. It is worth while here to

point out that the interaction of external force with bodies is not limited to the earth, but extends to all bodies like stars, planets and other mass aggregation.

For example, we may consider the motion of Earth around Sun that takes one year. For illustration purpose, let us approximate the path of motion of the earth as circle. Now, the natural tendency of the earth is to move linearly along the straight line in accordance with the Newton's first law of motion. But, the earth is made to change its direction continuously by the force of gravitation (shown with red arrow in the figure) that operates between the Earth and the Sun. The change in velocity in this simplified illustration is limited to the change in the direction of the velocity (shown in with blue arrow).

Motion of earth around sun



Gravitational force changes the direction of the motion

This example points to an interesting aspect of the change in velocity under the action of an external force. The change in velocity need not be a change in the magnitude of velocity alone, but may involve change of magnitude or direction or both. Also, the change in velocity (effect) essentially indicates the presence of a net external force (cause).

Acceleration

Acceleration

Acceleration is the rate of change of velocity with respect to time.

It is evident from the definition that acceleration is a vector quantity having both magnitude and direction, being a ratio involving velocity vector and scalar time. Mathematically,

$$\mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t}$$

The ratio denotes average acceleration, when the measurement involves finite time interval; whereas the ratio denotes instantaneous acceleration for infinitesimally small time interval, $\Delta t \rightarrow 0$.

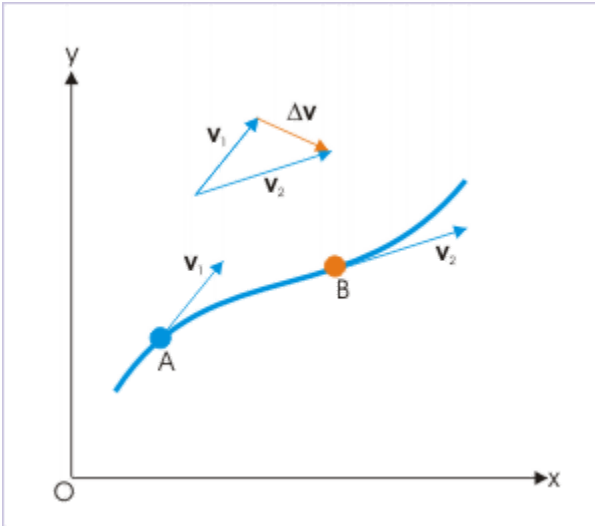
We know that velocity itself has the dimension of length divided by time; the dimension of the acceleration, which is equal to the change in velocity divided by time, involves division of length by squared time and hence its dimensional formula is LT^{-2} . The SI unit of acceleration is meter/second² i.e. m/s^2 .

Average acceleration

Average acceleration gives the overall acceleration over a finite interval of time. The magnitude of the average acceleration tells us the rapidity with which the velocity of the object changes in a given time interval.

$$\mathbf{a}_{\text{avg}} = \frac{\Delta \mathbf{v}}{\Delta t}$$

Average acceleration



The direction of acceleration is along the vector $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ and not required to be in the direction of either of the velocities. If the initial velocity is zero, then $\Delta \mathbf{v} = \mathbf{v}_2 = \mathbf{v}$ and average acceleration is in the direction of final velocity.

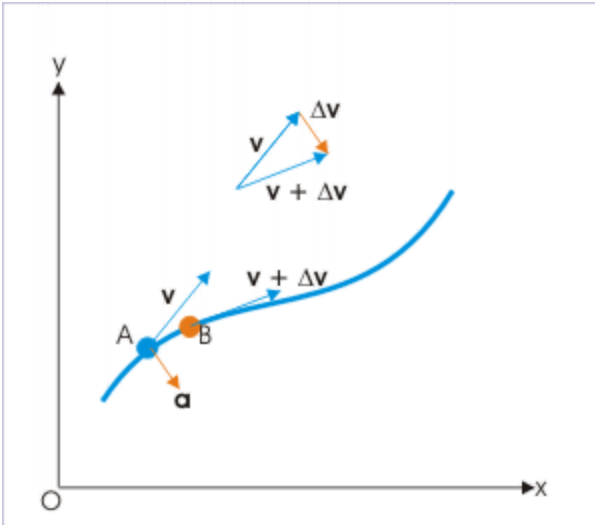
Note: Difference of two vectors $\mathbf{v}_2 - \mathbf{v}_1$ can be drawn conveniently following certain convention (We can take a mental note of the procedure for future use). We draw a straight line, starting from the arrow tip (i.e. head) of the vector being subtracted $\mathbf{v}_1$ to the arrow tip (i.e. head) of the vector from which subtraction is to be made $\mathbf{v}_2$. Then, from the triangle law of vector addition,

$$\begin{aligned}\mathbf{v}_1 + \Delta \mathbf{v} &= \mathbf{v}_2 \\ \Rightarrow \Delta \mathbf{v} &= \mathbf{v}_2 - \mathbf{v}_1\end{aligned}$$

Instantaneous acceleration

Instantaneous acceleration, as the name suggests, is the acceleration at a given instant, which is obtained by evaluating the limit of the average acceleration as $\Delta t \rightarrow 0$.

Instantaneous acceleration



$$\mathbf{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

As the point B approaches towards A, the limit of the ratio evaluates to a finite value. Note that the ratio evaluates not along the tangent to the curve as in the case of velocity, but along the direction shown by the red arrow. This is a significant result as it tells us that direction of acceleration is independent of the direction of velocity.

Instantaneous acceleration

Instantaneous acceleration is equal to the first derivative of velocity with respect to time.

It is evident that a body might undergo different phases of acceleration during the motion, depending on the external forces acting on the body. It means that accelerations in a given time interval may vary. As such, the average and the instantaneous accelerations need not be equal.

A general reference to the term acceleration ($\mathbf{a}$) refers to the instantaneous acceleration – not average acceleration. The absolute value of acceleration gives the magnitude of acceleration :

$$|\mathbf{a}| = a$$

Acceleration in terms of position vector

Velocity is defined as derivative of [position vector](#) :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

Combining this expression of velocity into the expression for acceleration, we obtain,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

Acceleration

Acceleration of a point body is equal to the second derivative of position vector with respect to time.

Now, position vector is represented in terms of [components](#) as :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Substituting in the expression of acceleration, we have :

$$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Example:

Acceleration

Problem : The position of a particle, in meters, moving in space is described by following functions in time.

$$x = 2t^2 - 2t + 3; y = -4t \text{ and } z = 5$$

Find accelerations of the particle at $t = 1$ and 4 seconds from the start of motion.

Solution : Here scalar components of accelerations in x,y and z directions are given as :

$$a_x = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2} (2t^2 - 2t + 3) = 4$$

$$a_y = \frac{d^2y}{dt^2} = \frac{d^2}{dt^2} (-4t) = 0$$

$$a_z = \frac{d^2z}{dt^2} = \frac{d^2}{dt^2} (5) = 0$$

Thus, acceleration of the particle is :

$$\mathbf{a} = 4\mathbf{i}$$

The acceleration of the particle is constant and is along x-direction. As acceleration is not a function of time, the accelerations at $t = 1$ and 4 seconds are same being equal to $4m/s$.

Understanding acceleration

The relationship among velocity, acceleration and force is central to the study of mechanics. Text book treatment normally divides the scope of study between kinematics (study of velocity and acceleration) and dynamics (study of acceleration and force). This approach is perfectly fine till we do not encounter some inherent problem as to the understanding of the subject matter.

One important short coming of this approach is that we tend to assume that acceleration depends on the state of motion i.e. velocity. Such inference is not all too uncommon as acceleration is defined in terms of velocities. Now the question is “Does acceleration actually depend on the velocity?”

There are many other such inferences that need to be checked. Following paragraphs investigate these fundamental issues in detail and enrich our understanding of acceleration.

Velocity, acceleration and force

Acceleration is related to external force. This relationship is given by Newton’s second law of motion. For a constant mass system,

$$\mathbf{F} = m\mathbf{a}$$

In words, Newton’s second law states that acceleration (effect) is the result of the application of net external force (cause). Thus, the relationship between the two quantities is that of cause and effect. Further, force is equal to the product of a scalar quantity, m , and a vector quantity, $\mathbf{a}$, implying that the direction of the acceleration is same as that of the net force. This means that acceleration is though the measurement of the change of velocity, but is strictly determined by the external force and the mass of the body; and expressed in terms of “change of velocity” per unit time.

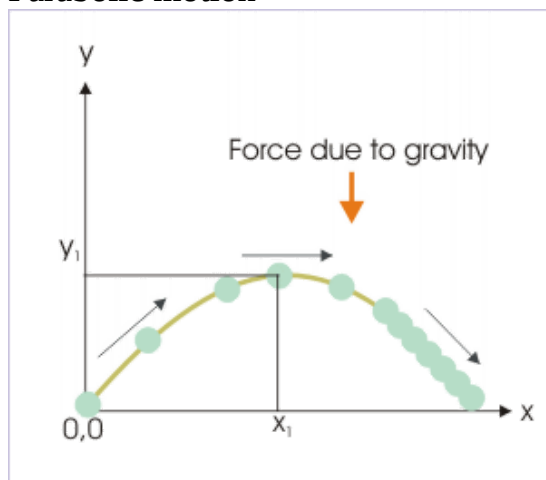
If we look around ourselves, we find that force modifies the state of motion of the objects. The immediate effect of a net force on a body is that the state of motion of the body changes. In other words, the velocity of the body changes in response to the application of net force. Here, the word “net” is important. The motion of the body responds to the net or resultant force. In this sense, acceleration is mere statement of the effect of the force as measurement of the rate of change of the velocity with time.

Now, there is complete freedom as to the magnitude and direction of force being applied. From our real time experience, we may substantiate this assertion. For example, we can deflect a foot ball, applying force as we wish (in both magnitude and direction). The motion of the ball has no bearing on how we apply force. Simply put : the magnitude and direction of the force (and that of acceleration) is not dependent on the magnitude and direction of the velocity of the body.

In the nutshell, **we conclude that force and hence acceleration is independent of the velocity of the body**. The magnitude and direction of the acceleration is determined by the magnitude and direction of the force and mass of the body. This is an important clarification.

To elucidate the assertion further, let us consider parabolic motion of a ball as shown in the figure. The important aspect of the parabolic motion is that the acceleration associated with motion is simply 'g' as there is no other force present except the force of gravity. The resultant force and mass of the ball together determine acceleration of the ball.

Parabolic motion



The resultant force and mass of the ball together determine acceleration of the ball

Example:
Acceleration

Problem : If the tension in the string is T , when the string makes an angle θ with the vertical. Find the acceleration of a pendulum bob, having a mass “ m ”.

Solution : As pointed out in our discussion, we need not study or consider velocities of bob to get the answer. Instead we should strive to know the resultant force to find out acceleration, using Newton’s second law of motion.

The forces, acting on the bob, are (i) force of gravity, mg , acting in the downward direction and (ii) Tension, T , acting along the string. Hence, the acceleration of the bob is determined by the resultant force, arising from the two forces.

Acceleration of a pendulum bob



Using parallelogram theorem for vector addition, the resultant force is :

$$F = \sqrt{\{m^2g^2 + T^2 + 2mgT \cos(180^\circ - \theta)\}} = \sqrt{(m^2g^2 + T^2 + 2mgT \cos \theta)}$$

The acceleration is in the direction of force as shown in the figure, whereas the magnitude of the acceleration, a , is given by :

$$a = \frac{\sqrt{(m^2g^2 + T^2 + 2mgT \cos \theta)}}{m}$$

External force and possible scenarios

The change in velocity, resulting from the application of external force, may occur in magnitude or direction or both.

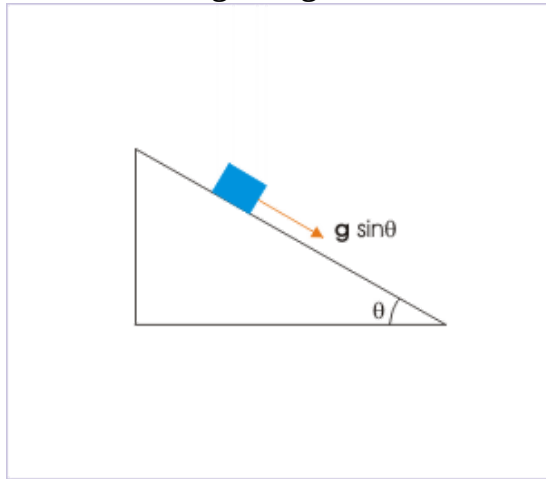
1: When force is applied in a given direction and the object is stationary.

Under this situation, the magnitude of velocity increases with time, while the body follows a linear path in the direction of force (or acceleration).

2: When force is applied in the direction of the motion, then it increases the magnitude of the velocity without any change in the direction.

Let us consider a block sliding on a smooth incline surface as shown in the figure. The component of the force due to gravity applies in the direction of motion. Under this situation, the magnitude of velocity increases with time, while the body follows a linear path. There is no change in the direction of motion.

A block sliding along smooth incline

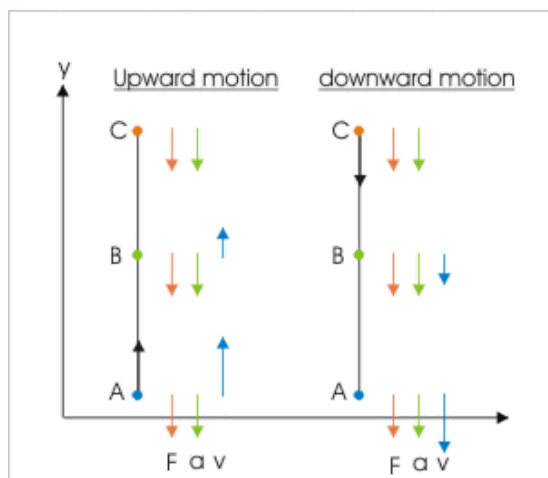


The block moves faster as it moves down the slope

3: When force is applied in the opposite direction to the motion, then it decreases the magnitude of the velocity without any change in the direction.

Take the example of a ball thrown vertically in the upward direction with certain velocity. Here, force due to gravity is acting downwards. The ball linearly rises to the maximum height till the velocity of ball reduces to zero.

A ball thrown in vertical direction



The ball reverses its direction during motion.

During upward motion, we see that the force acts in the opposite direction to that of the velocity. Under this situation, the magnitude of velocity decreases with time, while the body follows a linear path. There is no change in the direction of motion.

The velocity of the object at the point, where motion changes direction, is zero and force is acting downwards. This situation is same as the case 1. The object is at rest. Hence, object moves in the direction of force i.e. in the downward direction.

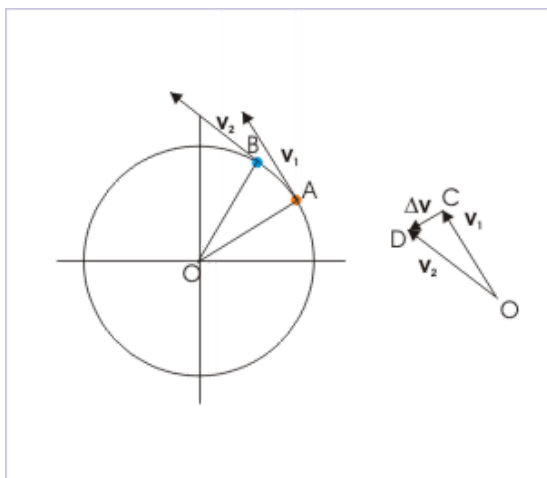
In the figure above, the vectors drawn at various points represent the direction and magnitude of force (**F**), acceleration (**a**) and velocity (**v**) during the motion. Note that both force and acceleration act in the downward direction during the motion.

4: When force is applied perpendicular to the direction of motion, then it causes change in the direction of the velocity.

A simple change of direction also constitutes change in velocity and, therefore, acceleration. Consider the case of a uniform circular motion in which a particle moves along a circular path at constant speed “**v**”. Let **v**₁ and **v**₂ be the velocities of the particle at two time instants, then

$$|\mathbf{v}_1| = |\mathbf{v}_2| = \text{a constant}$$

Uniform circular motion



A central force perpendicular to motion causes change in direction.

In the adjoining figure, the vector segments OC and OD represent $\mathbf{v}_1$ and $\mathbf{v}_2$. Knowing that vector difference $\Delta \mathbf{v}$ is directed from initial to final position, it is represented by vector CD. Using the adjoining vector triangle,

$$\mathbf{OC} + \mathbf{CD} = \mathbf{OD}$$

$$\mathbf{v} = \mathbf{CD} = \mathbf{v}_2 - \mathbf{v}_1$$

The important thing to realize here is that direction of $\Delta \mathbf{v}$ is along CD, which is directed towards the origin. This result is in complete agreement of what we know about uniform circular motion (The topic of uniform circular motion is covered in separate module). We need to apply a force (causing acceleration to the moving particle) across (i.e. perpendicular) to the motion to change direction. If the force (hence acceleration) is perpendicular to velocity, then magnitude of velocity i.e. speed remains same, whereas the direction of motion keeps changing.

Example:

Acceleration

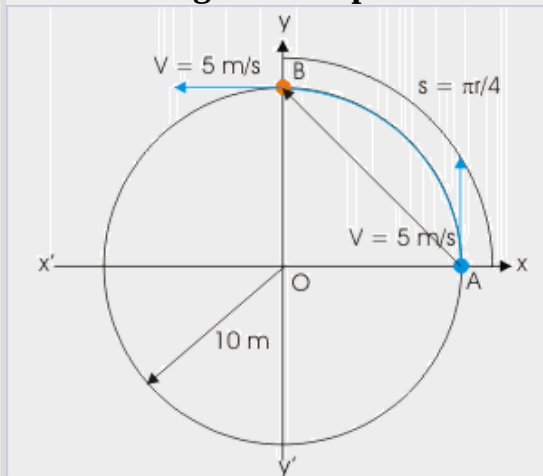
Problem : An object moves along a quadrant AB of a circle of radius 10 m with a constant speed 5 m/s. Find the average velocity and average acceleration in this interval.

Solution : Here, we can determine time interval from the first statement of the question. The particle covers a distance of $2\pi r/4$ with a speed v . Hence, time interval,

$$t = \frac{\text{distance}}{\text{speed}} = \frac{2\pi r}{4v} = \frac{3.14 \times 10}{2 \times 5} = 3.14 \text{ s}$$

Magnitude of average velocity is given by the ratio of the magnitude of displacement and time :

Motion along circular path

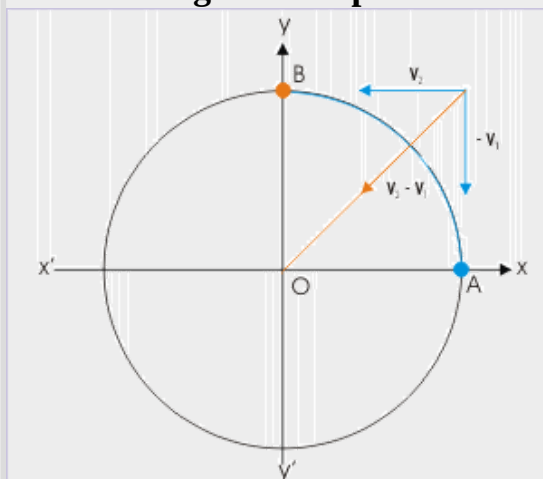


$$|\mathbf{v}_{\text{avg}}| = \frac{|\mathbf{AB}|}{\text{Time}} = \frac{\sqrt{(10^2 + 10^2)}}{3.14} = \frac{14.14}{3.14} = 4.5 \text{ m/s}$$

Average velocity is directed along vector AB i.e. in the direction of displacement vector.

Magnitude of average acceleration is given by the ratio :

Motion along circular path



Direction of acceleration

$$a_{\text{avg}} == \frac{|\Delta \mathbf{v}|}{\Delta t}$$

Here,

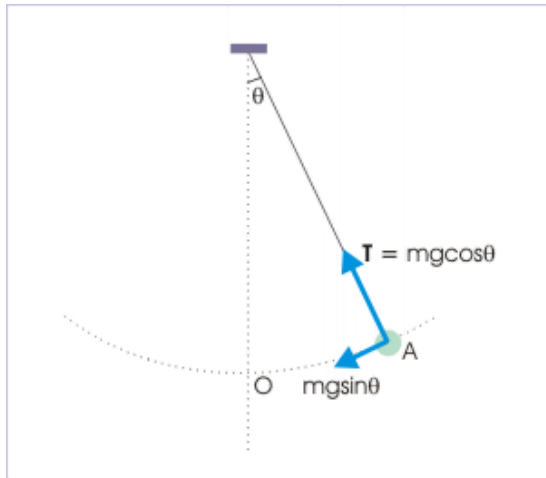
$$\begin{aligned} |\Delta \mathbf{v}| &= |\mathbf{v}_2 - \mathbf{v}_1| \\ |\Delta \mathbf{v}| &= \sqrt{(5^2 + 5^2)} = 5\sqrt{2} \text{ m/s} \\ \Rightarrow a_{\text{avg}} &= \frac{5\sqrt{2}}{3.14} = 3.1 \text{ m/s}^2 \end{aligned}$$

Average acceleration is directed along the direction of vector $\mathbf{v}_2 - \mathbf{v}_1$ i.e. directed towards the center of the circle (as shown in the figure).

5: When force is applied at a certain angle with the motion, then it causes change in both magnitude and direction of the velocity. The component of force in the direction of motion changes its magnitude (an increase or decrease), while the component of force perpendicular to the direction of motion changes its direction.

The motion of a small spherical mass “m”, tied to a fixed point with the help of a string illustrates the changes taking place in both the direction and magnitude of the velocity. Saving the details for discussion at a later point in the course, here tension in the string and tangential force provide for the change in the direction and magnitude of velocity respectively.

When the string makes an angle, θ , with the vertical, then :
Motion of a pendulum



Forces acting on the pendulum bob

(a) Tension in the string, T is :

$$T = mg \cos \theta$$

This force acts normal to the path of motion at all points and causes the body to continuously change its direction along the circular path. Note that this force acts perpendicular to the direction of velocity, which is along the tangent at any given point on the arc.

(b) Tangential force, F , is :

$$F = mg \sin \theta$$

This force acts tangentially to the path and is in the direction of velocity of the spherical mass, which is also along the tangent at that point. As the force and velocity are in the same direction, this force changes the magnitude of velocity. The magnitude of velocity increases when force acts in the direction of velocity and magnitude of velocity decreases when force acts in the opposite direction to that of velocity.

Acceleration and deceleration

Negative vector is a relative term - not an independent concept. We completely lose the significance of a negative vector when we consider it in isolation. A negative vector assumes meaning only in relation with another vector or some reference direction.

Acceleration and deceleration are terms which are generally considered to have opposite meaning. However, there is difference between literal and scientific meanings of these terms. In literal sense, acceleration is considered to describe an increase or positive change of speed or velocity. On the other hand, deceleration is considered to describe a decrease or negative change of speed or velocity. Both these descriptions are incorrect in physics. We need to form accurate and exact meaning of these two terms. In this module, we shall explore these terms in the context of general properties of vector and scalar quantities.

A total of six (6) attributes viz time, distance, displacement, speed, velocity and acceleration are used to describe motion. Three of these namely time, distance and speed are scalar quantities, whereas the remaining three attributes namely displacement, velocity and acceleration are vectors. Interpretations of these two groups are different with respect to (ii) negative and positive sign and (i) sense of increase and decrease. Further these interpretations are also affected by whether we consider these terms in one or two/three dimensional motion.

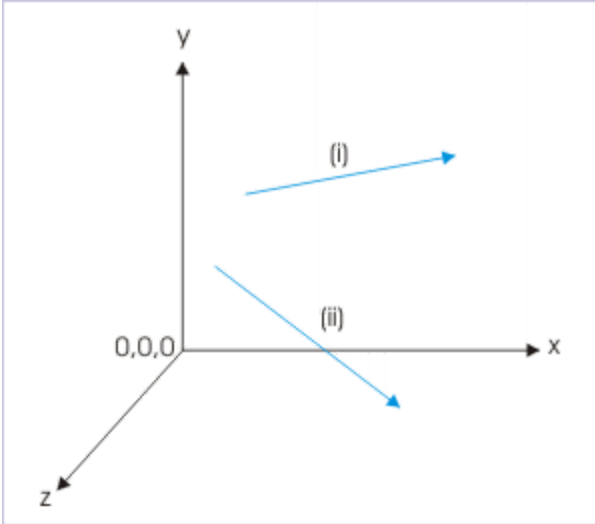
The meaning of scalar quantities is more or less clear. The scalar attributes have only magnitude and no sense of direction. The attributes “distance” and “speed” are positive quantities. There is no possibility of negative values for these two quantities. In general, time is also positive. However, it can be assigned negative value to represent a time instant that occurs before the start of observation. For this reason, it is entirely possible that we may get negative time as solution of kinematics consideration.

Negative vector quantities

A vector like acceleration may be directed in any direction in three dimensional space, which is defined by the reference coordinate system.

Now, why should we call vector (as shown in the figure below) represented by line (i) positive and that by line (ii) negative? What is the qualification for a vector being positive or negative? There is none. Hence, in pure mathematical sense, a negative vector can not be identified by itself.

Vector representation in three dimensional reference



Vectors in isolation have no sense of being negative.

In terms of component vectors, let us represent two accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ as :

$$\mathbf{a}_1 = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$$

$$\mathbf{a}_2 = -2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$

Why should we call one of the above vectors as positive and other as negative acceleration? Which sign should be considered to identify a positive or negative vector? Further, the negative of $\mathbf{a}_1$ expressed in component form is another vector given by $-\mathbf{a}_1$:

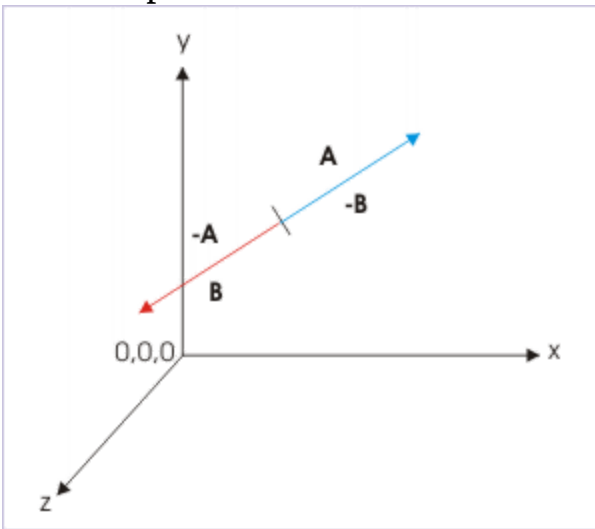
$$-\mathbf{a}_1 = -(-2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

So, what is the actual position? The concept of negative vector is essentially a relative concept. If we represent a vector $\mathbf{A}$ as shown in the figure below,

then negative to this vector $-\mathbf{A}$ is just another vector, which is directed in the opposite direction to that of vector $\mathbf{A}$ and has equal magnitude as that of $\mathbf{A}$.

In addition, if we denote “ $-\mathbf{A}$ ” as “ $\mathbf{B}$ ”, then “ $\mathbf{A}$ ” is “ $-\mathbf{B}$ ”. We, thus, completely lose the significance of a negative vector when we consider it in isolation. We can call the same vector either “ $\mathbf{A}$ ” or “ $-\mathbf{A}$ ”. We conclude, therefore, that a negative vector assumes meaning only in relation with another vector.

Vector representation in three dimensional reference



Vectors in isolation have no sense of being negative.

In one dimensional motion, however, it is possible to assign distinct negative values. In this case, there are only two directions; one of which is in the direction of reference axis (positive) and the other is in the opposite direction (negative). The significance of negative vector in one dimensional motion is limited to relative orientation with respect to reference direction. In the nutshell, sign of vector quantity in one dimensional motion represents the directional property of vector. It has only this meaning. We can not attach any other meaning for negating a vector quantity.

It is important to note that the sequence in “ $-5\mathbf{i}$ ” is misleading in the sense that a vector quantity can not have negative magnitude. The negative sign, as a matter of fact, is meant for unit vector “ $\mathbf{i}$ ”. The correct reading sequence would be “ $5 \times -\mathbf{i}$ ”, meaning that it has a magnitude of “5” and is directed in “ $-\mathbf{i}$ ” direction i.e. opposite to reference direction. Also, since we are free to choose our reference, the previously assigned “ $-5 \mathbf{i}$ ” can always be changed to “ $5 \mathbf{i}$ ” and vice-versa.

We summarize the discussion so far as :

- There is no independent meaning of a negative vector attribute.
- In general, a negative vector attribute is defined with respect to another vector attribute having equal magnitude, but opposite direction.
- In the case of one dimensional motion, the sign represents direction with respect to reference direction.

“Increase” and “decrease” of vectors quantities

The vector consist of both magnitude and direction. There can be infinite directions of a vector. On the other hand, increase and decrease are bi-directional and opposite concepts. Can we attach meaning to a phrase “increase in direction” or “decrease in direction”? There is no sense in saying that direction of the moving particle has increased or decreased. In the nutshell, we can associate the concepts of increase and decrease with quantities which are scalar – not quantities which are vector. Clearly, we can attach the sense of increase or decrease with the magnitudes of velocity or acceleration, but not with velocity and acceleration.

For this reason, we may recall that velocity is defined as the time rate of “change” – not “increase or decrease” in displacement. Similarly, acceleration is defined as time rate of change of velocity – not “increase or decrease” in velocity. It is so because the term “change” conveys the meaning of “change” in direction as well that of “change” as increase or decrease in the magnitude of a vector.

However, we see that phrases like “increase or decrease in velocity” or “increase or decrease in acceleration” are used frequently. We should be

aware that these references are correct only in very specific context of motion. If motion is unidirectional, then the vector quantities associated with motion is treated as either positive or negative scalar according as it is measured in the reference direction or opposite to it. Even in this situation, we can not associate concepts of increase and decrease to vector quantities. For example, how would we interpret two particles moving in negative x-direction with negative accelerations -10m/s^2 and -20m/s^2 respectively ? Which of the two accelerations is greater acceleration ? Algebraically, “-10” is greater than “-20”. But, we know that second particle is moving with higher rate of change in velocity. The second particle is accelerating at a higher rate than first particle. Negative sign only indicates that particle is moving in a direction opposite to a reference direction.

Clearly, the phrases like “increase or decrease in velocity” or “increase or decrease in acceleration” are correct only when motion is "unidirectional" and in "positive" reference direction. Only in this restricted context, we can say that acceleration and velocity are increasing or decreasing. In order to be consistent with algebraic meaning, however, we may prefer to associate relative measure (increase or decrease) with magnitude of the quantity and not with the vector quantity itself.

Deceleration

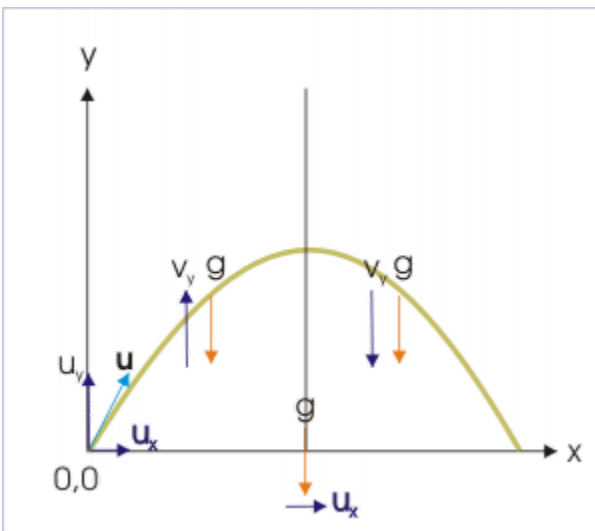
Acceleration is defined strictly as the time rate of change of velocity vector. Deceleration, on the other hand, is acceleration that causes reduction in "speed". Deceleration is not opposite of acceleration. It is certainly not negative time rate of change of velocity. It is a very restricted term as explained below.

We have seen that speed of a particle in motion decreases when component of acceleration is opposite to the direction of velocity. In this situation, we can say that particle is being decelerated. Even in this situation, we can not say that deceleration is opposite to acceleration. Here, only a component of acceleration is opposite to velocity – not the entire acceleration. However, if acceleration itself (not a component of it) is opposite to velocity, then deceleration is indeed opposite to acceleration.

If we consider motion in one dimension, then the deceleration occurs when signs of velocity and acceleration are opposite. A negative velocity and a positive acceleration mean deceleration; a positive velocity and a negative acceleration mean deceleration; a positive velocity and a positive acceleration mean acceleration; a negative velocity and a negative acceleration mean acceleration.

Take the case of projectile motion of a ball. We study this motion as two equivalent linear motions; one along x-direction and another along y-direction.

Parabolic motion



The projected ball undergoes deceleration in y-direction.

For the upward flight, velocity is positive and acceleration is negative. As such the projectile is decelerated and the speed of the ball in + y direction decreases (deceleration). For downward flight from the maximum height, velocity and acceleration both are negative. As such the projectile is accelerated and the speed of the ball in - y direction increases (acceleration).

In the nutshell, we summarize the discussion as :

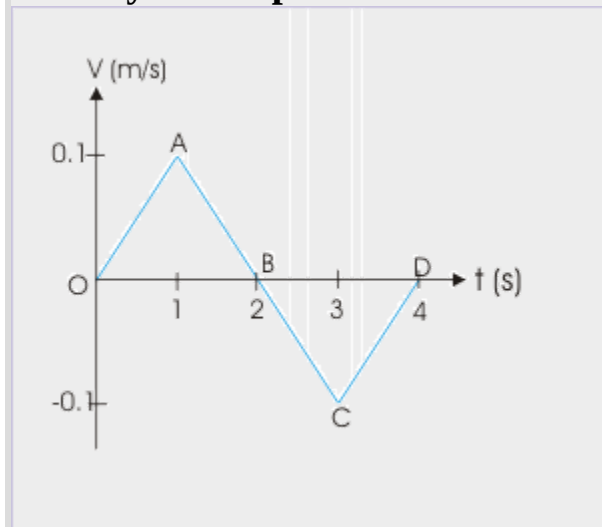
- Deceleration results in decrease in speed i.e magnitude of velocity.
- In one dimensional motion, the “deceleration” is defined as the acceleration which is opposite to the velocity.

Example:

Acceleration and deceleration

Problem : The velocity of a particle along a straight line is plotted with respect to time as shown in the figure. Find acceleration of the particle between OA and CD. What is acceleration at $t = 0.5$ second and 1.5 second. What is the nature of accelerations in different segments of motion? Also investigate acceleration at A.

Velocity – time plot



Solution : Average acceleration between O and A is given by the slope of straight line OA :

$$a_{OA} = \frac{v_2 - v_1}{t} = \frac{0.1 - 0}{1} = 0.1 \text{ m/s}^2$$

Average acceleration between C and D is given by the slope of straight line CD :

$$a_{CD} = \frac{v_2 - v_1}{t} = \frac{0 - (-0.1)}{1} = 0.1 \text{ m/s}^2$$

Accelerations at $t = 0.5$ second and 1.5 second are obtained by determining slopes of the curve at these time instants. In the example, the slopes at

these times are equal to the slope of the lines OA and AB.

Instantaneous acceleration at $t = 0.5 \text{ s}$:

$$a_{0.5} = a_{OA} = 0.1 \text{ m/s}^2$$

Instantaneous acceleration at $t = 1.5 \text{ s}$:

$$a_{1.5} = a_{AB} = \frac{v_2 - v_1}{t} = \frac{0 - 0.1}{1} = -0.1 \text{ m/s}^2$$

We check the direction of velocity and acceleration in different segments of the motion in order to determine deceleration. To enable comparison, we determine directions with respect to the assumed positive direction of velocity. In OA segment, both acceleration and velocity are positive (hence particle is accelerated). In AB segment, acceleration is negative, but velocity is positive (hence particle is decelerated). In BC segment, both acceleration and velocity are negative (hence particle is accelerated). In CD segment, acceleration is positive but velocity is negative (hence particle is decelerated).

Alternatively, the speed increases in segment OA and BC (hence acceleration); decreases in segments AB and CD (hence deceleration).

We note that it is not possible to draw an unique tangent at point A. We may draw infinite numbers of tangent at this point. In other words, limit of average acceleration can not be evaluated at A. Acceleration at A, therefore, is indeterminate.

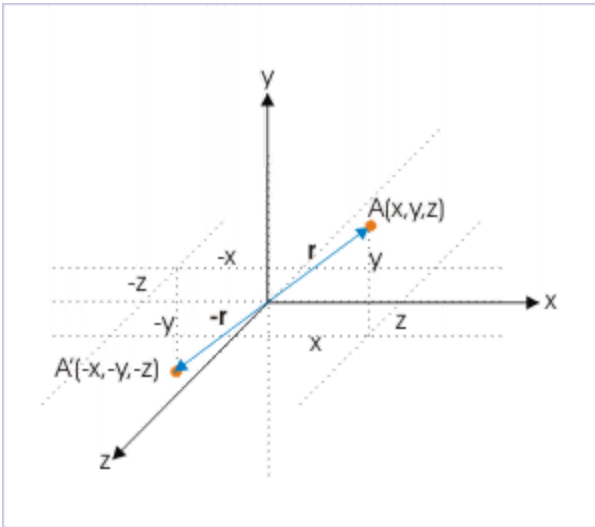
Graphical interpretation of negative vector quantities describing motion

Position vector

We may recall that position vector is drawn from the origin of reference to the position occupied by the body on a scale taken for drawing coordinate axes. This implies that the position vector is rooted to the origin of reference system and the position of the particle. Thus, we find that position vector is tied at both ends of its graphical representation.

Also if position vector ' $\mathbf{r}$ ' denotes a particular position (A), then " $-\mathbf{r}$ " denotes another position (A'), which is lying on the opposite side of the reference point (origin).

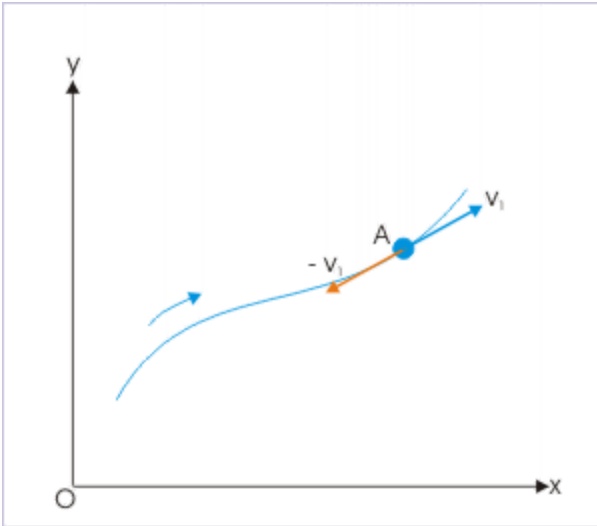
Position vector



Velocity vector

The velocity vector, on the other hand, is drawn on a scale from a particular position of the object with its tail and takes the direction of the tangent to the position curve at that point. Also, if velocity vector ' $\mathbf{v}$ ' denotes the velocity of a particle at a particular position, then " $-\mathbf{v}$ " denotes another velocity vector, which is reversed in direction with respect to the velocity vector, $\mathbf{v}$.

Velocity vector

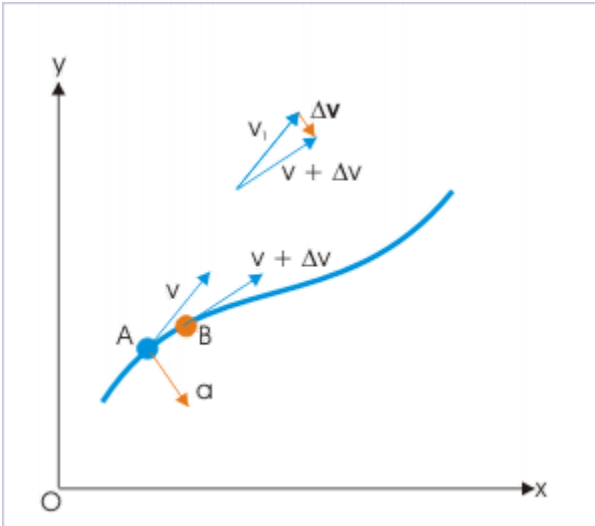


In either case (positive or negative), the velocity vectors originate from the position of the particle and are drawn along the tangent to the motion curve at that point. It must be noted that velocity vector, $\mathbf{v}$, is not rooted to the origin of the coordinate system like position vector.

Acceleration vector

Acceleration vector is drawn from the position of the object with its tail. It is independent of the origin (unlike position vector) and the direction of the tangent to the curve (unlike velocity vector). Its direction is along the direction of the force or alternatively along the direction of the vector representing change in velocities.

Acceleration vector



Further, if acceleration vector ' $\mathbf{a}$ ' denotes the acceleration of a particle at a particular position, then " $-\mathbf{a}$ " denotes another acceleration vector, which is reversed in direction with respect to the acceleration vector, $\mathbf{a}$.

Summarizing the discussion held so far :

- 1: Position vector is rooted to a pair of points i.e. the origin of the coordinate system and the position of the particle.
- 2: Velocity vector originates at the position of the particle and acts along the tangent to the curve, showing path of the motion.
- 3: Acceleration vector originates at the position of the particle and acts along the direction of force or equivalently along the direction of the change in velocity.
- 4: In all cases, negative vector is another vector of the same magnitude but reversed in direction with respect to another vector. The negative vector essentially indicates a change in direction and not the change in magnitude and hence, there may not be any sense of relative measurement (smaller or bigger) as in the case of scalar quantities associated with negative quantities. For example, -4° C is a smaller temperature than $+4^\circ \text{ C}$. Such is not the case with vector quantities. A 4 Newton force is as big as -4 Newton. Negative sign simply indicates the direction.

We must also emphasize here that we can shift these vectors laterally without changing direction and magnitude for vector operations like vector addition and multiplication. This independence is characteristic of vector operation and is not influenced by the fact that they are actually tied to certain positions in the coordinate system or not. Once vector operation is completed, then we can shift the resulting vector to the appropriate positions like the position of the particle (for velocity and acceleration vectors) or the origin of the coordinate system (for position vector).

Accelerated motion in one dimension

The nature of acceleration is determined by the net external force for constant mass system. Depending on the nature of force, there exists wide range of possibilities like zero, constant or varying accelerations in one dimensional motion.

Motion in one dimension is the basic component of all motion. A general three dimensional motion is equivalent to a system of three linear motion along three axes of a rectangular coordinate system. Thus, study of one dimensional accelerated motion forms the building block for studying accelerated motion in general. The basic defining differential equations for velocity and acceleration retain the form in terms of displacement and position, except that they consist of displacement or position component along a particular direction. In terms of representation, position vector $\mathbf{r}$ is replaced with $x\mathbf{i}$ or $y\mathbf{j}$ or $z\mathbf{k}$ in accordance with the direction of motion considered.

The defining equations of velocity and acceleration, in terms of position, for one dimensional motion are (say in x-direction) :

$$\mathbf{v} = \frac{dx}{dt} \mathbf{i}$$

and

$$\mathbf{a} = \frac{d^2x}{dt^2} \mathbf{i}$$

Only possible change of direction in one dimensional motion is reversal of motion. Hence, we can define velocity and acceleration in a particular direction, say x - direction, with equivalent scalar system, in which positive and negative values of scalar quantities defining motion represent the two possible direction.

The corresponding scalar form of the defining equations of velocity and acceleration for one dimensional motion are :

$$v = \frac{dx}{dt}$$

and

$$a = \frac{d^2x}{dt^2}$$

It must be clearly understood that the scalar forms are completely equivalent to vector forms. In the scalar form, the sign of various quantities describing motion serves to represent direction.

Example:

Problem : The displacement of a particle along x – axis is given by :

$$x = t^3 - 3t^2 + 4t - 12$$

Find the velocity when acceleration is zero.

Solution : Here, displacement is :

$$x = t^3 - 3t^2 + 4t - 12$$

We obtain expression for velocity by differentiating the expression of displacement with respect to time,

$$\Rightarrow v = \frac{dx}{dt} = 3t^2 - 6t + 4$$

Similarly, we obtain expression for acceleration by differentiating the expression of velocity with respect to time,

$$\Rightarrow a = \frac{dv}{dt} = 6t - 6$$

Note that acceleration is a function of time "t" and is not constant. For acceleration, $a = 0$,

$$\Rightarrow 6t - 6 = 0$$

$$\Rightarrow t = 1$$

Putting this value of time in the expression of velocity, we have :

$$\Rightarrow v = 3 t^2 - 6t + 4$$

$$\Rightarrow v = 3 \times 1^2 - 6 \times 1 + 4 = 1 \text{ m / s}$$

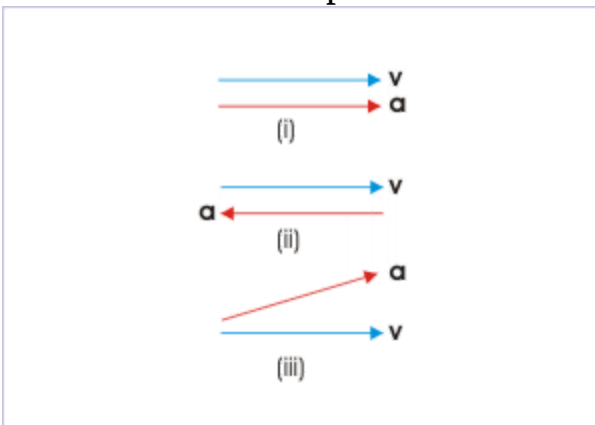
Nature of acceleration in one dimensional motion

One dimensional motion results from the action of net external force that applies along the direction of motion. It is a requirement for motion to be in one dimension. In case, force and velocity are at certain angle to each other, then there is sideways deflection of the object and the resulting motion is no more in one dimension.

If velocity and force are in the same direction, then magnitude of velocity increases; If velocity and force are in the opposite direction, then magnitude of velocity decreases.

The valid combination (i and ii) and invalid combination (iii) of velocity and acceleration for one dimensional motion are shown in the figure.

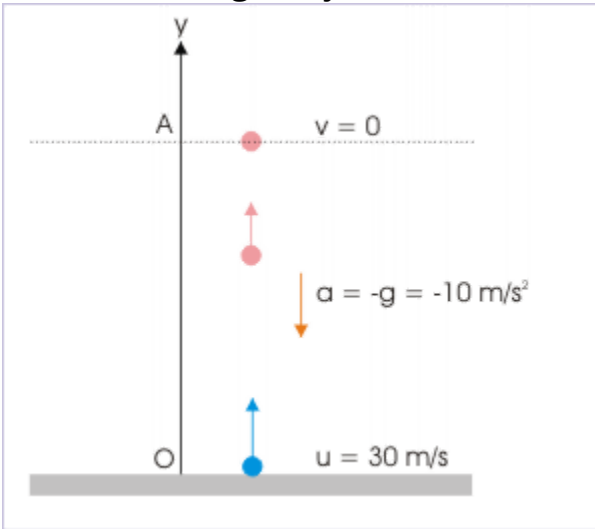
Acceleration – time plot



The requirement of one dimensional motion characterizes the nature of acceleration involved. The acceleration may vary in magnitude only. No sideways directional change in acceleration of the motion is possible for a given external force. We must emphasize that there may be reversal of motion i.e. velocity even without any directional change in acceleration. A projectile, thrown up in vertical direction, for example, returns to ground

with motion reversed at the maximum height, but acceleration at all time during the motion is directed downwards and there is no change in the direction of acceleration.

Motion under gravity



In mathematical parlance, if $\mathbf{v} = A\mathbf{i}$, then $\mathbf{a} = B\mathbf{i}$, where A and B are positive or negative numbers. For one dimensional motion, no other combination of unit vectors is possible. For example, acceleration can not be $\mathbf{a} = B\mathbf{j}$ or $\mathbf{a} = B(\mathbf{i} + \mathbf{j})$.

We summarize the discussion as :

- The velocity and force (hence acceleration) are directed along a straight line.
- For a given external force, the direction of acceleration remains unchanged in one dimension.

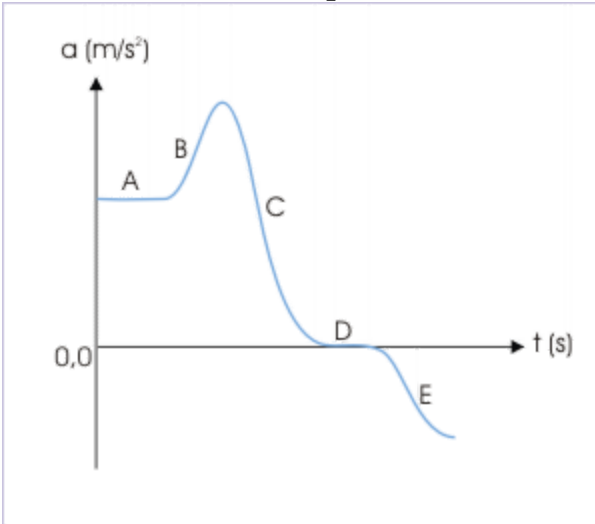
Graphs of one dimensional motion

Graphs are signature of motion. Here, we shall discuss broad categories of motion types in terms of acceleration and velocity.

Acceleration – time plot

Acceleration can be zero, constant or varying, depending upon the net external force and mass of the body. It is imperative that a single motion such as the motion of a car on the road may involve all kinds of variations in acceleration. A representation of an arbitrary real time acceleration - time variation may look like :

Acceleration – time plot



We can interpret this plot if we know the direction of velocity. We consider that positive direction of velocity is same as that of acceleration. Section, A, in the figure, represents constant acceleration. Section B represents an increasing magnitude of acceleration, whereas section C represents a decreasing magnitude of acceleration. Nonetheless, all of these accelerations result in the increase of speed with time as both velocity and acceleration are positive (hence in the same direction). The section E, however, represents deceleration as velocity (positive) and acceleration (negative) are in opposite direction. There is no acceleration during motion corresponding to section D. This section represents uniform motion.

For section A : Constant acceleration : $a = \frac{dv}{dt} = \text{Constant}$

For section B and C : Positive acceleration : $a = \frac{dv}{dt} > 0$

For section D : Zero acceleration : $a = \frac{dv}{dt} = 0$

For section E : Negative acceleration : $a = \frac{dv}{dt} < 0$

Area under acceleration - time plot

We know that :

$$a = \frac{dv}{dt}$$
$$\Rightarrow dv = a dt$$

Integrating both sides, we have :

$$\Delta v = a \Delta t$$

Thus, areas under acceleration – time plot gives the change in velocity in a given interval.

Velocity – time plot

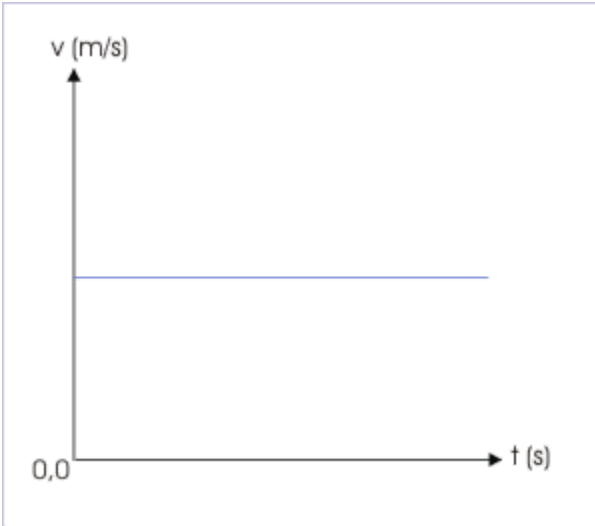
Here, we discuss velocity – time plot for various scenarios of motion in one dimension :

1: Acceleration is zero

If acceleration is zero, then velocity remains constant. As such the velocity time plot is a straight line parallel to time axis.

$$a = \frac{dv}{dt} = 0$$
$$\Rightarrow dv = 0$$
$$\Rightarrow v = \text{Constant}$$

Velocity – time plot



2: Acceleration is constant

Constant acceleration means that instantaneous acceleration at all points during motion is same. It, therefore, implies that instantaneous acceleration at any time instant and average acceleration in any time interval during motion are equal.

$$a_{\text{avg}} = a$$

Velocity of the object under motion changes by an equal value in equal time interval. It implies that the velocity – time plot for constant acceleration should be a straight line. Here,

$$\frac{dv}{dt} = \text{constant} = a$$

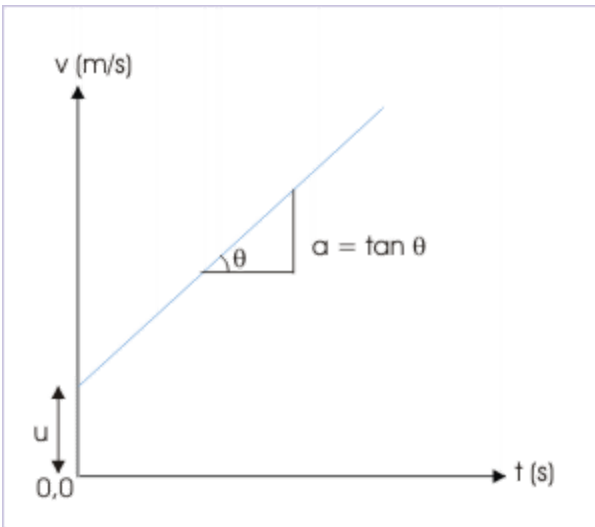
$$\Rightarrow dv = a dt$$

Integrating both sides,

$$\Rightarrow v = at + u$$

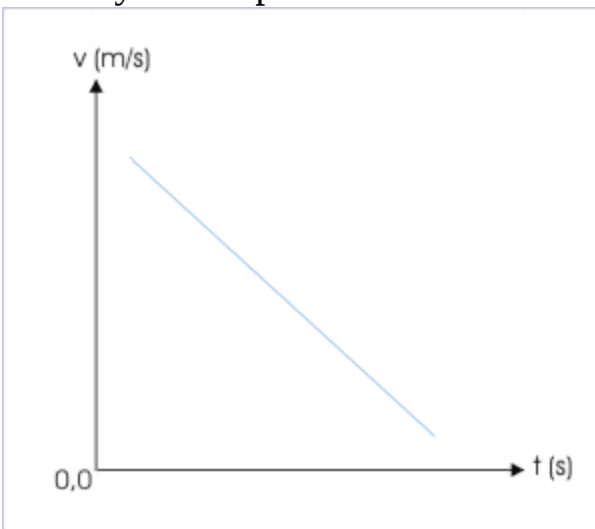
This is a linear equation in time “ t ” representing a straight line, where “ a ” is acceleration and is equal to the slope of the straight line and “ u ” is the intercept on velocity axis, representing velocity at $t = 0$. The plot of velocity with respect to time, therefore, is a straight line as shown in the figure here.

Velocity – time plot



For constant deceleration, the velocity – time plot has negative slope. Here, speed decreases with the passage of time :

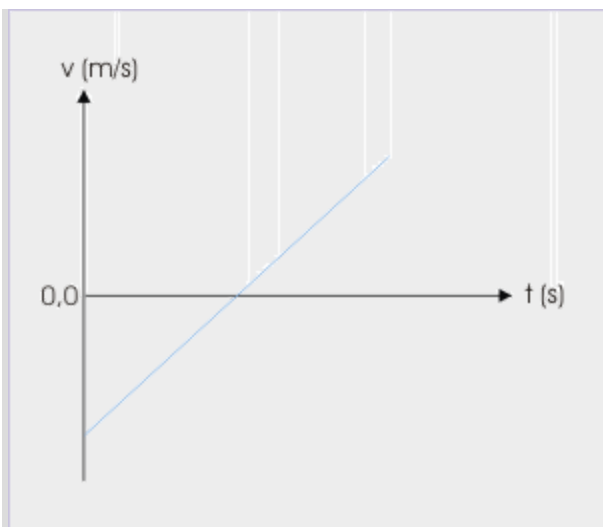
Velocity – time plot



Example:

Problem : A velocity – time plot describing motion of a particle in one dimension is shown in the figure.

Velocity – time plot



Determine (i) Whether the direction of velocity is reversed during motion?
(ii) Does the particle stop? (iii) Does the motion involve deceleration and
(iv) Is the acceleration constant?

Solution : (i) We see that velocity in the beginning is negative and changes to positive after some time. Hence, there is reversal of the direction of motion.

(ii) Yes. Velocity of the particle becomes zero at a particular time, when plot crosses time axis. This observation indicates an interesting aspect of reversal of direction of motion (i.e. velocity): A reversal of direction of a motion requires that the particle is stopped before the direction is reversed.

(iii) Yes. Occurance of deceleration is determined by comparing directions of velocity and acceleration. In the period before the particle comes to rest, velocity is negative, whereas acceleration is always positive with respect to the positive direction of velocity (slope of the line is positive on velocity - time plot). Thus, velocity and acceleration are in opposite direction for this part of motion and is decelerated. Further, it is also seen that speed of the particle is decreasing in this period.

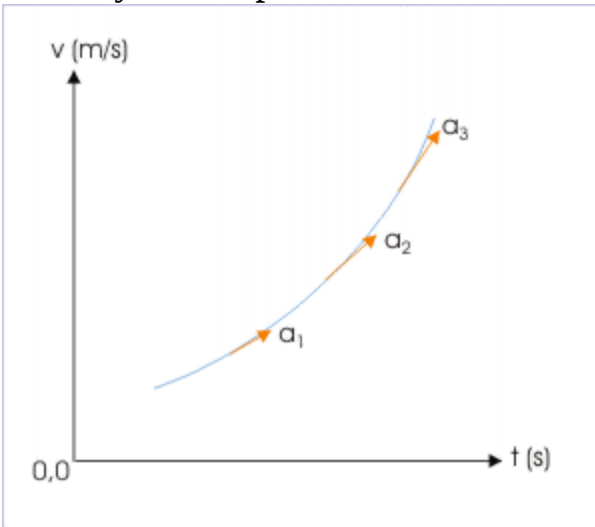
(iv) The plot is straight line with a constant slope. Thus, motion is under constant acceleration.

3: The magnitude of acceleration is increasing

Since slope of velocity – time curve is equal to the acceleration at that instant, it is expected that velocity – time plot should be a curve, whose

slope increases with time as shown in the figure.

Velocity – time plot



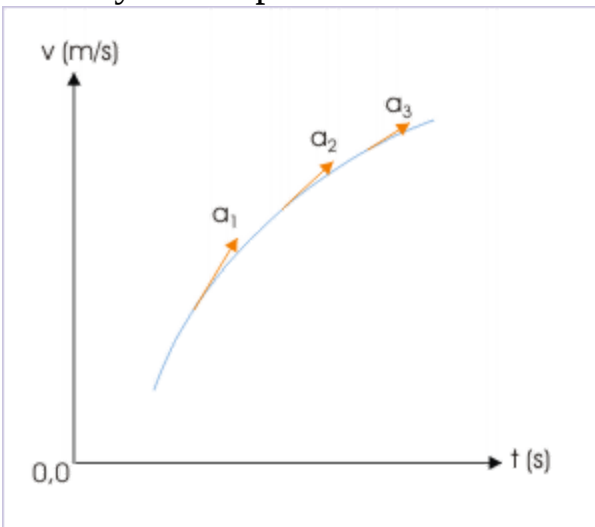
Here,

$$a_1 < a_2 < a_3$$

4: The magnitude of acceleration is decreasing

The velocity – time plot should be a curve, whose slope decreases with time as shown in the figure here :

Velocity – time plot



Here,

$$a_1 > a_2 > a_3$$

Example:

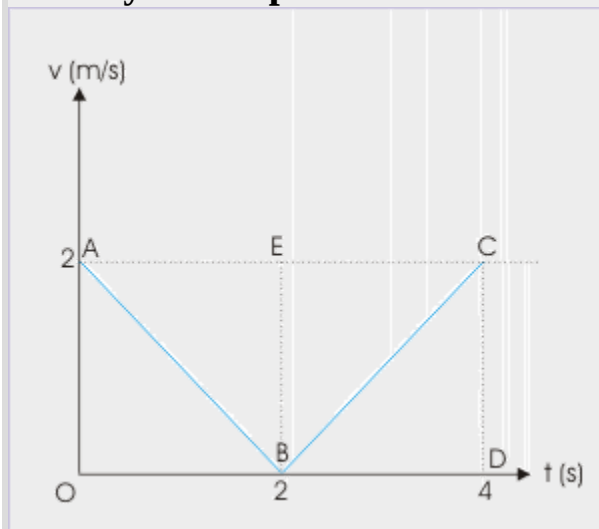
Problem : A person walks with a velocity given by $|t - 2|$ along a straight line. Find the distance and displacement for the motion in the first 4 seconds. What are the average velocity and acceleration in this period? Discuss the nature of acceleration.

Solution : Here, velocity is equal to the modulus of a function in time. It means that velocity is always positive. The representative values at the end of every second in the interval of 4 seconds are tabulated as below :

	Time (s)	Velocity (m/s)
	0	2
	1	1
	2	0
	3	1
	4	2

An inspection of the values in the table reveals that velocity linearly decreases for the first 2 second from 2 m/s to zero. It, then, increases from zero to 2 m/s. In order to obtain distance and displacement, we draw the plot between displacement and time as shown.

Velocity – time plot



Now, area under the plot gives displacement.

$$\text{Displacement} = \Delta OAB + \Delta BCD$$

$$\Rightarrow \text{Displacement} = \frac{1}{2} \times OB \times OA + \frac{1}{2} \times BD \times CD$$

$$\Rightarrow \text{Displacement} = \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 2 \times 2 = 4 \text{ m}$$

We see that velocity does not change its sign. Thus, motion is unidirectional apart from being rectilinear. As such, distance is equal to displacement and is also 4 m.

Now, average velocity is given by :

$$v_{\text{avg}} = \frac{\text{Displacement}}{\text{Time}} = \frac{4}{4} = 1 \text{ m/s}$$

Similarly, average acceleration during the motion is :

$$v_{\text{avg}} = \frac{\Delta v}{\Delta t} = \frac{0}{4} = 0$$

Though average acceleration is zero, the instantaneous acceleration of the motion during whole period is not constant as the velocity – time plot is not a single straight line. An inspection of the plot reveals that velocity – time has two line segments AB and BC, each of which separately represents constant acceleration.

Magnitude of constant acceleration (instantaneous) in the first part of the motion is equal to the slope of the displacement – time plot AB :

$$a_{\text{AB}} = \frac{\text{AO}}{\text{OB}} = \frac{-2}{2} = -1 \text{ m/s}^2$$

Similarly, acceleration in the second part of the motion is :

$$a_{\text{AB}} = \frac{\text{DC}}{\text{BD}} = \frac{2}{2} = 1 \text{ m/s}^2$$

Thus, the acceleration of the motion is negative for the first part of motion and positive for the second of motion with respect to velocity on velocity-time plot. We can, therefore, conclude that acceleration is not constant.

Constant acceleration

A constant acceleration is a special case of the accelerated motion.

The constant acceleration is a special case of accelerated motion. There are vast instances of motions, which can be approximated to be under constant force and hence constant acceleration. Two of the most important forces controlling motions in our daily life are force due to gravity and friction force. Incidentally, these two forces are constant in the range of motions of bodies in which we are interested. For example, force due to gravity is given by :

$$F = \frac{GMm}{r^2} = mg$$

where "G" is the universal constant, "M" is the mass of earth, "m" is the mass of the body and "r" is the distance between center of earth and the body.

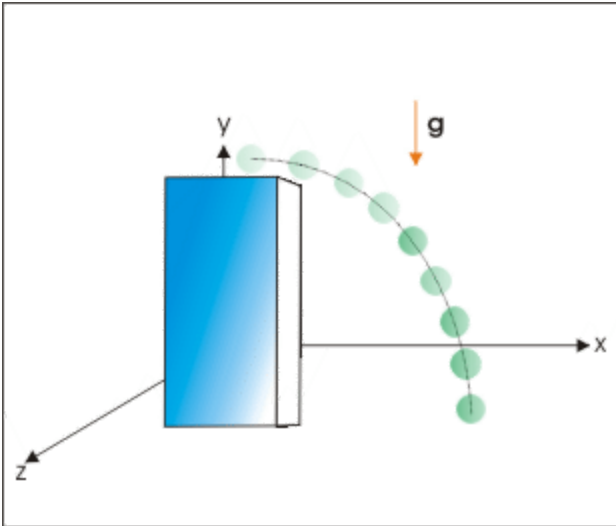
The resulting acceleration due to gravity, g, is a constant in the immediate neighborhood of the earth surface and is given by :

$$g = \frac{GM}{r^2}$$

The only variable for a given mass is "r", which changes with the position of the body. The distance "r", however, is equal to Earth's radius for all practical purposes as any difference arising from the position of body on earth can be ignored. Therefore, acceleration due to gravity can safely be considered to be constant for motions close to the surface of the earth.

Significantly acceleration due to gravity is a constant irrespective of the mass "m" of the body.

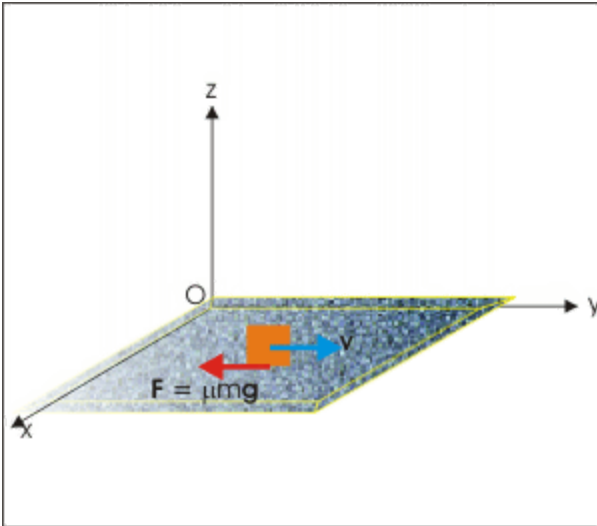
A ball kicked from a height



The ball follows a parabolic path
as it approaches ground

The figure above shows the motion of a ball kicked from the top of a tower. The ball moves under constant acceleration of gravity during its flight to the ground. The constant acceleration, therefore, assumes significance in relation to the motion that takes place under the influence of gravity. In the same manner, motion on a rough plane is acted upon by the force of friction in the direction opposite to the motion. The force of friction is a constant force for the moving body and characteristic of the surfaces in contact. As a result, the object slows down at a constant rate.

Motion influenced by friction force



The block decelerates due to friction.

Understanding constant acceleration

The constant acceleration means that the acceleration is independent of time and is equal to a constant value. The implication of a constant acceleration is discussed here as under :

1: As acceleration is same at all instants during the motion, it follows that average acceleration is equal to instantaneous acceleration during the motion. Mathematically,

$$\mathbf{a}_{\text{avg}} = \mathbf{a}$$

$$\Rightarrow \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

2: When $\Delta t = 1$ second, then

$$\mathbf{a}_{\text{avg}} = \mathbf{a} = \Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$$

This means that initial velocity , on an average, is changed by the acceleration vector after every second.

Example:**Constant acceleration**

Problem : The position of a particle, in meters, moving in the coordinate space is described by the following functions in time.

$$x = 2t^2 - 4t + 3; y = -2t; \text{ and } z = 5$$

Find the velocity and acceleration at $t = 2$ seconds from the start of motion. Also, calculate average acceleration in the first four seconds.

Solution : The component velocities in three directions are :

$$v_x = \frac{dx}{dt} = \frac{d}{dt} (2t^2 - 4t + 3) = 4t - 4$$

$$v_y = \frac{dy}{dt} = \frac{d}{dt} (-2t) = -2$$

$$v_z = \frac{dz}{dt} = \frac{d}{dt} (5) = 0$$

and the velocity is given by :

$$\begin{aligned}\mathbf{v} &= v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \\ \Rightarrow \mathbf{v} &= (4t - 4) \mathbf{i} - 2 \mathbf{j}\end{aligned}$$

Thus, velocity at $t = 2$ seconds,

$$\mathbf{v}_2 = 4 \mathbf{i} - 2 \mathbf{j}$$

Acceleration of the particle along three axes are given as :

$$a_x = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2} (2t^2 - 4t + 3) = 4$$

$$a_y = \frac{d^2y}{dt^2} = \frac{d^2}{dt^2} (-2t) = 0$$

$$a_z = \frac{d^2z}{dt^2} = \frac{d^2}{dt^2} (5) = 0$$

The resultant acceleration is given by :

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$$

$$\Rightarrow \mathbf{a} = 4\mathbf{i} \text{ m/s}^2$$

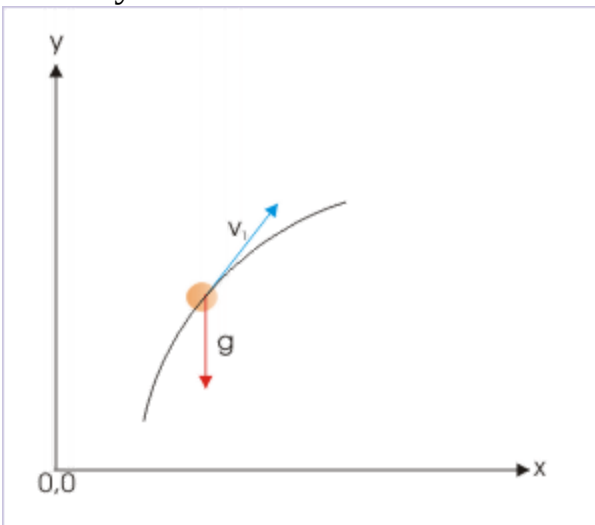
which is a constant and is independent of time. The accelerations at all time instants are, therefore, same. We know that the average and instantaneous accelerations are equal when acceleration is constant. Hence, $a_{\text{avg}} = 4\mathbf{i}$. We can check the result calculating average acceleration for the first four seconds as :

$$\mathbf{a}_{\text{avg}} = \frac{\mathbf{v}_4 - \mathbf{v}_0}{4} = \frac{12\mathbf{i} - 2\mathbf{j} + 4\mathbf{i} + 2\mathbf{j}}{4} = 4\mathbf{i} \text{ m/s}^2$$

The important fall out of a constant acceleration is that its magnitude has a constant value and its direction is fixed. A change in either of the two attributes, constituting acceleration, shall render acceleration variable. This means that acceleration is along a straight line. But does this linear nature of acceleration mean that the associated motion is also linear? Answer is no.

Reason is again the “disconnect” between acceleration and velocity. We know that magnitude and direction of acceleration are solely determined by the mass of the object and net external force applied on it. Thus, a constant acceleration only indicates that the force i.e the cause that induces change in motion is linear. It does not impose any restriction on velocity to be linear.

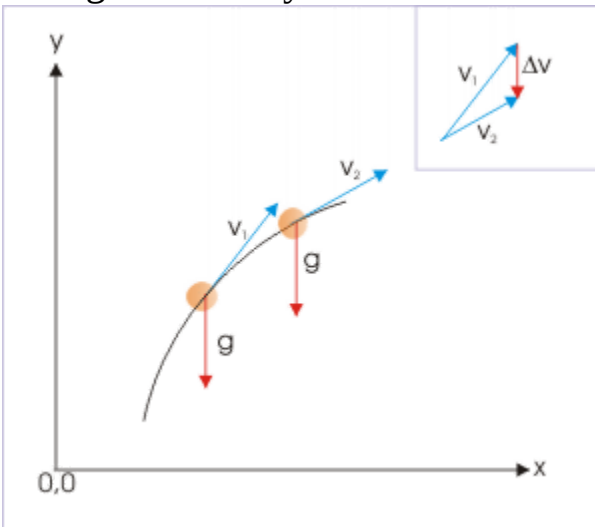
Velocity and acceleration



Force and velocity are in two
different directions

It is imperative that if the initial velocity of the object is not aligned with linear constant acceleration like in the figure above, then the immediate effect of the applied force, causing acceleration, is to change the velocity. Since acceleration is defined as the time rate of change in velocity, the resulting velocity would be so directed and its magnitude so moderated that the change in velocity (not the resulting velocity itself) is aligned in the direction of force.

Change in velocity

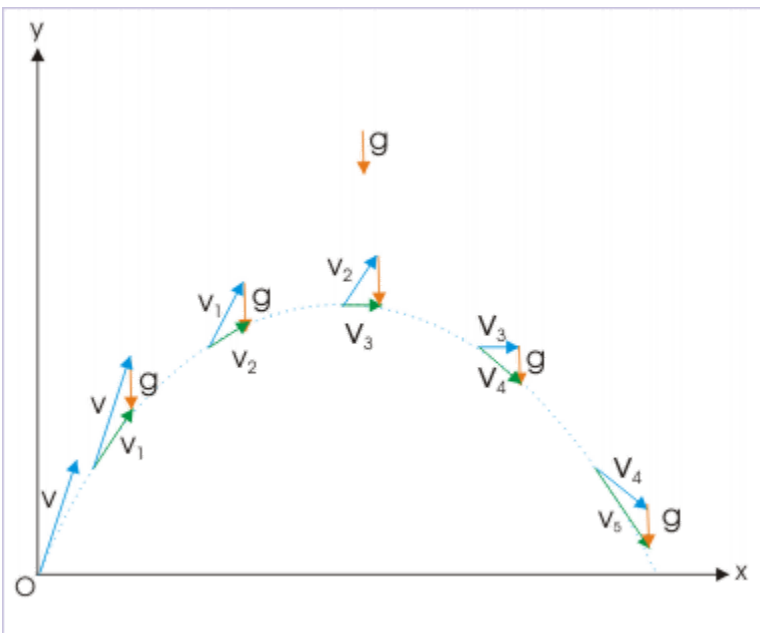


The change in velocity has the
same direction as that of
acceleration.

As the resulting velocity may not be aligned with the direction of force (acceleration), the resulting motion may not be linear either. For motion being linear, it is essential that the initial velocity and the force applied (and the resulting acceleration) are aligned along a straight line.

Examples of motions in more than one dimension with constant acceleration abound in nature. We have already seen that motion of a projectile in vertical plane has constant acceleration due to gravity, having constant magnitude, g , and fixed downward direction. If we neglect air resistance, we can assume that all non-propelled projectile motions above ground are accelerated with constant acceleration. In the nutshell, we can say that constant acceleration is unidirectional and linear, but the resulting velocity may not be linear. Let us apply this understanding to the motion of a projectile, which is essentially a motion under constant acceleration due to gravity.

Parabolic motion

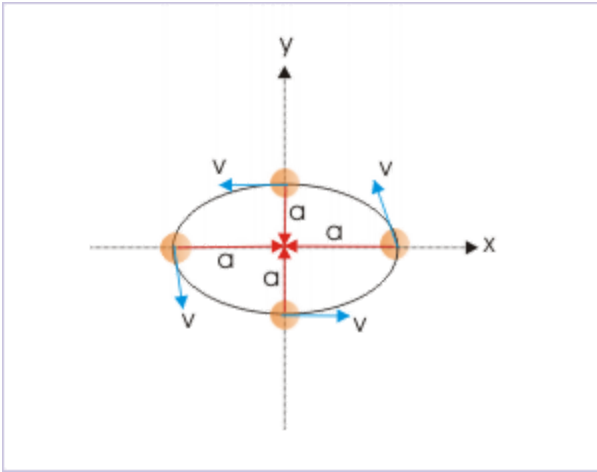


The velocity of the ball is moderated by acceleration.

In the figure, see qualitatively, how the initial velocity vector, v , is modified by the constant acceleration vector, g , at the end of successive seconds. Note that combined change in both magnitude and direction of the velocity is taking place at a constant rate and is in vertically downward direction.

In the context of constant acceleration, we must also emphasize that both magnitude and direction are constant. A constant acceleration in magnitude only is not sufficient. For constant acceleration, the direction of acceleration should also be same (i.e constant). We can have a look at a uniform circular motion in horizontal plane, which follows a horizontal circular path with a constant speed.

Uniform circular motion



The object moves along a circular path constant speed.

Notwithstanding the constant magnitude, acceleration of uniform circular motion is a variable acceleration in the horizontal plane, because direction of radial centripetal acceleration (shown with red arrow) keeps changing with time. Therefore, the acceleration of the motion keeps changing and is not independent of time as required for acceleration to be constant.

Equation of motion

The motion under constant acceleration allows us to describe accelerated motion, using simple mathematical construct. Here, we set out to arrive at these relationships in the form of equations. In these equations, “**u**” stands for initial velocity, “**v**” for final velocity, “**a**” for constant acceleration and “**t**” for time interval of motion under consideration. This is short of

convention followed by many text books and hence the convention has been retained here.

First equation

$$1: \quad \mathbf{v} = \mathbf{u} + \mathbf{a}t$$

This relation can be established in many ways. One of the fundamental ways is to think that velocity is changed by constant acceleration (vector) at the end of successive seconds. Following this logic, the velocity at the end of t successive seconds are as under :

$$\text{At the end of 1}^{\text{st}} \text{ second} \quad : \quad \mathbf{u} + \mathbf{a}$$

$$\text{At the end of 2}^{\text{nd}} \text{ second} \quad : \quad \mathbf{u} + 2\mathbf{a}$$

$$\text{At the end of 3}^{\text{rd}} \text{ second} \quad : \quad \mathbf{u} + 3\mathbf{a}$$

$$\text{At the end of 4}^{\text{th}} \text{ second} \quad : \quad \mathbf{u} + 4\mathbf{a}$$

$$\text{At the end of } t^{\text{th}} \text{ second} \quad : \quad \mathbf{u} + t\mathbf{a} = \mathbf{u} + \mathbf{a}t$$

We can also derive this equation, using the defining concept of constant acceleration. We know that :

$$\mathbf{a} = \mathbf{a}_{\text{avg}}$$

$$\Rightarrow \mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v} - \mathbf{u}}{t}$$

$$\Rightarrow \mathbf{a}t = \mathbf{v} - \mathbf{u}$$

$$\Rightarrow \mathbf{v} = \mathbf{u} + \mathbf{a}t$$

Alternatively (using calculus), we know that :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$\Rightarrow d\mathbf{v} = \mathbf{a}dt$$

Integrating on both sides, we have :

$$\Delta \mathbf{v} = \mathbf{a}\Delta t$$

$$\Rightarrow \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{a}t$$

$$\Rightarrow \mathbf{v} - \mathbf{u} = \mathbf{a}t$$

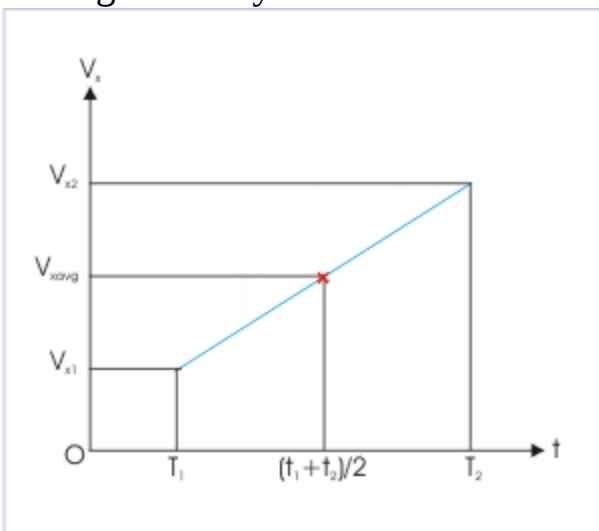
$$\Rightarrow \mathbf{v} = \mathbf{u} + \mathbf{a}t$$

Second equation

$$2: \mathbf{v}_{\text{avg}} = \frac{\mathbf{u} + \mathbf{v}}{2}$$

The equation $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ is a linear relationship. Hence, average velocity is arithmetic mean of initial and final velocities :

Average velocity



Average velocity is equal to mean velocity.

$$\Rightarrow \mathbf{v}_{\text{avg}} = \frac{\mathbf{u} + \mathbf{v}}{2}$$

Third equation

$$3: \quad \mathbf{s} = \Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

This equation is derived by combining the two expressions available for the average velocity.

$$\mathbf{v}_{\text{avg}} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t}$$

and

$$\mathbf{v}_{\text{avg}} = \frac{(\mathbf{v}_1 + \mathbf{v}_2)}{2} = \frac{(\mathbf{v} + \mathbf{u})}{2}$$

Combining two expressions of average acceleration and rearranging, we have :

$$\Rightarrow (\mathbf{r}_2 - \mathbf{r}_1) = \frac{(\mathbf{v} + \mathbf{u})t}{2}$$

Using the relation, $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ and substituting for v, we have :

$$\begin{aligned}\Rightarrow (\mathbf{r}_2 - \mathbf{r}_1) &= \frac{(\mathbf{u} + \mathbf{a}t + \mathbf{u})t}{2} \\ \Rightarrow \mathbf{s} = \Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 &= \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2\end{aligned}$$

Alternatively, we can derive this equation using calculus. Here,

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ \Rightarrow d\mathbf{r} &= \mathbf{v}dt\end{aligned}$$

Integrating between the limits on both sides,

$$\int d\mathbf{r} = \int \mathbf{v}dt$$

Now substituting v,

$$\begin{aligned}
\Rightarrow \int d\mathbf{r} &= \int (\mathbf{u} + \mathbf{a}t) dt \\
\Rightarrow \Delta\mathbf{r} &= \int \mathbf{u} dt + \int \mathbf{a}t dt \\
\Rightarrow \mathbf{s} = \Delta\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2
\end{aligned}$$

The expression for the displacement has two terms : one varies linearly ($\mathbf{u}t$) with the time and the other ($\frac{1}{2}\mathbf{a}t^2$) varies with the square of time. The first term is equal to the displacement due to non-accelerated motion i.e the displacement when the particle moves with uniform velocity, $\mathbf{u}$. The second term represents the contribution of the acceleration (change in velocity) towards displacement.

This equation is used for determining either displacement($\Delta\mathbf{r}$) or position($\mathbf{r}_1$). A common simplification, used widely, is to consider beginning of motion as the origin of coordinate system so that

$$\begin{aligned}
\mathbf{r}_1 &= 0 \\
\Rightarrow \Delta\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r} \text{ (say)}
\end{aligned}$$

In this case, both final position vector and displacement are equal. This simplification, therefore, allows us to represent both displacement and position with a single vector variable $\mathbf{r}$.

Graphical interpretation of equations of motion

The three basic equations of motion with constant acceleration, as derived above, are :

$$\begin{aligned}
1: \mathbf{v} &= \mathbf{u} + \mathbf{a}t \\
2: \mathbf{v}_{\text{avg}} &= \frac{(\mathbf{u} + \mathbf{v})}{2} \\
3: \mathbf{s} = \Delta\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2
\end{aligned}$$

These three equations completely describe motion of a point like mass, moving with constant acceleration. We need exactly five parameters to describe the motion under constant acceleration : $\mathbf{u}$, $\mathbf{v}$, $\mathbf{r}_1$, $\mathbf{r}_2$ and t .

It can be emphasized here that we can not use these equations if the acceleration is not constant. We should use basic differentiation or integration techniques for motion having variable acceleration (non-uniform acceleration). These equations serve to be a ready to use equations that avoids differentiation and integration. Further, it is evident that equations of motion are vector equations, involving vector addition. We can evaluate a motion under constant acceleration, using either graphical or algebraic method based on components.

Here, we interpret these vector equations, using graphical technique. For illustration purpose, we apply these equations to a motion of an object, which is thrown at an angle θ from the horizontal. The magnitude of acceleration is "g", which is directed vertically downward. Let acceleration vector be represented by corresponding bold faced symbol $\mathbf{g}$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities at time instants t_1 and t_2 respectively and corresponding position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$.

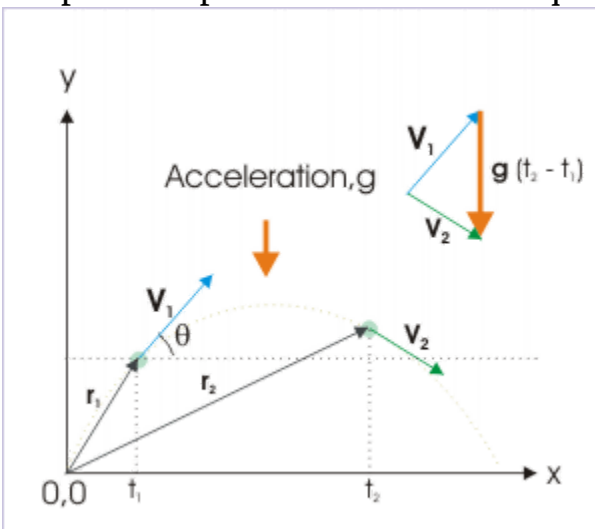
The final velocity at time instant t_2 , is given by :

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\Rightarrow \mathbf{v}_2 = \mathbf{v}_1 + \mathbf{g}(t_2 - t_1)$$

Graphically, the final velocity is obtained by modifying initial vector $\mathbf{v}_1$ by the vector $\mathbf{g}(t_2 - t_1)$.

Graphical representation of first equation



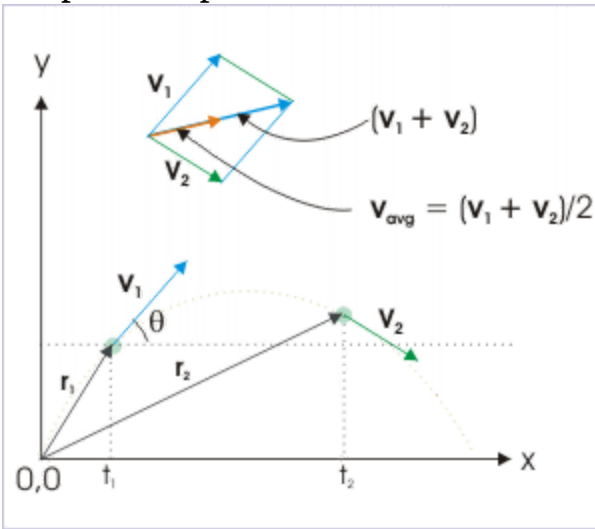
Now, we discuss graphical representation of second equation of motion.
The average velocity between two time instants or two positions is given by :

$$\mathbf{v}_{\text{avg}} = \frac{(\mathbf{u} + \mathbf{v})}{2}$$

$$\Rightarrow \mathbf{v}_{\text{avg}} = \frac{(\mathbf{v}_1 + \mathbf{v}_2)}{2}$$

The vector addition involved in the equation is graphically represented as shown in the figure. Note that average velocity is equal to half of the vector sum $\mathbf{v}_1 + \mathbf{v}_2$.

Graphical representation of second equation

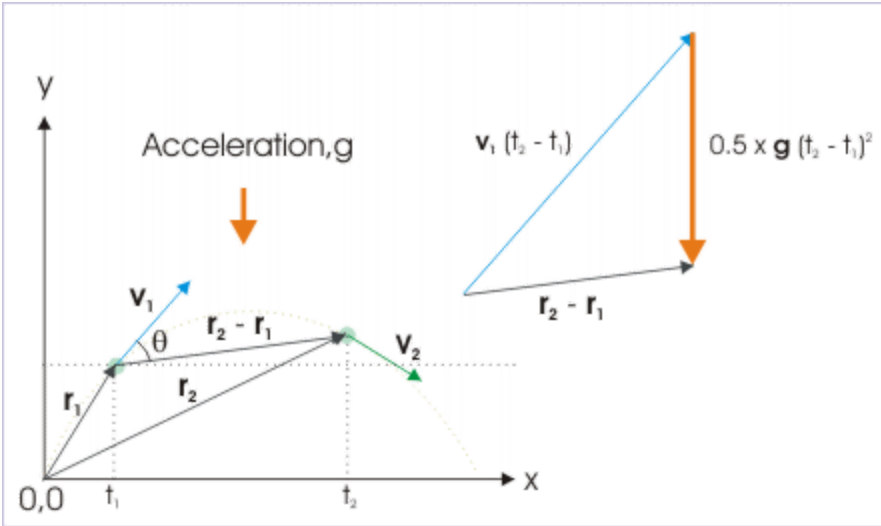


Third equation of motion provides for displacement in terms of two vector quantities - initial velocity and acceleration. The displacement, $\mathbf{s}$, is equal to addition of two vector terms :

$$\mathbf{s} = \Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\Rightarrow \mathbf{s} = \Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{v}_1(t_2 - t_1) + \frac{1}{2}\mathbf{g}(t_2 - t_1)^2$$

Graphical representation of third equation



Equations of motion in component form

The application of equations of motion graphically is tedious. In general, we use component representation that allows us to apply equations algebraically. We use equations of motion, using component forms of vector quantities involved in the equations of motion. The component form of the various vector quantities are :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\Delta\mathbf{r} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}$$

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

$$\mathbf{v} = v_{avgx}\mathbf{i} + v_{avgy}\mathbf{j} + v_{avgz}\mathbf{k}$$

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Using above relations, equations of motion are :

$$1: v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) + (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) t$$

$$2: v_{avgx}\mathbf{i} + v_{avgy}\mathbf{j} + v_{avgz}\mathbf{k} = \frac{(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) + (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})}{2}$$

$$3: \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k} = (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) t + \frac{1}{2} (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) t^2$$

Example:**Acceleration in component form**

Problem : A particle is moving with an initial velocity $(8\mathbf{i} + 2\mathbf{j}) \text{ m/s}$, having an acceleration $(0.4\mathbf{i} + 0.3\mathbf{j}) \text{ m/s}^2$. Calculate its speed after 10 seconds.

Solution : The particle has acceleration of $(0.4\mathbf{i} + 0.3\mathbf{j}) \text{ m/s}^2$, which is a constant acceleration. Its magnitude is $\sqrt{(0.4^2 + 0.3^2)} = 0.5 \text{ m/s}^2$ making an angle with x-direction. $\theta = \tan^{-1}\left(\frac{3}{4}\right)$. Time interval is 10 seconds. Thus, applying equation of motion, we have :

$$\begin{aligned}v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} &= (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) + (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) t \\ \Rightarrow \mathbf{v} &= (8\mathbf{i} + 2\mathbf{j}) + (0.4\mathbf{i} + 0.3\mathbf{j}) \times 10 = 12\mathbf{i} + 5\mathbf{j}\end{aligned}$$

Speed i.e. magnitude of velocity is :

$$\Rightarrow v = \sqrt{(12^2 + 5^2)} = 13 \text{ m/s}$$

Note: This example illustrates the basic nature of the equations of motion. If we treat them as scalar equations, we may be led to wrong answers. For example, magnitude of initial velocity i.e. speed is

$\sqrt{(8^2 + 2^2)} = 8.25 \text{ m/s}$, Whereas magnitude of acceleration is $\sqrt{(0.4^2 + 0.3^2)} = 0.5 \text{ m/s}^2$. Now, using equation of motion as scalar equation, we have :

$$v = u + at = 8.25 + 0.5 \times 10 = 13.25 \text{ m/s}$$

Equivalent scalar system of equations of motion

We have discussed earlier that a vector quantity in one dimension can be conveniently expressed in terms of an equivalent system of scalar representation. The advantage of linear motion is that we can completely do away with vector notation with an appropriate scheme of assigning plus or minus signs to the quantities involved. The equivalent scalar representation takes advantage of the fact that vectors involved in linear motion has only two possible directions. The one in the direction of chosen axis is considered positive and the other against the direction of the chosen axis is considered negative.

At the same time, the concept of component allows us to treat a motion into an equivalent system of the three rectilinear motions in the mutually perpendicular directions along the axes. The two concepts, when combined together, renders it possible to treat equations of motion in scalar terms in mutually three perpendicular directions.

Once we follow the rules of equivalent scalar representation, we can treat equations of motion as scalar equations in the direction of an axis, say x - axis, as :

$$1x: v_x = u_x + a_x t$$

$$2x: v_{avgx} = \frac{(u_x + v_x)}{2} = \frac{(x_2 - x_1)}{t}$$

$$3x: \Delta x = x_2 - x_1 = u_x t + \frac{1}{2} a_x t^2$$

We have similar set of equations in the remaining two directions. We can obtain the composite interpretation of the motion by combining the individual result in each direction. In order to grasp the method, we rework the earlier example.

Example:

Acceleration in scalar form

Problem : A particle is moving with an initial velocity $(8\mathbf{i} + 2\mathbf{j}) \text{ m/s}$, having an acceleration $(0.4\mathbf{i} + 0.3\mathbf{j}) \text{ m/s}^2$. Calculate its speed after 10 seconds.

Solution : The motion in x – direction :

$$u_x = 8 \text{ m/s} ; a_x = 0.4 \text{ m/s}^2; 10 \text{ s and,}$$

$$v_x = u_x + a_x t$$

$$\Rightarrow v_x = 8 + 0.4 \times 10 = 12 \text{ m/s}$$

The motion in y – direction :

$$u_y = 2 \text{ m/s} ; a_y = 0.3 \text{ m/s}^2; 10 \text{ s and,}$$

$$v_y = u_y + a_y t$$

$$\Rightarrow v_y = 2 + 0.3 \times 10 = 5 \text{ m/s}$$

Therefore, the velocity is :

$$\Rightarrow \mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$$

$$\Rightarrow v = \sqrt{(12^2 + 5^2)} = 13 \text{ m/s}$$

Constant acceleration (application)

Solving problems is an essential part of the understanding

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1. Though acceleration is constant and hence one – dimensional, but the resulting motion can be one, two or three dimensional – depending on the directional relation between velocity and acceleration.
2. Identify : what is given and what is required. Establish relative order between given and required attribute.
3. Use differentiation method to get a higher order attribute in the following order : displacement (position vector) $\rightarrow$ velocity $\rightarrow$ acceleration.
4. Use integration method to get a lower order attribute in the following order : acceleration $\rightarrow$ velocity $\rightarrow$ displacement (position vector).

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the motion with constant acceleration. The questions are categorized in terms of the characterizing features of the subject matter :

- Average velocity
- Differentiation and Integration method
- Components of constant acceleration
- Rectilinear motion with constant acceleration
- Equations of motion

Average velocity

Example:

Problem : A particle moves with an initial velocity “**u**” and a constant acceleration “**a**”. What is average velocity in the first “**t**” seconds?

Solution : The particle is moving with constant acceleration. Since directional relation between velocity and acceleration is not known, the motion can have any dimension. For this reason, we shall be using vector form of equation of motion. Now, the average velocity is given by :

$$\mathbf{v}_a = \frac{\Delta \mathbf{r}}{\Delta t}$$

The displacement for motion with constant acceleration is given as :

$$\Delta \mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

Thus, average velocity is :

$$\mathbf{v}_a = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{u}t + \frac{1}{2}\mathbf{a}t^2}{t} = \mathbf{u} + \frac{1}{2}\mathbf{a}t$$

Differentiation and Integration methods

Example:

Problem : A particle is moving with a velocity $2\mathbf{i} + 2t\mathbf{j}$ in m/s. Find (i) acceleration and (ii) displacement at $t = 1$ s.

Solution : Since velocity is given as a function in “**t**”, we can find acceleration by differentiating the function with respect to time.

$$\mathbf{a} = \frac{d}{dt} (2\mathbf{i} + 2t\mathbf{j}) = 2\mathbf{j}$$

Thus, acceleration is constant and is directed in y-direction. However, as velocity and acceleration vectors are not along the same direction, the motion is in two dimensions. Since acceleration is constant, we can employ equation of motion for constant acceleration in vector form,

$$\Delta \mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\Delta \mathbf{r} = (2\mathbf{i} + 2t\mathbf{j})t + \frac{1}{2} \times 2\mathbf{j} \times t^2$$

For $t = 1 \text{ s}$

$$\Delta \mathbf{r} = (2\mathbf{i} + 2 \times 1\mathbf{j}) \times 1 + \frac{1}{2} \times 2\mathbf{j} \times 1^2$$

$$\Delta \mathbf{r} = 2\mathbf{i} + 3\mathbf{j}$$

Note 1 : We should remind ourselves that we obtained displacement using equation of motion for constant acceleration. Had the acceleration been variable, then we would have used integration method to find displacement.

Note 2 : A constant acceleration means that neither its magnitude or direction is changing. Therefore, we may be tempted to think that a constant acceleration is associated with one dimensional motion. As we see in the example, this is not the case. A constant acceleration can be associated with two or three dimensional motion as well. It is the relative directions of acceleration with velocity that determines the dimension of motion – not the dimension of acceleration itself.

Example:

Problem : The coordinates of a particle (m/s) in a plane at a given time “ t ” is $2t$, t^2 . Find (i) path of motion (ii) velocity at time “ t ” and (iii) acceleration at time “ t ”.

Solution : Clearly, the position of the particle is a function of time and the particle moves in a two dimensional xy - plane. Here,

$$x = 2t$$

$$y = t^2$$

In order to find the relation between “ x ” and “ y ”, we substitute “ t ” from the first equation in to second as :

$$y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$$

$$\Rightarrow x^2 = 4y$$

Hence, path of motion is parabolic. Now, the position vector is :

$$\mathbf{r} = 2t\mathbf{i} + t^2\mathbf{j}$$

Differentiating with respect to time, the velocity of the particle is :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + 2t\mathbf{j} \text{ m/s}$$

Further differentiating with respect to time, the acceleration of the particle is :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \text{ m/s}^2$$

Components of acceleration

Example:

Problem : At a certain instant, the components of velocity and acceleration are given as :

$$v_x = 4 \text{ m/s}; \quad v_y = 3 \text{ m/s}; \quad a_x = 2 \text{ m/s}^2; \quad a_y = 1 \text{ m/s}^2.$$

What is the rate of change of speed?

Solution : Here, phrasing of question is important. We are required to find the rate of change of speed – not the rate of change of velocity or magnitude of rate of change of velocity. Let us have a look at the subtle differences in the meaning here :

1: The rate of change of velocity

The rate of change of velocity is equal to acceleration. For the given two dimensional motion,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} (a_x \mathbf{i} + a_y \mathbf{j})$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{i} + 1\mathbf{j}$$

2: The magnitude of rate of change of velocity

The magnitude of rate of change of velocity is equal to magnitude of acceleration. For the given two dimensional motion,

$$|\mathbf{a}| = \left| \frac{d\mathbf{v}}{dt} \right| = \sqrt{(2^2 + 1^2)} = \sqrt{5} \text{ m/s}^2$$

3: The rate of change of speed

The rate of change of speed (dv/dt) is not equal to the magnitude of acceleration, which is equal to the absolute value of the rate of change of velocity. It is so because speed is devoid of direction, whereas acceleration consists of both magnitude and direction.

Let “v” be the instantaneous speed, which is given in terms of its component as :

$$v^2 = v_x^2 + v_y^2$$

We need to find rate of change of speed i.e dv/dt , using the values given in the question. Therefore, we need to differentiate speed with respect to time,

$$\Rightarrow 2v \frac{dv}{dt} = 2v_x \left(\frac{dv_x}{dt} \right) + 2v_y \left(\frac{dv_y}{dt} \right)$$

If we ponder a bit, then we would realize that when we deal with component speed or magnitude of component velocity then we are essentially dealing with unidirectional motion. No change in direction is possible as components are aligned to a fixed axis. As such, equating rate of change in speed with the magnitude of acceleration in component direction is valid. Now, proceeding ahead,

$$\Rightarrow \frac{dv}{dt} = \frac{v_x a_x + v_y a_y}{v}$$

$$\Rightarrow \frac{dv}{dt} = \frac{v_x a_x + v_y a_y}{\sqrt{(v_x^2 + v_y^2)}}$$

Putting values, we have :

$$\Rightarrow \frac{dv}{dt} = \frac{4 \times 2 + 3 \times 1}{\sqrt{(4^2 + 3^2)}}$$

$$\Rightarrow \frac{dv}{dt} = \frac{11}{5} = 2.2 \text{ m/s}$$

Note: This is an important question as it brings out differences in interpretation of familiar terms. In order to emphasize the difference, we summarize the discussion as hereunder :

i: In general (i.e two or three dimensions),

$$\Rightarrow \frac{dv}{dt} \neq \left| \frac{d\mathbf{v}}{dt} \right|$$

$$\Rightarrow \frac{dv}{dt} \neq |\mathbf{a}|$$

$$\Rightarrow \frac{dv}{dt} \neq a$$

ii: In the case of one dimensional motion, the inequality as above disappears.

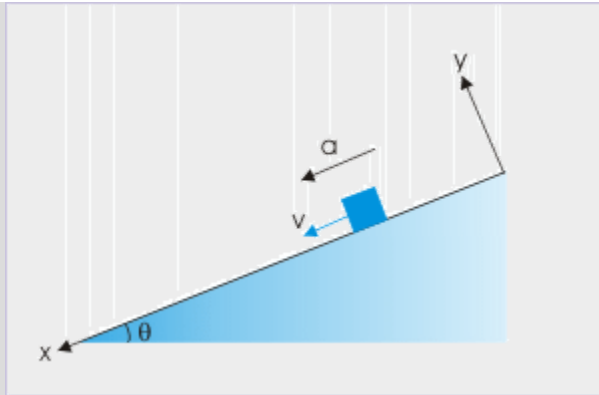
$$\Rightarrow \frac{dv}{dt} = a$$

Rectilinear motion with constant acceleration

Example:

Problem : A block is released from rest on a smooth inclined plane. If S_n denotes the distance traveled by it from $t = n - 1$ second to $t = n$ seconds, then find the ratio :

Motion along an incline



The block moves with a constant acceleration.

$$A = \frac{S_n}{S_{n+1}}$$

Solution : It must be noted that the description of linear motion is governed by the equations of motion whether particle moves on a horizontal surface (one dimensional description) or on an inclined surface (two dimensional description). Let us orient our coordinates so that the motion can be treated as one dimensional unidirectional motion. This allows us to use equations of motion in scalar form,

$$\Rightarrow S_n = u + \frac{a}{2} (2n - 1)$$

Here, $u = 0$, thus

$$\Rightarrow S_n = \frac{a}{2} (2n - 1)$$

Following the description of term S_n as given by the question, we can define S_{n+1} as the linear distance from $t = n$ second to $t = n + 1$ seconds. Thus, substituting “ n ” by “ $n+1$ ” in the formulae, we have :

$$\Rightarrow S_{n+1} = \frac{a}{2} (2n + 2 - 1)$$

$$\Rightarrow S_{n+1} = \frac{a}{2} (2n + 1)$$

The required ratio is :

$$\Rightarrow A = \frac{S_n}{S_{n+1}} = \frac{(2n-1)}{(2n+1)}$$

Equations of motion

Example:

Problem : A force of 2 N is applied on a particle of mass 1 kg, which is moving with a velocity 4 m/s in a perpendicular direction. If the force is applied all through the motion, then find displacement and velocity after 2 seconds.

Solution : It is a two dimensional motion, but having a constant acceleration. Notably, velocity and accelerations are not in the same direction. In order to find the displacement at the end of 2 seconds, we shall use algebraic method. Let the direction of initial velocity and acceleration be along “x” and “y” coordinates (they are perpendicular to each other). Also, let “A” be the initial position and “B” be the final position of the particle. The displacement between A (position at time $t = 0$) and B (position at time $t = 2$ s) is given as :

$$\mathbf{AB} = \mathbf{ut} + \frac{1}{2}\mathbf{at}^2$$

For time $t = 2$ s,

$$\mathbf{AB}_{t=2} = 2\mathbf{u} + \frac{1}{2}\mathbf{a} \times 2^2 = 2(\mathbf{u} + \mathbf{a})$$

This is a vector equation involving sum of two vectors at right angles. According to question,

$$u = 4 \text{ m/s}; a = \frac{F}{m} = \frac{2}{1} = 2 \text{ m/s}^2$$

Since $\mathbf{u}$ and $\mathbf{a}$ perpendicular to each other, the magnitude of the vector sum ($\mathbf{u} + \mathbf{a}$) is :

$$|\mathbf{u} + \mathbf{a}| = \sqrt{(u^2 + a^2)} = \sqrt{(4^2 + 2^2)} = 2\sqrt{5}$$

Hence, magnitude of displacement is :

$$AB_{t=2} = 2|\mathbf{u} + \mathbf{a}| = 2 \times 2\sqrt{5} = 4\sqrt{5} \text{ m}$$

Let the displacement vector makes an angle “ θ ” with the direction of initial velocity.

$$\tan \theta = \frac{a}{u} = \frac{2}{4} = \frac{1}{2}$$

Let the direction of initial velocity and acceleration be along “x” and “y” coordinates (they are perpendicular to each other). Then,

$$u = 4\mathbf{i}$$

$$a = 2\mathbf{j}$$

Using equation of motion for constant acceleration, the final velocity is :

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\Rightarrow \mathbf{v} = 4\mathbf{i} + 2\mathbf{j} \times 1 = 4(\mathbf{i} + \mathbf{j})$$

The magnitude of velocity is :

$$v = |\mathbf{v}| = 4\sqrt{(1^2 + 1^2)} = 4\sqrt{2} \text{ m}$$

Let the final velocity vector makes an angle “ θ ” with the direction of initial velocity.

$$\tan \theta = \frac{1}{1} = 1$$

$$\theta = 45^\circ$$

One dimensional motion with constant acceleration

Free falling bodies under gravity represents typical case of motion in one dimension with constant acceleration. A body projected vertically upwards is also a case of constant acceleration in one dimension, but with the difference that body undergoes reversal of direction as well after reaching the maximum height. Yet another set of examples of constant accelerations may include object sliding on an incline plane, motion of an object impeded by rough surfaces and many other motions under the influence of gravitational and frictional forces.

The defining differential equations of velocity and acceleration involve only one position variable (say x). In the case of motion under constant acceleration, the differential equation defining acceleration must evaluate to a constant value.

$$\mathbf{v} = \frac{dx}{dt} \mathbf{i}$$

and

$$\mathbf{a} = \frac{d^2x}{dt^2} \mathbf{i} = k\mathbf{i}$$

where k is a positive or negative constant.

The corresponding scalar form of the defining equations of velocity and acceleration for one dimensional motion with constant acceleration are :

$$v = \frac{dx}{dt}$$

and

$$a = \frac{d^2x}{dt^2} = k$$

Example:
Constant acceleration

Problem : The position “x” in meter of a particle moving in one dimension is described by the equation :

$$t = \sqrt{x} + 1$$

where “t” is in second.

1. Find the time when velocity is zero.
2. Does the velocity changes its direction?
3. Locate position of the particle in the successive seconds for first 3 seconds.
4. Find the displacement of the particle in first three seconds.
5. Find the distance of the particle in first three seconds.
6. Find the displacement of the particle when the velocity becomes zero.
7. Determine, whether the particle is under constant or variable force.

Solution : Velocity is equal to the first differential of the position with respect to time, while acceleration is equal to the second differential of the position with respect to time. The given equation, however, expresses time, t, in terms of position, x. Hence, we need to obtain expression of position as a function in time.

$$\begin{aligned} t &= \sqrt{x} + 1 \\ \Rightarrow \sqrt{x} &= t - 1 \end{aligned}$$

Squaring both sides, we have :

$$\Rightarrow x = t^2 - 2t + 1$$

This is the desired expression to work upon. Now, taking first differential w.r.t time, we have :

$$v = \frac{dx}{dt} = \frac{d}{dt} (t^2 - 2t + 1) = 2t - 2$$

1: When $v = 0$, we have $v = 2t - 2 = 0$

$$\Rightarrow t = 1s$$

2. Velocity is expressed in terms of time as :

$$v = 2t - 2$$

It is clear from the expression that velocity is negative for $t < 1$ second, while positive for $t > 1$. As such velocity changes its direction during motion.

3: Positions of the particle at successive seconds for first three seconds are :

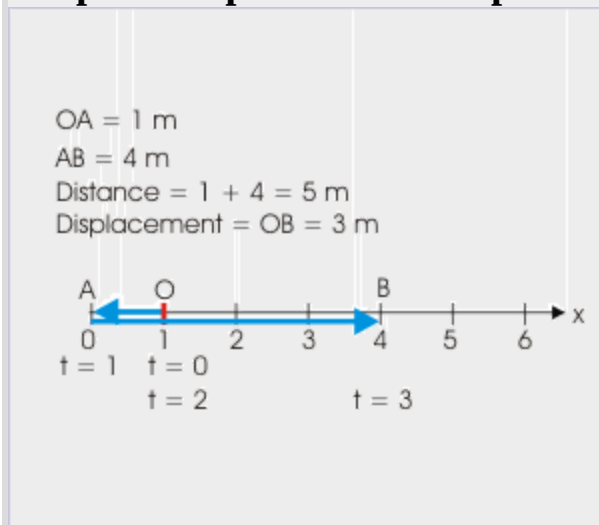
$$t = 0; x = t^2 - 2t + 1 = 0 - 0 + 1 = 1 \text{ m}$$

$$t = 1; x = t^2 - 2t + 1 = 1 - 2 + 1 = 0 \text{ m}$$

$$t = 2; x = t^2 - 2t + 1 = 4 - 4 + 1 = 1 \text{ m}$$

$$t = 3; x = t^2 - 2t + 1 = 9 - 6 + 1 = 4 \text{ m}$$

Graphical representation of position



4: Positions of the particle at $t = 0$ and $t = 3$ s are 1 m and 4 m from the origin.

Hence, displacement in first three seconds is $4 - 1 = 3$ m

5: The particle moves from the start position, $x = 1$ m, in the negative direction for 1 second. At $t = 1$, the particle comes to rest. For the time interval from 1 to 3 seconds, the particle moves in the positive direction. Distance in the interval $t = 0$ to 1 s is :

$$s_1 = 1 - 0 = 1 \text{ m}$$

$$s_2 = 4 - 0 = 4 \text{ m}$$

Total distance is $1 + 4 = 5 \text{ m}$.

6: Velocity is zero, when $t = 1 \text{ s}$. In this period, displacement is 1 m .

7: In order to determine the nature of force on the particle, we first determine the acceleration as :

$$a = \frac{dv}{dt} = \frac{d}{dt}(2t - 2) = 2 \text{ m/s}^2$$

Acceleration of the motion is constant, independent of time. Hence, force on the particle is also constant during the motion.

Equation of motion for one dimensional motion with constant acceleration

The equation of motions in one dimension for constant acceleration is obtained from the equations of motion established for the general case i.e. for the three dimensional motion. In one dimension, the equation of motion is simplified ($\mathbf{r}$ is replaced by x or y or z with corresponding unit vector).

The three basic equations of motions are (say in x - direction) :

$$1: \mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$2: \mathbf{v}_{\text{avg}} = \frac{(\mathbf{u} + \mathbf{v})}{2}$$

$$3: \Delta x \mathbf{i} = \mathbf{u}t + \frac{1}{2} \mathbf{a}t^2$$

Significantly, we can treat vector equivalently as scalars with appropriate sign. Following is the construct used for this purpose :

Sign convention

1. Assign an axis along the motion. Treat direction of axis as positive.
2. Assign the origin with the start of motion or start of observation. It is, however, a matter of convenience and is not a requirement of the construct.

3. Use all quantities describing motion in the direction of axis as positive.
4. Use all quantities describing motion in the opposite direction of axis as negative.

Once, we follow the rules as above, we can treat equations of motion as scalar equations as :

$$1: v = u + at$$

$$2: v_{\text{avg}} = \frac{(u+v)}{2}$$

$$3: \Delta x = ut + \frac{1}{2}at^2$$

Example:

Constant acceleration

Problem : A car moving with constant acceleration covers two successive kilometers in 20 s and 30 s respectively. Find the acceleration of the car.

Solution : Let "u" and "a" be the initial velocity and acceleration of the car. Applying third equation of motion for first kilometer, we have :

$$\begin{aligned} 1000 &= u \times 20 + \frac{1}{2}a20^2 = 20u + 200a \\ \Rightarrow 100 &= 2u + 20a \end{aligned}$$

At the end of second kilometer, total displacement is 2 kilometer (=2000 m) and total time is 20 + 30 = 50 s. Again applying third equation of motion,

$$\begin{aligned} 2000 &= u \times 50 + \frac{1}{2}a50^2 = 50u + 2500a \\ \Rightarrow 200 &= 5u + 125a \end{aligned}$$

Solving two equations,

$$a = -2.86 \text{ m/s}^2$$

Note that acceleration is negative to the positive direction (direction of velocity) and as such it is termed “deceleration”. This interpretation is

valid as we observe that the car covers second kilometer in longer time than for the first kilometer, which means that the car has slowed down.

It is important to emphasize here that mere negative value of acceleration does not mean it to be deceleration. The deciding criterion for deceleration is that acceleration should be opposite to the direction of velocity.

Motion under gravity

We have observed that when a feather and an iron ball are released from a height, they reach earth surface with different velocity and at different times. These objects are under the action of different forces like gravity, friction, wind and buoyancy force. In case forces other than gravity are absent like in vacuum, the bodies are only acted by the gravitational pull towards earth. In such situation, acceleration due to gravity, denoted by g , is the only acceleration.

The acceleration due to gravity near the earth surface is nearly constant and equal to 9.8 m/s^2 . Value of ' g ' is taken as 10 m/s^2 as an approximation to facilitate ease of calculation.

When only acceleration due to gravity is considered, neglecting other forces, each of the bodies (feather and iron ball) starting from rest is accelerated at the same rate. Velocity of each body increases by 9.8 m/s at the end of every second. As such, the feather and the iron ball reach the surface at the same time and at the same velocity.

Additional equations of motion

A close scrutiny of three equations of motion derived so far reveals that they relate specific quantities, which define the motion. There is possibility that we may encounter problems where inputs are not provided in the manner required by equation of motion.

For example, a problem involving calculation of displacement may identify initial velocity, final velocity and acceleration as input. Now, the

equation of motion for displacement is expressed in terms of initial velocity, time and acceleration. Evidently, there is a mis-match between what is given and what is required. We can, no doubt, find out time from the set of given inputs, using equation for velocity and then we can solve equation for the displacement. But what if we develop a relation-ship which relates the quantities as given in the input set! This would certainly help.

From first equation :

$$\begin{aligned}v &= u + at \\ \Rightarrow v - u &= at \\ \Rightarrow t &= \frac{(v-u)}{a} = at\end{aligned}$$

From second equation :

$$\begin{aligned}\frac{(u+v)}{2} &= \frac{(x_2-x_1)}{t} \\ \Rightarrow (u + v) &= \frac{2(x_2-x_1)}{t}\end{aligned}$$

Eliminating 't',

$$\begin{aligned}\Rightarrow (u + v) &= \frac{2a(x_2-x_1)}{(v-u)} \\ \Rightarrow v^2 - u^2 &= 2a(x_2 - x_1)\end{aligned}$$

Equation:

$$\Rightarrow v^2 - u^2 = 2a(x_2 - x_1)$$

This equation relates initial velocity, final velocity, acceleration and displacement.

Also, we observe that equation for displacement calculates displacement when initial velocity, acceleration and time are given. If final velocity - instead of initial velocity - is given, then displacement can be obtained with slight modification.

$$\Delta x = x_2 - x_1 = ut = \frac{1}{2}at^2$$

Using $v = u + at$,

$$\Rightarrow \Delta x = x_2 - x_1 = (v - at)t + \frac{1}{2}at^2$$

Equation:

$$\Rightarrow \Delta x = x_2 - x_1 = vt - \frac{1}{2}at^2$$

Example:

Constant acceleration in one dimensional motion

Problem : A train traveling on a straight track moves with a speed of 20 m/s. Brake is applied uniformly such that its speed is reduced to 10 m/s, while covering a distance of 200 m. With the same rate of deceleration, how far will the train go before coming to rest.

Solution : To know the distance before train stops, we need to know the deceleration. We can find out deceleration from the first set of data. Here, $u = 20 \text{ m/s}$; $v = 10 \text{ m/s}$; $x = 200 \text{ m}$; $a = ?$ An inspection of above data reveals that none of the first three equations of motion fits the requirement in hand, while the additional form of equation $v^2 - u^2 = 2ax$ serves the purpose :

$$a = \frac{v^2 - u^2}{2x} = \frac{10^2 - 20^2}{2 \times 200} = -\frac{3}{4} \text{ m/s}^2$$

For the train to stop, we have

$$u = 20 \text{ m/s}; v = 0 \text{ m/s}; a = -\frac{3}{4} \text{ m/s}^2; x = ?$$

and,

$$x = \frac{v^2 - u^2}{2a} = \frac{0^2 - 20^2}{2 \times (-\frac{3}{4})} = 266.67 \text{ m}$$

Displacement in a particular second

The displacement in a second is obtained by subtracting two displacements in successive seconds. We calculate displacements in n^{th} second and $(n-1)^{\text{th}}$ seconds. The difference of two displacements is the displacement in the n^{th} second. Now, the displacements at the end of n and $(n-1)$ seconds as measured from origin are given by :

$$x_n = un + \frac{1}{2}an^2$$
$$x_{n-1} = u(n-1) + \frac{1}{2}a(n-1)^2$$

The displacement in the n^{th} second, therefore, is :

$$s_n = x_n - x_{n-1} = un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n-1)^2$$
$$\Rightarrow s_n = x_n - x_{n-1} = u + \frac{1}{2}a(2n-1)$$

Equation:

$$\Rightarrow s_n = x_n - x_{n-1} = u + \frac{a}{2}(2n-1)$$

Example:

Problem : The equation of motion for displacement in n^{th} second is given by :

$$x_n = u + \frac{a}{2}(2n-1)$$

This equation is dimensionally incompatible, yet correct. Explain.

Solution : Since “ n ” is a number, the dimension of the terms of the equation is indicated as :

$$\begin{aligned}
[x_n] &= [\text{Displacement}] = [L] \\
[u] &= [\text{velocity}] = [LT^{-1}] \\
\left[\frac{a}{2}(2n-1)\right] &= [\text{acceleration}] = [LT^{-2}]
\end{aligned}$$

Clearly, dimensions of the terms are not same and hence equation is apparently incompatible in terms of dimensions.

In order to resolve this apparent incompatibility, we need to show that each term of right hand side of the equation has dimension that of displacement i.e. length. Now, we know that the relation is derived for a displacement for time equal to “1” second. As multiplication of “1” or “1²” with any term is not visible, the apparent discrepancy has appeared in otherwise correct equation. We can, therefore, re-write the equation with explicit time as :

$$\begin{aligned}
x_n &= u \times 1 + \frac{a}{2}(2n-1) \times 1^2 \\
\Rightarrow x_n &= u \times t + \frac{a}{2}(2n-1) \times t^2
\end{aligned}$$

Now, the dimensions of the first and second terms on the right side are :

$$\begin{aligned}
[u \times t] &= [LT^{-1}T] = [L] \\
\left[\frac{a}{2}(2n-1) \times t^2\right] &= [LT^{-2}T^2] = [L]
\end{aligned}$$

Thus, we see that the equation is implicitly correct in terms of dimensions.

Exercise:

Problem:

If a particle, moving in straight line, covers distances “x”, “y” and “z” in p^{th} , q^{th} and r^{th} seconds respectively, then prove that :

$$(p-q)z + (r-p)y + (q-r)x = 0$$

Solution:

This is a motion with constant acceleration in one dimension. Let the particle moves with a constant acceleration, “a”. Now, the distances in particular seconds as given in the question are :

$$x = u + \frac{a}{2}(2p - 1)$$

$$y = u + \frac{a}{2}(2q - 1)$$

$$z = u + \frac{a}{2}(2r - 1)$$

Subtracting, second from first equation,

$$x - y = \frac{a}{2}(2p - 1 - 2q + 1) = a(p - q)$$

$$\Rightarrow (p - q) = \frac{(x-y)}{a}$$

$$\Rightarrow (p - q)z = \frac{(x-y)z}{a}$$

Similarly,

$$\Rightarrow (r - p)y = \frac{(z-x)y}{a}$$

$$\Rightarrow (q - r)x = \frac{(y-z)x}{a}$$

Adding three equations,

$$(p - q)z + (r - p)y + (q - r)x = \frac{1}{a}(xy - yz + yz - xy + xy - xz)$$

$$\Rightarrow (p - q)z + (r - p)y + (q - r)x = 0$$

Average acceleration

Average acceleration is ratio of change in velocity and time. This is an useful concept where acceleration is not constant throughout the motion. There may be motion, which has constant but different values of acceleration in different segments of motion. Our job is to find an equivalent constant acceleration, which may be used to determine attributes

for the whole of motion. Clearly, the single value equivalent acceleration should be such that it yields same value of displacement and velocity for the entire motion. This is the underlying principle for determining equivalent or average acceleration for the motion.

Example:

Problem : A particle starting with velocity “u” covers two equal distances in a straight line with accelerations a_1 and a_2 . What is the equivalent acceleration for the complete motion?

Solution : The equivalent acceleration for the complete motion should yield same value for the attributes of the motion at the end of the journey. For example, the final velocity at the end of the journey with the equivalent acceleration should be same as the one calculated with individual accelerations.

Here initial velocity is "u". The velocity at the end of half of the distance (say “x”) is :

$$v_1^2 = u^2 + 2a_1x$$

Clearly, v_1 is the initial velocity for the second leg of motion,

$$v_2^2 = v_1^2 + 2a_2x$$

Adding above two equations, we have :

$$v_2^2 = u^2 + 2(a_1 + a_2)x$$

This is the square of velocity at the end of journey. Now, let “a” be the equivalent acceleration, then applying equation of motion for the whole distance (2x),

$$v_2^2 = u^2 + 2a(2x)$$

Comparing equations,

$$a = \frac{a_1 + a_2}{2}$$

Interpretation of equations of motion

One dimensional motion facilitates simplified paradigm for interpreting equations of motion. Description of motion in one dimension involves mostly the issue of “magnitude” and only one aspect of direction. The only possible issue of direction here is that the body undergoing motion in one dimension may reverse its direction during the course of motion. This means that the body may either keep moving in the direction of initial velocity or may start moving in the opposite direction of the initial velocity at certain point of time during the motion. This depends on the relative direction of initial velocity and acceleration. Thus, there are two paradigms :

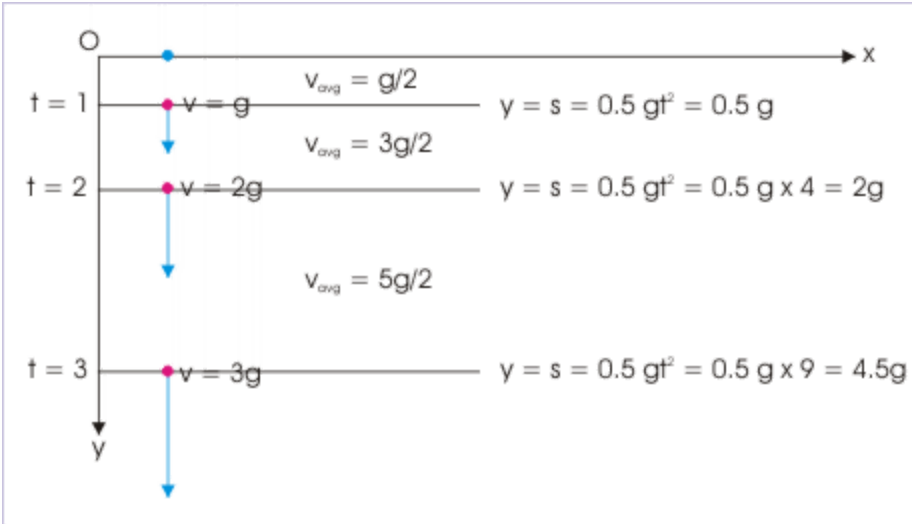
1. Constant force is applied in the direction of initial velocity.
2. Constant force is applied in the opposite direction of initial velocity.

Irrespective of the above possibilities, one fundamental attribute of motion in one dimension is that all parameters defining motion i.e initial velocity, final velocity and acceleration act along a straight line.

Constant acceleration (force) is applied in the direction of velocity

The magnitude of velocity increases by the magnitude of acceleration at the end of every second (unit time interval). In this case, final velocity at any time instant is greater than velocity at an earlier instant. The motion is not only in one dimension i.e. linear , but also unidirectional. Take the example of a ball released (initial velocity is zero) at a certain height ‘h’ from the surface. The velocity of the ball increases by the magnitude of ‘g’ at the end of every second. If the body has traveled for 3 seconds, then the velocity after 3 seconds is $3g$ ($v = 0 + 3 \times g = 3g \text{ m/s}$).

Attributes of motion



Attributes of motion as the ball falls under gravity

In this case, all parameters defining motion i.e initial velocity, final velocity and acceleration not only act along a straight line, but also in the same direction. As a consequence, displacement is always increasing during the motion like distance. This fact results in one of the interesting aspect of the motion that magnitude of displacement is equal to distance. For this reason, average speed is also equal to the magnitude of average velocity.

$$s = |x|$$

and

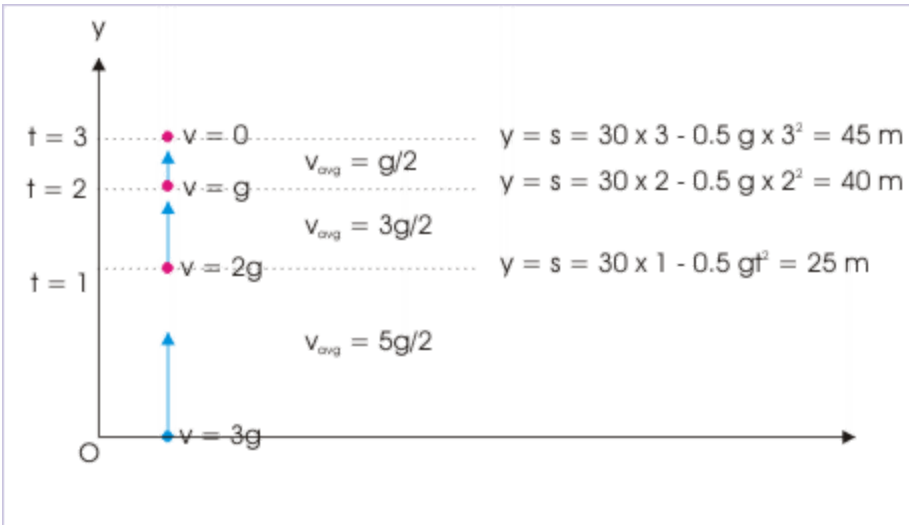
$$\frac{\Delta s}{\Delta t} = \left| \frac{\Delta x}{\Delta t} \right|$$

Constant acceleration (force) is applied in the opposite direction of velocity

The magnitude of velocity decreases by the magnitude of acceleration at the end of every second (unit time interval). In this case, final velocity at any time instant is either less than velocity at an earlier instant or has reversed

its direction. The motion is in one dimension i.e. linear, but may be unidirectional or bidirectional. Take the example of a ball thrown (initial velocity is ,say, 30 m/s) vertically from the surface. The velocity of the ball decreases by the magnitude of 'g' at the end of every second. If the body has traveled for 3 seconds, then the velocity after 3 seconds is $30 - 3g = 0$ (assume $g = 10 \text{ m/s}^2$).

Attributes of motion



Attributes of motion as the ball moves up against gravity

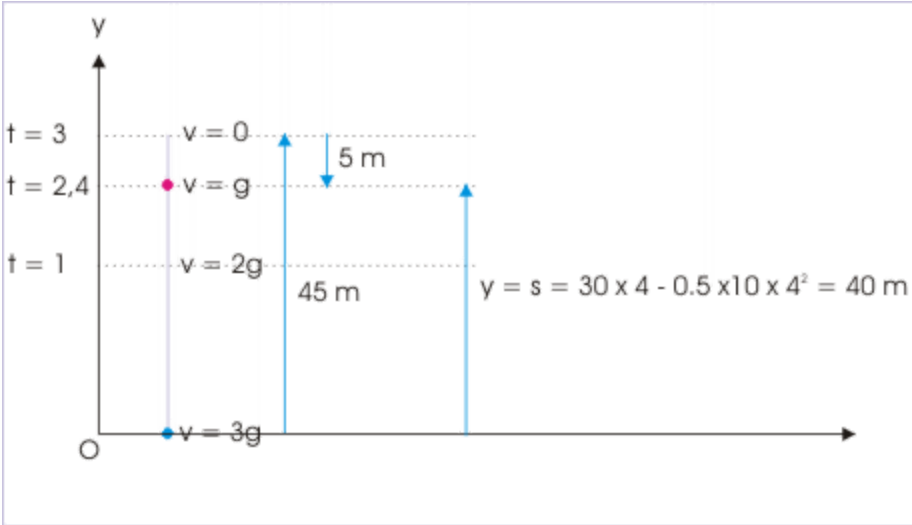
During upward motion, velocity and acceleration due to gravity are in opposite direction. As a result, velocity decreases till it achieves the terminal velocity of zero at the end of 3rd second. Note that displacement during the motion is increasing till the ball reaches the maximum height.

At the maximum height, the velocity of the ball is zero and is under the action of force due to gravity as always during the motion. As such, the ball begins moving in downward direction with the acceleration due to gravity. The directions of velocity and acceleration, in this part of motion, are same. Note that displacement with respect to point of projection is decreasing.

In the overall analysis of motion when initial velocity is against acceleration, parameters defining motion i.e initial velocity, final velocity

and acceleration act along a straight line, but in different directions. As a consequence, displacement may either be increasing or decreasing during the motion. This means that magnitude of displacement may not be equal to distance. For example, consider the motion of ball from the point of projection, A, to maximum height, B, to point, C, at the end of 4 seconds. The displacement is 40 m, while distance is $45 + 5 = 50$ m as shown in the figure below.

Attributes of motion



For this reason, average speed is not always equal to the magnitude of average velocity.

$$s \neq |x|$$

and

$$\frac{\Delta s}{\Delta t} \neq \left| \frac{\Delta x}{\Delta t} \right|$$

Exercises

Exercise:

Problem:

Two cyclists start off a race with initial velocities 2 m/s and 4 m/s respectively. Their linear accelerations are 2 and 1 m/s^2 respectively. If they reach the finish line simultaneously, then what is the length of the track?

Solution:

This is a case of one dimensional motion with constant acceleration. Since both cyclists cross the finish line simultaneously, they cover same displacement in equal times. Hence,

$$x_1 = x_2$$

$$u_1 t + \frac{1}{2} a_1 t^2 = u_2 t + \frac{1}{2} a_2 t^2$$

Putting values as given in the question, we have :

$$2t + \frac{1}{2} \times 2 \times t^2 = 4t + \frac{1}{2} \times 1 \times t^2$$

$$\Rightarrow t^2 - 4t = 0$$

$$\Rightarrow t = 0 \text{ s}, 4 \text{ s}$$

Neglecting zero value,

$$\Rightarrow t = 4 \text{ s}$$

The linear distance covered by the cyclist is obtained by evaluating the equation of displacement of any of the cyclists as :

$$x = 2t + \frac{1}{2} \times 2 \times t^2 = 2 \times 4 + \frac{1}{2} \times 2 \times 4^2 = 24 \text{ m}$$

Exercise:

Problem:

Two cars are flagged off from the starting point. They move with accelerations a_1 and a_2 respectively. The car “A” takes time “t” less than car “B” to reach the end point and passes the end point with a difference of speed, “v”, with respect to car “B”. Find the ratio v/t.

Solution:

Both cars start from rest. They move with different accelerations and hence take different times to reach equal distance, say t_1 and t_2 for cars A and B respectively. Their final speeds at the end points are also different, say v_1 and v_2 for cars A and B respectively. According to the question, the difference of time is “t”, whereas difference of speeds is “v”.

As car A is faster, it takes lesser time. Here, $t_2 > t_1$. The difference of time, “t”, is :

$$t = t_2 - t_1$$

From equation of motion,

$$\begin{aligned} x &= \frac{1}{2} a_1 t_1^2 \\ \Rightarrow t_1 &= \sqrt{\left(\frac{2x}{a_1} \right)} \end{aligned}$$

Similarly,

$$\Rightarrow t_2 = \sqrt{\left(\frac{2x}{a_2} \right)}$$

Hence,

$$\Rightarrow t = t_2 - t_1 = \sqrt{\left(\frac{2x}{a_2} \right)} - \sqrt{\left(\frac{2x}{a_1} \right)}$$

Car A is faster. Hence, $v_1 > v_2$. The difference of time, “v”, is :

$$\Rightarrow v = v_1 - v_2$$

From equation of motion,

$$\Rightarrow v_1^2 = 2a_1x$$

$$\Rightarrow v_1 = \sqrt{(2a_1x)}$$

Similarly,

$$\Rightarrow v_2 = \sqrt{(2a_2x)}$$

Hence,

$$v = v_1 - v_2 = \sqrt{(2a_1x)} - \sqrt{(2a_2x)}$$

The required ratio, therefore, is :

$$\frac{v}{t} = \frac{\sqrt{(2a_1x)} - \sqrt{(2a_2x)}}{\sqrt{\left(\frac{2x}{a_2}\right)} - \sqrt{\left(\frac{2x}{a_1}\right)}} = \frac{\{\sqrt{(2a_1)} - \sqrt{(2a_2)}\}\sqrt{a_1}\sqrt{a_2}}{\{\sqrt{(2a_1)} - \sqrt{(2a_2)}\}}$$

$$\frac{v}{t} = \sqrt{(a_1a_2)}$$

Exercise:

Problem:

Two particles start to move from same position. One moves with constant linear velocity, “v”; whereas the other, starting from rest, moves with constant acceleration, “a”. Before the second catches up with the first, what is maximum separation between two?.

Solution:

One of the particles begins with a constant velocity, “v” and continues to move with that velocity. The second particle starts with zero velocity and continues to move with a constant acceleration, “a”. At a given instant, “t”, the first covers a linear distance,

$$x_1 = vt$$

In this period, the second particle travels a linear distance given by :

$$x_2 = \frac{1}{2}at^2$$

First particle starts with certain velocity as against second one, which is at rest. It means that the first particle will be ahead of second particle in the beginning. The separation between two particles is :

$$\Delta x = x_1 - x_2$$

$$\Delta x = vt - \frac{1}{2}at^2$$

For the separation to be maximum, its first time derivative should be equal to zero and second time derivative should be negative. Now, first and second time derivatives are :

$$\frac{d(\Delta x)}{dt} = v - at$$

$$\frac{d^2(\Delta x)}{dt^2} = -a < 0$$

For maximum separation,

$$\Rightarrow \frac{d(\Delta x)}{dt} = v - at = 0$$

$$\Rightarrow t = \frac{v}{a}$$

The separation at this time instant,

$$\Delta x = vt - \frac{1}{2}at^2 = v \times \frac{v}{a} - \frac{1}{2}a\left(\frac{v}{a}\right)^2$$

$$\Delta x = \frac{v^2}{2a}$$

Graphs of motion with constant acceleration

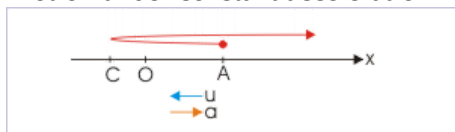
Graphical analysis of motion with constant acceleration in one dimension is based on the equation defining position and displacement, which is a quadratic function. Nature of motion under constant acceleration and hence its graph is characterized by the relative orientation of initial velocity and acceleration. The relative orientation of these parameters controls the nature of motion under constant acceleration. If initial velocity and acceleration are in the same direction, then particle is accelerated such that speed of the particle keeps increasing with time. However, if velocity and acceleration are directed opposite to each other, then particle comes to rest momentarily. As a result, the motion is divided in two segments - one with deceleration in which particle moves with decreasing speed and second with acceleration in which particle moves with increasing speed.

We study various scenarios of motion of constant acceleration by analyzing equation of position, which is quadratic expression in time. The position of particle is given by :

$$x = x_0 + ut + \frac{1}{2}at^2$$
$$\Rightarrow x = \frac{1}{2}at^2 + ut + x_0$$

Clearly, the equation for position of the particle is a quadratic expression in time, “t”. A diagram showing a generalized depiction of motion represented by quadratic expression is shown here :

Motion under constant acceleration



Motion under constant
acceleration

In case initial position of the particle coincides with origin of reference, then initial position $x_0 = 0$ and in that case x denotes position as well as displacement :

$$\Rightarrow x = ut + \frac{1}{2}at^2$$

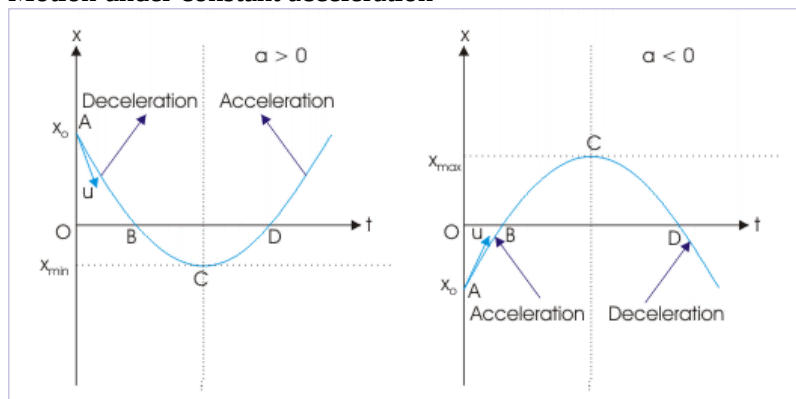
Nature of graphs

The nature of quadratic polynomial is determined by two controlling factors (i) nature of coefficient of squared term, t^2 and (ii) nature of discriminant of the quadratic equation, which is formed by equating quadratic expression to zero.

Nature of coefficient of squared term

The coefficient of squared term is “ $a/2$ ”. Thus, its nature is completely described by the nature of “ a ” i.e. acceleration. If “ a ” is positive, then graph is a parabola opening up. On the other hand, if “ a ” is negative, then graph is a parabola opening down. These two possibilities are shown in the picture.

Motion under constant acceleration



Motion under constant acceleration

It may be interesting to know that sign of acceleration is actually a matter of choice. A positive acceleration, for example, is negative acceleration if we reverse the reference direction. Does it mean that mere selection of reference direction will change the nature of motion? As expected, it is not so. The parabola comprises two symmetric sections about a line passing through the minimum or maximum point (C as shown in the figure). One section represents a motion in which speed of the particle is decreasing as velocity and acceleration are in opposite directions and second section represents a motion in which speed of the particle is increasing as velocity and acceleration are in the same direction. In other words, one section represents deceleration, whereas other section represents acceleration. The two sections are simply exchanged in two graphs. As such, a particular motion of acceleration is described by different sections of two graphs when we change the sign of acceleration. That is all. The nature of motion remains same. Only the section describing motion is exchanged.

Nature of discriminant

Comparing with general quadratic equation $Ax^2 + Bx + C = 0$, we see that determinant of the corresponding quadratic equation is given by :

$$D = B^2 - 4AC = u^2 - 4X \frac{a}{2} X x_0$$

$$\Rightarrow D = u^2 - 2ax_0$$

The important aspect of discriminant is that it comprises of three variable parameters. However, simplifying aspect of the discriminant is that all parameters are rendered constant by the “initial” setting of motion. Initial position, initial velocity and acceleration are all set up by the initial conditions of motion.

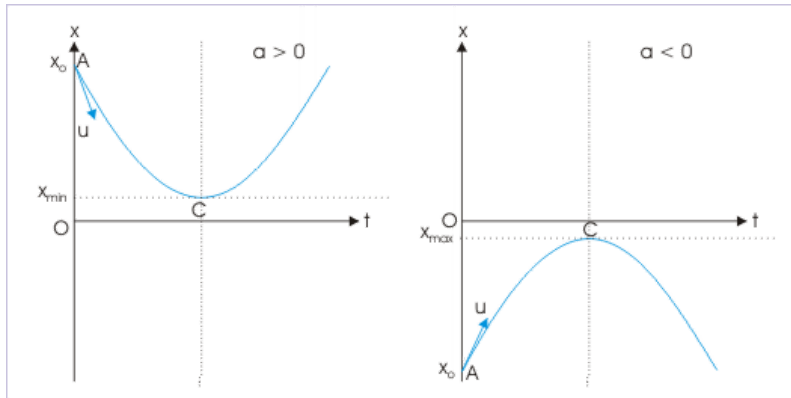
The points on the graph intersecting t-axis gives the time instants when particle is at the origin i.e. $x=0$. The curve of the graph intersects t-axis when corresponding quadratic equation (quadratic expression equated to zero) has real roots. For this, discriminant of the corresponding quadratic equation is non-negative (either zero or positive). It means :

$$D = u^2 - 2ax_0 \geq 0$$

$$\Rightarrow 2ax_0 \leq u^2$$

Note that squared term u^2 is always positive irrespective of sign of initial velocity. Thus, this condition is always fulfilled if the signs of acceleration and initial position are opposite. However, if two parameters have same sign, then above inequality should be satisfied for the curve to intersect t-axis. In earlier graphs, we have seen that parabola intersects t-axis at two points corresponding two real roots of corresponding quadratic equation. However, if discriminant is negative, then parabola does not intersect t-axis. Such possibilities are shown in the figure :

Motion under constant acceleration



Motion under constant acceleration

Clearly, motion of particle is limited by the minimum or maximum positions. It is given by :

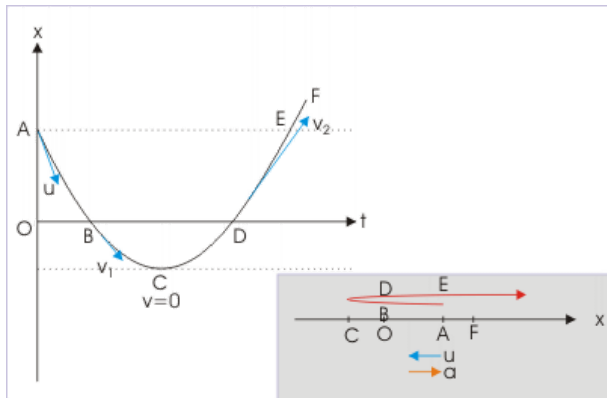
$$x = -\frac{D}{4A} = -\frac{u^2 - 2ax_0}{4 \times \frac{a}{2}} = -\frac{u^2 - 2ax_0}{2a}$$

It may be noted that motion of particle may be restricted to some other reference points as well. Depending on combination of initial velocity and acceleration, a particle may not reach a particular point.

Reading of graph

Our consideration, here, considers only positive values of time. It means that we are discussing motion since the start of observation at $t=0$ and subsequently as the time passes by. Mathematically, the time parameter, “t” is a non-negative number. It can be either zero or positive, but not negative. In the following paragraphs, we describe critical segments or points of a typical position – time graph as shown in the figure above :

Motion under constant acceleration



Motion under constant acceleration

Point A : This is initial position at $t = 0$. The position corresponding to this time is denoted as x_0 . This point is the beginning of graph, which lies on x-axis. The tangent to the curve at this point is the direction of initial velocity, “u”. Note that initial position and origin of reference may be different. Further, this point is revisited by the particle as it reaches E. Thus, A and E denotes the same start position - though represented separately on the graph.

Curve AC : The tangent to the curve part AB has negative slope. Velocity is directed in negative x-direction. The slope to the curve keeps decreasing in magnitude as we move from A to C. It means that speed of the particle keeps decreasing in this segment. In other words, particle is decelerated during motion in this segment.

Points B, D : The curve intersects t-axis. The particle is at the origin of the reference, O, chosen for the motion. The time corresponding to these points (B and D) are real roots of the quadratic equation, which is obtained by equating quadratic expression to zero.

Point C : The slope of tangent to the curve at this point is zero. It means that the speed of the particle has reduced to zero. The particle at this point is at rest. The slopes of the tangent to curve about this point changes sign. It means that velocity is oppositely directed about this point. The point B, therefore, is a point, where reversal of direction of motion occurs. Note that particle can reverse its direction only once during its motion under constant acceleration.

Point E : The particle reaches start point E (i.e. A) again in its motion after reversal of motion at C.

Point F : This is the end point of motion.

The graph of a motion under constant acceleration is bounded by set of parameters defining a motion. In a particular case, we may be considering only a segment of the curve starting from point A and ending at point F – not necessarily covering the whole of graph as shown in the figure. The point F may lie anywhere on the graph. Further, nature of curve will be determined by values and signs of various parameters like initial position, initial velocity and acceleration. Here, we have divided our study in two categories based on the sign of acceleration : (i) acceleration is positive and (ii) acceleration is negative.

Acceleration is positive (in the reference direction)

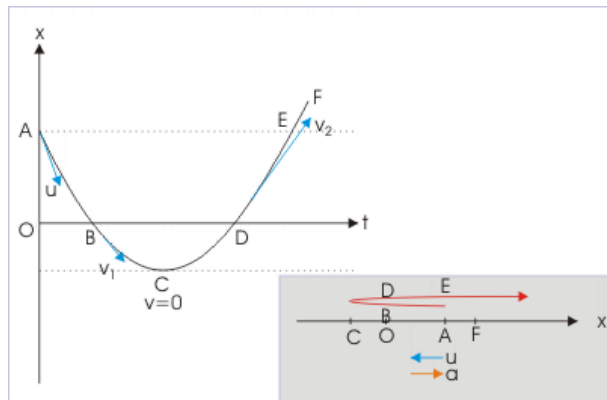
The graph of quadratic equation is a parabola opening upwards as coefficient of squared term t^2 is positive i.e. $a > 0$. The minimum value of expression i.e. x is :

$$x_{\min} = -\frac{D}{4A} = -\frac{u^2 - 2ax_0}{4 \times \frac{a}{2}} = -\frac{u^2 - 2ax_0}{2a}$$

The various possibilities of initial positions of the particle with time are discussed here :

Case 1: Initial position and origin of reference are different. Initial position is positive. Initial velocity is negative i.e. it is directed in negative reference direction. The velocity and acceleration are oppositely directed.

Motion under constant acceleration



Motion under constant acceleration

Initially particle is decelerated as the speed of the particle keeps on decreasing till it becomes zero at point C. This is indicated by the diminishing slope (magnitude) of the tangents to the curve. Subsequently, particle is accelerated so long force causing acceleration is applied on the particle.

The segment DF with origin at D is typical graph of free fall of particle under gravity, considering DF as the height of fall and downward direction as positive direction. Only differing aspect is that particle has initial velocity. Nevertheless, the nature of curve of free fall is similar. Note that speed of the particle keeps increasing till it hits the ground.

Example:

Problem : Given $x_0 = 10 \text{ m}$; $u = -15 \text{ m/s}$; $a = 10 \text{ m/s}^2$

Find the position when particle reverses its motion.

Solution : Acceleration is negative. The position-time graph is a parabola opening up. Therefore, the particle changes its direction of motion at its minimum position, which is given as :

$$x_{\min} = -\frac{u^2 - 2ax_0}{2a}$$

$$\Rightarrow x_{\min} = -\frac{(-15)^2 - 2 \times 10 \times 10}{2 \times 10} = -\frac{25}{20} = -1.25 \text{ m}$$

Alternatively,

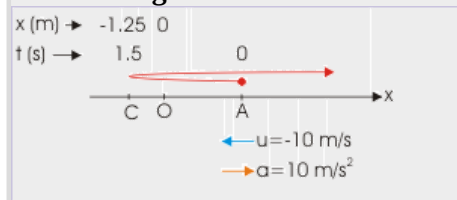
At the point of reversal, velocity is zero. Using $v = u + at$:

$$\Rightarrow 0 = -15 + 10t$$

$$\Rightarrow t = \frac{3}{2} = 1.5 \text{ s}$$

Position of particle in 1.5 s is :

Motion diagram



Motion diagram

$$x = x_0 + ut + \frac{1}{2}at^2$$

$$\Rightarrow x = 10 + (-15) \times 1.5 + \frac{1}{2} \times 10 \times 2.25 = 10 - 22.5 + 5 \times 2.25 = -12.5 + 11.25 = -1.25 \text{ m}$$

Exercise:

Problem: Given : $x_0 = 10 \text{ m}$; $u = -15 \text{ m/s}$; $a = 10 \text{ m/s}^2$

Find the time instants when the particle is at origin of reference and initial position. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = x_0 + ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put $x=0$,

$$0 = 10 - 15t + \frac{1}{2} \times 10t^2$$

$$\Rightarrow t^2 - 3t + 2 = 0$$

$$\Rightarrow t^2 - t - 2t + 2 = 0$$

$$\Rightarrow t(t - 1) - 2(t - 1) = 0$$

$$\Rightarrow t = 1 \text{ s and } 2 \text{ s}$$

In order to determine time instant when particle is at initial position, we put $x=10$,

$$\Rightarrow 10 = 10 - 15t + \frac{1}{2} \times 10t^2$$

$$\Rightarrow 5t^2 - 15t = 0$$

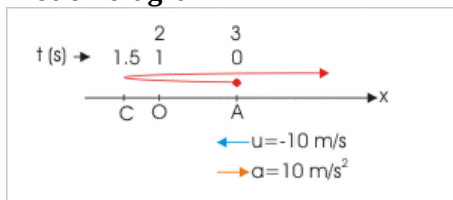
$$\Rightarrow t = 0 \text{ and } 3 \text{ s}$$

The particle is at the initial position twice, including start of motion. Now, at the point of reversal of direction, speed of the particle is zero. Putting $v = 0$ and using $v=u+at$,

$$\Rightarrow 0 = -15 + 10t$$

$$\Rightarrow t = 1.5 \text{ s}$$

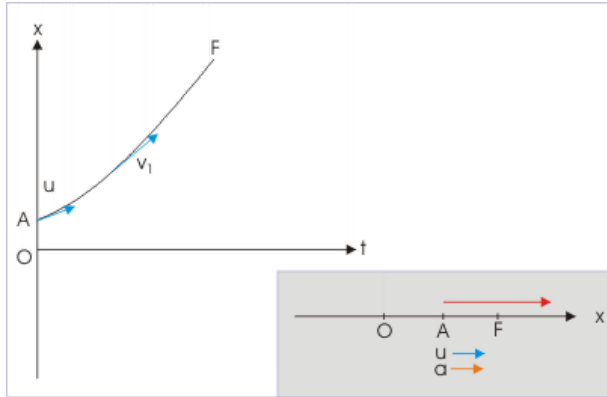
Motion diagram



Motion diagram

Case 2: Initial position and origin of reference are different. Initial position is positive. Initial velocity is positive i.e. it is directed in reference direction. The velocity and acceleration are in same direction.

Motion under constant acceleration



Motion under constant acceleration

The particle is accelerated so long force causing acceleration is applied on the particle. The segment AF with origin at D is typical graph of free fall of particle under gravity, considering AF as the height of fall and downward direction as positive direction. Only differing aspect is that particle has initial velocity. Nevertheless, the nature of curve of free fall is similar. Note that speed of the particle keeps increasing till it hits the ground.

Exercise:

Problem: Given $x_0 = 10 \text{ m}$; $u = 15 \text{ m/s}$; $a = 10 \text{ m/s}^2$

Find the time instants when the particle is at origin of reference and initial position. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = x_0 + ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put $x=0$,

$$\Rightarrow 0 = 10 + 15t + \frac{1}{2} \times 10t^2$$

$$\Rightarrow t^2 + 3t + 2 = 0$$

$$\Rightarrow t^2 + t + 2t + 2 = 0$$

$$\Rightarrow t = -1 \text{ s} \text{ and } -2 \text{ s}$$

We neglect both these negative values and deduce that the particle never reaches point of origin. In order to determine time instant when particle is at initial position, we put $x=10$,

$$\Rightarrow 10 = 10 + 15t + \frac{1}{2} \times 10t^2$$

$$\Rightarrow 5t^2 + 15t = 0$$

$$\Rightarrow t = 0 \quad \text{and} \quad -3 \quad s$$

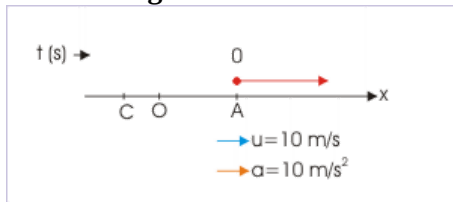
We neglect negative value. The particle is at the initial position only at the start of motion. Now, at the point of reversal of direction, speed of the particle is zero. Putting $v = 0$ and using $v = u + at$,

$$\Rightarrow 0 = 15 + 10t$$

$$\Rightarrow t = -1.5 \quad s$$

The particle never changes its direction.

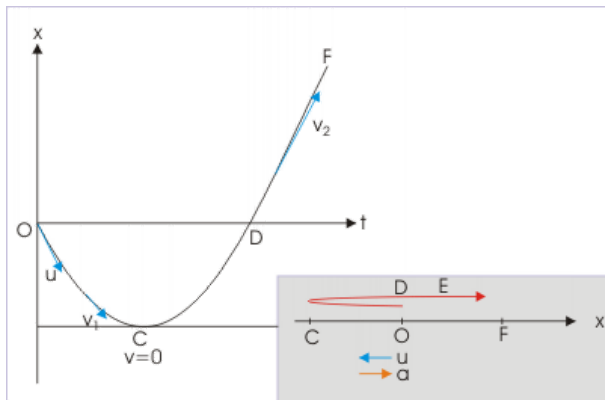
Motion diagram



Motion diagram

Case 3: Initial position and origin of reference are same. Initial velocity is negative i.e. it is directed in negative reference direction. The velocity and acceleration are oppositely directed.

Motion under constant acceleration



Motion under constant acceleration

Initially particle is decelerated as the speed of the particle keeps on decreasing till it becomes zero at point C. This is indicated by the diminishing slope (magnitude) of the tangents to the curve.

Subsequently, particle is accelerated so long force causing acceleration is applied on the particle.

Exercise:

Problem: Given $x_0 = 0 \text{ m}$; $u = -15 \text{ m/s}$; $a = 10 \text{ m/s}^2$

Find the time instants when the particle is at origin of reference. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put $x=0$,

$$\Rightarrow 0 = -15t + \frac{1}{2} \times 10t^2$$

$$\Rightarrow t^2 - 3t = 0$$

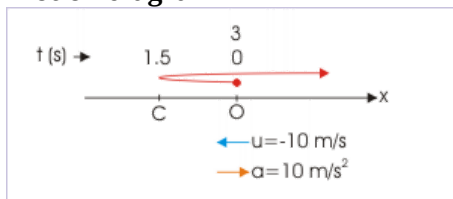
$$\Rightarrow t = 0 \text{ and } 3 \text{ s}$$

The particle is at the origin of reference twice, including start of motion. Now, at the point of reversal of direction, speed of the particle is zero. Putting $v = 0$ and using $v=u+at$,

$$\Rightarrow 0 = -15 + 10t$$

$$\Rightarrow t = 1.5 \text{ s}$$

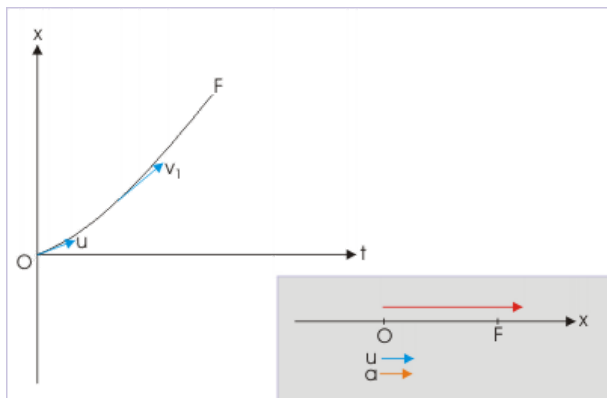
Motion diagram



Motion diagram

Case 4: Initial position and origin of reference are same. Initial velocity is positive i.e. it is directed in reference direction. The velocity and acceleration are in same direction.

Motion under constant acceleration



Motion under constant acceleration

The particle is accelerated so long force causing acceleration is applied on the particle.

The segment OF is typical graph of free fall of particle under gravity, considering OF as the height of fall and downward direction as positive direction. Only differing aspect is that particle has initial velocity. Nevertheless, the nature of curve of free fall is similar. Note that speed of the particle keeps increasing till it hits the ground.

Exercise:

Problem: Given $x_0 = 0$ m; $u = 15$ m/s; $a = 10$ m/s²

Find the time instants when the particle is at origin of reference. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put $x=0$,

$$0 = 15t + \frac{1}{2} \times 10t^2$$

$$\Rightarrow t^2 + 3t = 0$$

$$\Rightarrow t = 0 \quad \text{and} \quad -3 \text{ s}$$

Neglecting negative value of time,

$$\Rightarrow t = 0$$

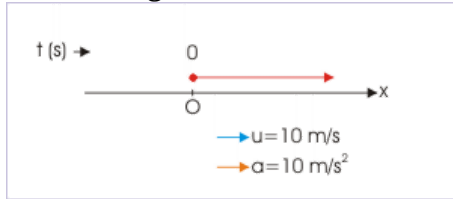
The particle is at the origin of reference only at the start of motion. Now, at the point of reversal of direction, speed of the particle is zero. Putting $v = 0$ and using $v=u+at$,

$$\Rightarrow 0 = 15 + 10t$$

$$\Rightarrow t = -1.5 \text{ s}$$

We neglect negative value and deduce that particle never ceases to move and as such there is no reversal of motion.

Motion diagram



Motion diagram

Acceleration is negative (opposite to the reference direction)

In this case, the graph of quadratic equation is a parabola opening downwards as coefficient of squared term t^2 is negative i.e. $a < 0$. The maximum value of expression i.e. x is :

$$x_{\max} = -\frac{D}{4A} = -\frac{u^2 - 2ax_0}{4 \times \frac{a}{2}} = -\frac{u^2 - 2ax_0}{2a}$$

The analysis for this case is similar to the first case. We shall, therefore, not describe this case here.

Exercise:

Problem: Given $x_0 = 10 \text{ m}$; $u = 15 \text{ m/s}$; $a = -10 \text{ m/s}^2$

Find the time instants when the particle is at origin of reference and initial position. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = x_0 + ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put $x=0$,

$$\Rightarrow 0 = 10 + 15t + \frac{1}{2}(-10)t^2$$

$$\Rightarrow -t^2 + 3t + 2 = 0$$

$$\Rightarrow t = \frac{-3 \pm \sqrt{\{9 - (4 \times -1 \times 2)\}}}{2 \times -1}$$

$$\Rightarrow t = \frac{-3 \pm 4.12}{-2} = -0.56 \text{ s} \text{ or } 3.56 \text{ s}$$

We neglect negative value of time. The particle reaches origin of reference only once at $t = 2 \text{ s}$. In order to determine time instant when particle is at initial position, we put $x=10$,

$$\Rightarrow 10 = 10 + 15t + \frac{1}{2}X - 10t^2$$

$$\Rightarrow -t^2 + 3t = 0$$

$$\Rightarrow t = 0, \quad 3 \text{ s}$$

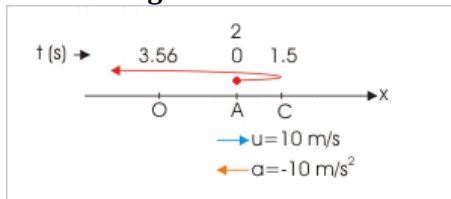
The particle is at the initial position twice, including start point. Now, at the point of reversal of direction, speed of the particle is zero. Putting $v = 0$ and using $v=u+at$,

$$\Rightarrow 0 = 15 - 10t$$

$$\Rightarrow t = 1.5 \text{ s}$$

We neglect negative value and deduce that particle never ceases to move and as such there is no reversal of motion.

Motion diagram



Motion diagram

Example

Example:

Problem : A particle's velocity in "m/s" is given by a function in time "t" as :

$$v = 40 - 10t$$

If the particle is at $x = 0$ at $t = 0$, find the time (s) when the particle is 60 m away from the initial position.

Solution : The velocity is given as a function in time "t". Thus, we can know its acceleration by differentiating with respect to time :

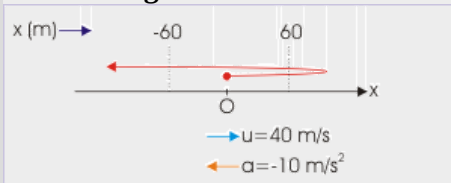
$$a = \frac{dv}{dt} = -10 \text{ m/s}^2$$

Alternatively, we can see that expression of velocity has the form $v = u + at$. Evidently, the motion has a constant acceleration of -10 m/s^2 . Also, note that origin of reference and initial position are same. Applying equation of motion for position as :

$$x = ut + \frac{1}{2}at^2 = 40t + \frac{1}{2} \times -10 \times t^2 = 40t - 5t^2$$

According to question, we have to find the time when particle is 60 m away from the initial position. Since it is one dimensional motion, the particle can be 60 m away either in the positive direction of the reference or opposite to it. Considering that it is 60 m away in the positive reference direction, we have :

Motion diagram



Motion diagram

$$60 = 40t - 5t^2$$

Re-arranging,

$$t^2 - 8t + 12 = 0$$

$$t = 2 \text{ s or } 6 \text{ s}$$

We observe here that the particle is ultimately moving in the direction opposite to reference direction. As such it will again be 60 away from the initial position in the negative reference direction. For considering that the particle is 60 m away in the negative reference direction,

$$-60 = 40t - 5t^2$$

$$\Rightarrow t^2 - 8t - 12 = 0$$

$$t = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 1 \times (-12)}}{2} = -1.29 \text{ s}, 9.29 \text{ s}$$

Neglecting negative value of time,

$$\Rightarrow t = 9.29 \text{ s}$$

Note : We can also solve the quadratic equation for zero displacement to find the time for the particle to return to initial position. This time is found to be 8 seconds. We note here that particle takes 2 seconds to reach the linear distance of 60 m in the positive direction for the first time, whereas it takes only $9.29 - 8 = 1.29$ second to reach 60 m from the initial position in the negative direction.

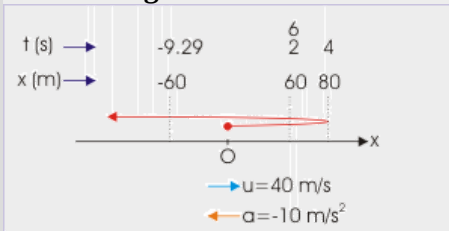
The particle is decelerated while moving in positive direction as velocity is positive, but acceleration is negative. On the other hand, both velocity and acceleration are negative while going away from the initial position in the negative direction and as such particle is accelerated. Therefore, the particle

takes lesser time to travel same linear distance in the negative direction than in the positive direction from initial position..

Hence, the particle is 60 m away from the initial position at $t = 2$ s, 6 s and 9.29 s.

t (s) 0 2 4 6 8 9.29 x (m) 0 60 80 60 0 -60 v (m/s) 40 20 0 -20 -40 -52.9

Motion diagram



Motion diagram

Example:

Problem : A particle moves along x-axis with a velocity 9 m/s and acceleration -2 m/s^2 . Find the distance covered in 5th second.

Solution : The displacement in 5th second is :

$$x_n = u + \frac{a}{2} (2n - 1)$$

$$x_n = 9 + \frac{-2}{2} (2 \times 5 - 1)$$

$$x_n = 0$$

The displacement in the 5th second is zero. A zero displacement, however, does not mean that distance covered is zero. We can see here that particle is decelerated at the rate of -2 m/s^2 and as such there is reversal of direction when $v = 0$.

$$v = u + at$$

$$0 = 9 - 2t$$

$$t = 4.5 \text{ s}$$

This means that particle reverses its motion at $t = 4.5$ s i.e in the period when we are required to find distance. The particle, here, travels in the positive direction from $t = 4$ s to 4.5 s and then travels in the negative direction from $t = 4.5$ s to 5 s. In order to find the distance in 5th second, we need to find displacement in each of these time intervals and then sum their magnitude to find the required distance.

Motion along a straight line

[missing_resource: odm1a.gif]

The particle reverses its direction in 5th second .

The displacement for the period $t = 0$ s to 4 s is :

$$x = ut + \frac{1}{2}at^2 = 9 \times 4 + \frac{1}{2} \times 2 \times 4^2 = 36 - 16 = 20 \text{ m}$$

The displacement for the period $t = 0$ s to 4.5 s is :

$$x = ut + \frac{1}{2}at^2 = 9 \times 4.5 + \frac{1}{2} \times 2 \times 4.5^2 = 40.5 - 20.25 = 20.25 \text{ m}$$

The displacement for the period $t = 4$ s to 4.5 s is :

$$\Delta x_1 = 20.25 - 20 = 0.25 \text{ m}$$

Since particle is moving with constant acceleration in one dimension, it travels same distance on its return for the same period. It means that it travels 0.25 m in the period from $t = 4.5$ s to $t = 5$ s.

$$\Delta x_2 = 0.25 \text{ m}$$

Thus, total distance covered between $t = 4$ s to $t = 5$ s i.e. in the 5^{th} second is :

$$\Delta x = \Delta x_1 + \Delta x_2 = 0.25 + 0.25 = 0.5 \text{ m}$$

Vertical motion under gravity

Vertical motion under gravity is a specific case of one dimensional motion with constant acceleration. Here, acceleration is always directed in vertically downward direction and its magnitude is "g".

As the force due to gravity may be opposite to the direction of motion, there exists the possibility that the body under force of gravity reverses its direction. It is, therefore, important to understand that the quantities involved in the equations of motion may evaluate to positive or negative values with the exception of time (t). We must appropriately assign sign to various inputs that goes into the equation and correctly interpret the result with reference to the assumed positive direction. Further, some of them evaluate to two values one for one direction and another of reversed direction.

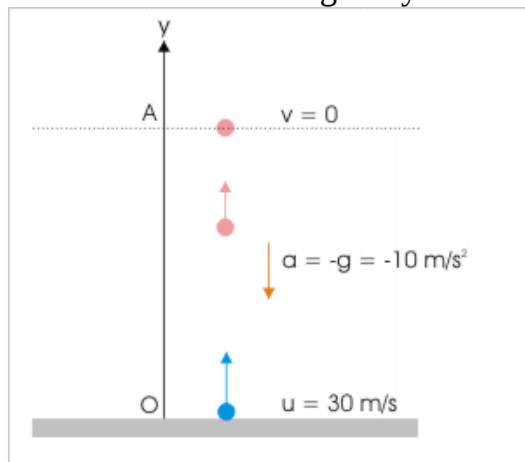
As pointed out earlier in the course, we must also realize that a change in reference direction may actually change the sign of the attributes, but their physical interpretation remains same. What it means that an attribute such as velocity, for example, can be either 5 m/s or -5 m/s, conveying the same velocity. The interpretation must be done with respect to the assigned positive reference direction.

Velocity

Let us analyze the equation " $v = u + at$ " for the vertical motion under gravity with the help of an example. We consider a ball thrown upwards from ground with an initial speed of 30 m/s. In the frame of reference with upward direction as positive,

$$u = 30 \text{ m/s and } a = -g = -10 \text{ m/s}^2$$

Vertical motion under gravity



The ball reaches maximum height when its velocity

height when its velocity
becomes zero

Putting this value in the equation, we have :

$$v = 30 - 10 t$$

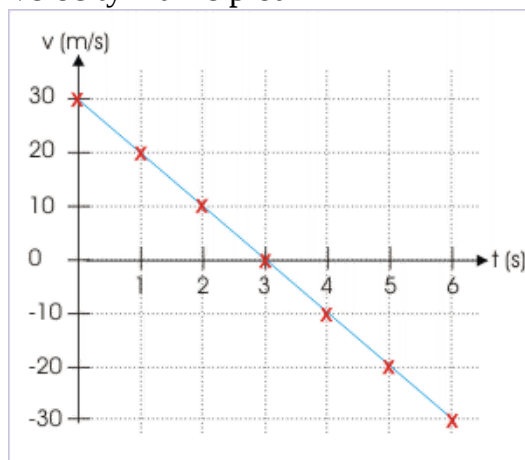
The important aspect of this equation is that velocity evaluates to both positive and negative values; positive for upward motion and negative for downward motion. The final velocity (v) is positive for $t < 3$ seconds, zero for $t = 3$ seconds and negative for $t > 3$ seconds. The total time taken for the complete up and down journey is 3 (for upward motion) + 3 (for downward motion) = 6 seconds.

The velocities of the ball at successive seconds are :

Time (t)		Final velocity	
(v) in seconds	in m/s		
0.0	30	1.0	20
2.0	10	3.0	0
4.0	-10	5.0	-20
6.0	-30		

The corresponding velocity – time plot looks like as shown in the figure.

Velocity – time plot



We notice following important characteristics of the motion :

- 1: The velocity at the maximum height is zero ($v=0$).
- 2: The time taken by the ball to reach maximum height is obtained as :

$$\begin{aligned}
 &\text{For } v = 0, \\
 &v = u + at = u - gt = 0 \\
 &\Rightarrow u = gt \\
 &\Rightarrow t = \frac{u}{g}
 \end{aligned}$$

3: The ball completely regains its speed when it returns to ground, but the motion is directed in the opposite direction i.e.

$$v = -u$$

4: The time taken for the complete round trip is :

$$\begin{aligned}
 &\text{For } v = -u, \\
 &v = u + at = u - gt \\
 &\Rightarrow -u = u - gt \\
 &\Rightarrow t = \frac{2u}{g}
 \end{aligned}$$

The time taken for the complete journey is twice the time taken to reach the maximum height. It means that the ball takes equal time in upward and downward journey. Thus, the total motion can be considered to be divided in two parts of equal duration.

5: The velocity of the ball is positive in the first half of motion; Zero at the maximum height; negative in the second of the motion.

6: The velocity is decreasing all through the motion from a positive value to less positive value in the first half and from a less negative value to more negative value in the second half of the motion. This renders acceleration to be always negative (directed in -y direction), which is actually the case.

7: The velocity (positive) and acceleration (negative) in the first part are opposite in direction and the resulting speed is decreasing. On the other hand, the velocity (negative) and acceleration (negative) in the second part are in the same direction and the resulting speed is increasing.

Displacement and distance

Let us analyze the equation $\Delta x = x_2 - x_1 = ut + \frac{1}{2}at^2$ for the vertical motion under gravity with the help of earlier example. If we choose initial position as the origin, then $x_1 = 0$, $\Delta x = x_2 = x$ (say) and $x = ut + \frac{1}{2}at^2$, where x denotes position and displacement as well. In the frame of reference with upward direction as positive,

$$u = 30 \text{ m/s and } a = -g = -10 \text{ m/s}^2$$

Putting these values in the equation, we have :

$$\Rightarrow x = 30t - 5t^2$$

The important aspect of this equation is that it is a quadratic equation in time “t”. This equation yields two values of time “t” for every position and displacement. This outcome is in complete agreement with the actual motion as the ball reaches a given position twice (during upward and downward motion). Only exception is point at the maximum height, which is reached only once. We have seen earlier that ball reaches maximum height at t = 3 s. Therefore, maximum height,H, is given as :

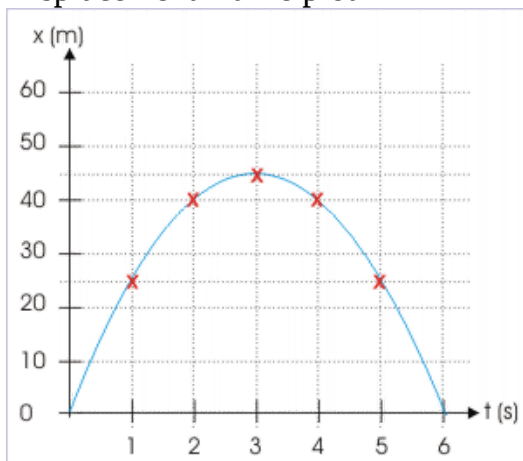
$$\Rightarrow H = 30 \times 3 - 5 \times 9 = 45 \text{ m}$$

The displacement values for the motion at successive seconds are :

Time (t) in seconds													Displacement (x) in meters														
0.0	0	0	0	1.0	30	5	25	2.0	60	20	40	3.0	90	45	45	4.0	120	80	40	5.0	150	125	25	6.0	180	180	0

The corresponding displacement – time plot looks like as shown in the figure.

Displacement – time plot



We notice following important characteristics of the motion :

1: The ball retraces every position during motion except the point at maximum height.

2: The net displacement when ball return to initial position is zero. Thus, the total time of journey (T) is obtained using displacement, $x = 0$,

$$\begin{aligned} \text{For } x &= 0, \\ x &= ut + \frac{1}{2}at^2 = uT - \frac{1}{2}gT^2 = 0 \\ \Rightarrow 2uT - gT^2 &= 0 \\ \Rightarrow T &= \frac{2u}{g} \end{aligned}$$

Here, we neglect $T = 0$, which corresponds to initial position.

3: The “x” in equation $x = ut + \frac{1}{2}at^2$ denotes displacement and not distance. Hence, it is not possible to use this equation directly to obtain distance, when motion is not unidirectional.

Let us answer the question with respect to the motion of the ball under consideration : what is the distance traveled in first 4 seconds? Obviously, the ball travels 30 m in the upward direction to reach maximum height in 3 seconds and then it travels 5 m in the 4th second in downward direction. Hence, the total distance traveled is $45 + 5 = 50$ m in 4 s. This means that we need to apply the equation of motion in two parts : one for the upward motion and the second for the downward motion. Thus, we find displacement for each segment of the motion and then we can add their magnitude to obtain distance.

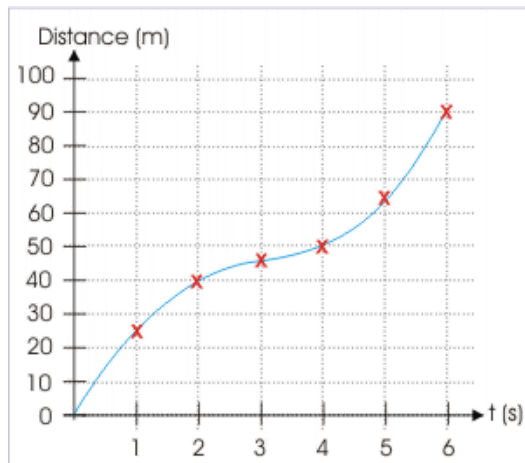
The distance values for the motion at successive seconds are :

```

-- Time (t) ut 5t*t Displacement Distance in seconds or
position in meters (x) in meters -----
--
-- 0.0 0 0 0 0 1.0 30 5
25 25 2.0 60 20 40 40 3.0 90 45 45 45 4.0 120 80 40 50 5.0
150 125 25 65 6.0 180 180 0 90 -----
--

```

Distance – time plot



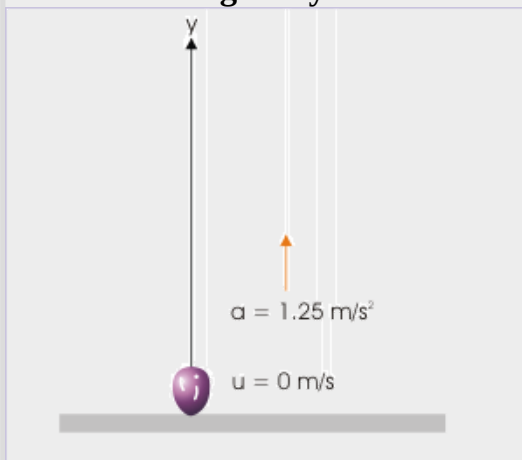
Example:

Constant acceleration

Problem : A balloon starts rising from the ground with an acceleration of 1.25 m/s^2 . After 8 second, a stone is released from the balloon. Starting from the release of stone, find the displacement and distance traveled by the stone on reaching the ground. Also, find the time taken to reach the ground (take $g = 10 \text{ m/s}^2$).

Solution : This question raises few important issues. First the rise of balloon is at a constant acceleration of 1.25 m/s^2 . This acceleration is the “measured” acceleration, which is net of the downward acceleration due to gravity. This means that the balloon rises with this net vertical acceleration of 1.25 m/s^2 in the upward direction.

Motion under gravity



Here, $u = 0$; $a = 1.25 \text{ m/s}^2$; $t = 8 \text{ s}$. Let the balloon rises to a height “h” during this time, then (considering origin on ground and upward direction as positive) the displacement of the balloon after 8 seconds is :

$$y = u t + \frac{1}{2} a t^2 = 0 \times 8 + 0.5 \times 1.25 \times 8^2 = 40 \text{ m}$$

Now, we know that, the body released from moving body acquires the velocity but not the acceleration of the container body under motion. The velocity of the balloon at the instant of separation is equal to the velocity of the balloon at that instant.

$$v = u + a t = 0 + 1.25 \times 8 = 10 \text{ m / s}$$

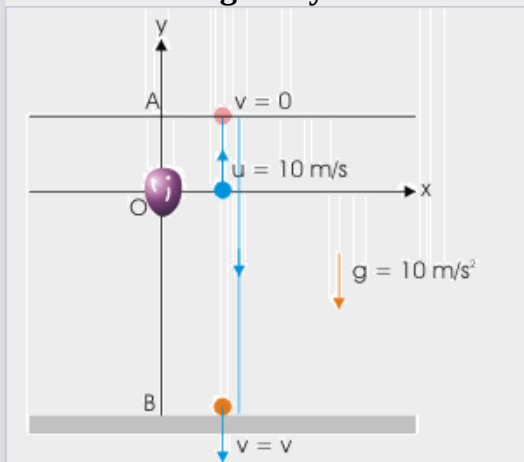
Thus, this is the initial velocity of the stone and is directed upward as that of the velocity of balloon. Once released, the stone is acted upon by the force of gravity alone. The role of the acceleration of the balloon is over. Now, the acceleration for the motion of stone is equal to the acceleration due to gravity, g .

The path of motion of the stone is depicted in the figure. Stone rises due to its initial upward velocity to a certain height above 40 m where it was released till its velocity is zero. From this highest vertical point, the stone falls freely under gravity and hits the ground.

1: In order to describe motion of the stone once it is released, we realize that it would be easier for us if we shift the origin to the point where stone is released. Considering origin at the point of release and upward direction as positive as shown in the figure, the displacement during the motion of stone is :

$$y = OB = -40 \text{ m}$$

Motion under gravity



2: Distance, on the other hand, is equal to :

$$s = OA + AO + OB = 2 OA + OB$$

In order to obtain, OA, we consider this part of rectilinear motion (origin at the point of release and upward direction as positive as shown in the figure).

Here, $u = 10 \text{ m/s}$; $a = -10 \text{ m/s}^2$ and $v = 0$. Applying equation of motion, we have :

$$v^2 = u^2 + 2ay$$

$$y = \frac{v^2 - u^2}{2a} = \frac{0^2 - 10^2}{-2 \times 10} = 5 \text{ m}$$

Hence, OA = 5 m, and distance is :

$$s = 2 OA + OB = 2 \times 5 + 40 = 50 \text{ m}$$

3: The time of the journey of stone after its release from the balloon is obtained using equation of motion (origin at the point of release and upward direction as positive as shown in the figure).

Here, $u = 12 \text{ m/s}$; $a = -10 \text{ m/s}^2$ and $y = -40 \text{ m}$.

$$\begin{aligned}y &= ut + \frac{1}{2}at^2 \\ \Rightarrow -40 &= 10t - 0.5 \times 10t^2 \\ \Rightarrow 5t^2 - 10t - 40 &= 0\end{aligned}$$

This is a quadratic equation in “t”. Its solution is :

$$\begin{aligned}\Rightarrow 5t^2 + 10t - 20t - 40 &= 0 \\ \Rightarrow 5t(t + 2) - 20(t + 2) &= 0 \\ \Rightarrow (5t - 20)(t + 2) &= 0 \\ \Rightarrow 4, -2 \text{ s}\end{aligned}$$

As negative value of time is not acceptable, time to reach the ground is 4s.

Note: It is important to realize that we are at liberty to switch origin or direction of reference after making suitable change in the sign of attributes.

Position

We use the equation $\Delta x = x_2 - x_1 = ut + \frac{1}{2}at^2$ normally in the context of displacement, even though the equation is also designed to determine initial (x_1) or final position (x_2). In certain situations, however, using this equation to determine position rather than displacement provides more elegant adaptability to the situation.

Let us consider a typical problem highlighting this aspect of the equation of motion.

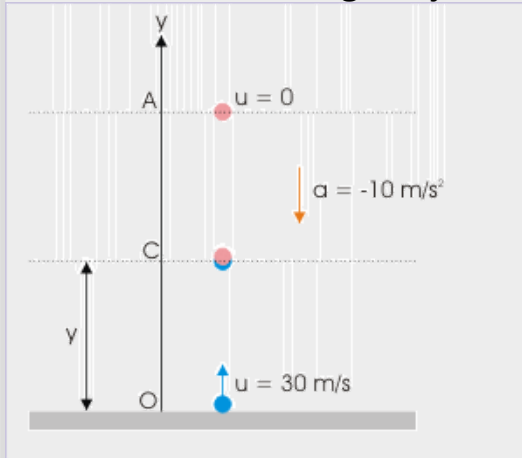
Example:

Problem : A ball is thrown vertically from the ground at a velocity 30 m/s, when another ball is dropped along the same line, simultaneously from the top of tower 120

m in height. Find the time (i) when the two balls meet and (ii) where do they meet.

Solution : This question puts the position as the central concept. In addition to equal time of travel for each of the balls, the coordinate positions of the two balls are also same at the time they meet. Let this position be “y”. Considering upward direction as the positive reference direction, we have :

Vertical motion under gravity



The balls have same coordinate value when they meet.

For ball thrown from the ground :

$$u = 30 \text{ m/s} , a = -10 \text{ m/s}^2 , y_1 = 0 , y_1 = y$$

$$y_2 - y_1 = ut + \frac{1}{2}at^2$$

$$\Rightarrow y - 0 = 30t - \frac{1}{2}10 \times t^2$$

Equation:

$$\Rightarrow y = 30t - 5 \times t^2$$

For ball dropped from the top of the tower :

$$u = 0 \text{ m/s} , a = -10 \text{ m/s}^2 , y_1 = 120 , y_2 = y$$

$$y_2 - y_1 = ut + \frac{1}{2}at^2$$

Equation:

$$\Rightarrow y - 120 = -5 \times t^2$$

Now, deducting equation (2) from (1), we have :

$$30t = 120$$

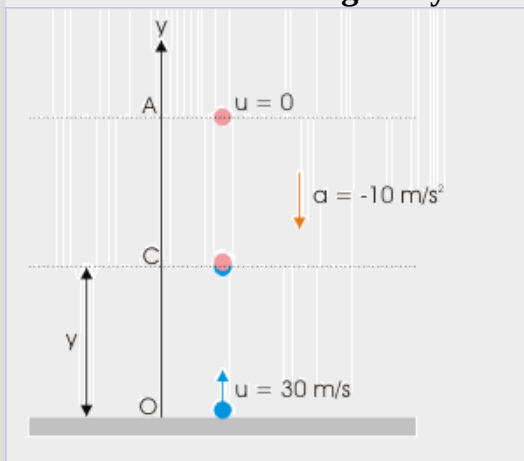
$$\Rightarrow t = 4 \text{ s}$$

Putting this value in equation – 1, we have :

$$\Rightarrow y = 30 \times 4 - 5 \times 4^2 = 120 - 80 = 40 \text{ m}$$

One interesting aspect of this simultaneous motion of two balls is that the ball dropped from the tower meets the ball thrown from the ground, when the ball thrown from the ground is actually returning from after attaining the maximum height in 3 seconds. For maximum height of the ball thrown from the ground,

Vertical motion under gravity



When returning from the maximum height, the ball thrown up from the ground is hit by the ball dropped from towers.

$$u = 30 \text{ m/s}, a = -10 \text{ m/s}^2 \text{ and } v = 0$$

$$v = u + at$$

$$= 30 - 10t$$

$$t = 3 \text{ s}$$

This means that this ball has actually traveled for 1 second ($4 - 3 = 1 \text{ s}$) in the downward direction, when it is hit by the ball dropped from the tower!

Exercises

Exercise:

Problem:

A ball is thrown up in vertical direction with an initial speed of 40 m/s. Find acceleration of the ball at the highest point.

Solution:

The velocity of the ball at the highest point is zero. The only force on the ball is due to gravity. The acceleration of ball all through out its motion is acceleration due to gravity “g”, which is directed downwards. The acceleration of the ball is constant and is not dependent on the state of motion - whether it is moving or is stationary.

Exercise:

Problem:

A ball is released from a height of 45 m. Find the magnitude of average velocity during its motion till it reaches the ground.

Solution:

The average velocity is ratio of displacement and time. Here, displacement is given. We need to find the time of travel. For the motion of ball, we consider the point of release as origin and upward direction as positive.

$$\begin{aligned}y &= ut + \frac{1}{2}at^2 \\ \Rightarrow -45 &= 0Xt + \frac{1}{2}X - 10Xt^2 \\ \Rightarrow t^2 &= \frac{45}{5} = 9 \\ \Rightarrow t &= \pm 3 \quad s\end{aligned}$$

Neglecting negative time, $t = 3$ s. Magnitude of average velocity is :

$$v_{\text{avg}} = \frac{45}{3} = 15 \quad m/s$$

Exercise:**Problem:**

A ball is released from an elevator moving upward with an acceleration 3 m/s^2 . What is the acceleration of the ball after it is released from the elevator ?

Solution:

On separation, ball acquires the velocity of elevator – not its acceleration. Once it is released, the only force acting on it is that due to gravity. Hence, acceleration of the ball is same as that due to gravity.

$$a = 10 \frac{m}{s^2} \text{ downward}$$

Exercise:**Problem:**

A ball is released from an elevator moving upward with an acceleration 5 m/s^2 . What is the acceleration of the ball with respect to elevator after it is released from the elevator ?

Solution:

On separation, ball acquires the velocity of elevator – not its acceleration. Once it is released, the only force acting on it is that due to gravity. Hence, acceleration of the ball is same as that due to gravity. The relative acceleration of the ball (considering downward direction as positive) :

$$\begin{aligned} a_{\text{rel}} &= a_{\text{ball}} - a_{\text{elevator}} \\ \Rightarrow a_{\text{rel}} &= 10 - (-5) = 15 \frac{m}{s^2} \end{aligned}$$

Exercise:**Problem:**

A balloon ascends vertically with a constant speed for 5 second, when a pebble falls from it reaching the ground in 5 s. Find the speed of balloon.

Solution:

The velocity of balloon is constant and is a measured value. Let the ball moves up with a velocity u . At the time of release, the pebble acquires velocity of balloon. For the motion of pebble, we consider the point of release as origin and upward direction as positive. Here,

$$y = vt = -u \times 5 = -5u; \quad u = u; \quad a = -g; \quad t = 5 \text{ s}$$

Using equation for displacement :

$$\begin{aligned} y &= ut + \frac{1}{2}at^2 \\ \Rightarrow -5u &= u \times 5 + \frac{1}{2}(-g) \times 5^2 \\ \Rightarrow 10u &= 5 \times 25 \quad u = 5 \frac{m}{s} \end{aligned}$$

Exercise:

Problem:

A balloon ascends vertically with a constant speed of 10 m/s. At a certain height, a pebble falls from it reaching the ground in 5 s. Find the height of balloon when pebble is released from the balloon.

Solution:

The velocity of balloon is constant. Let the ball moves up with a velocity u . At the time of release, the pebble acquires velocity of balloon. For the motion of pebble, we consider the point of release as origin and upward direction as positive. Here,

$$y = ?; \quad u = 10 \text{ m/s}; \quad a = -g; \quad t = 5 \text{ s}$$

Using equation for displacement :

$$\begin{aligned} y &= ut + \frac{1}{2}at^2 \\ \Rightarrow y &= 10 \times 5 + \frac{1}{2}(-g) \times 5^2 \\ \Rightarrow y &= 50 - 5 \times 25 \\ \Rightarrow y &= -75 \text{ m} \end{aligned}$$

Height of ballon when pebble is released from it,

$$\Rightarrow H = 75 \text{ m}$$

Exercise:

Problem:

A ball is released from a top. Another ball is dropped from a point 15 m below the top, when the first ball reaches a point 5 m below the top. Both balls reach the ground simultaneously. Determine the height of the top.

Solution:

We compare motion of two balls under gravity, when second ball is dropped. At that moment, two balls are 10 m apart. The first ball moves with certain velocity, whereas first ball starts with zero velocity.

Let us consider downward direction as positive. The velocity of the first ball when it reaches 10 m below the top is :

$$\begin{aligned}v^2 &= u^2 + 2ax \\ \Rightarrow v^2 &= 0 + 2 \times 10 \times 5 \\ \Rightarrow v &= 10 \text{ m/s}\end{aligned}$$

Let the balls take time “t” to reach the gorund. First ball travels 10 m more than second ball. Let 1 and 2 denote first and second ball, then,

$$\begin{aligned}\Rightarrow 10 + 10t + \frac{1}{2} \times 10t^2 &= \frac{1}{2} \times 10t^2 \\ 10t &= 10 \\ t &= 1 \text{ s}\end{aligned}$$

In this time, second ball travels a distance given by :

$$\Rightarrow y = \frac{1}{2} \times 10t^2 = 5 \times 1^2 = 5 \text{ m}$$

But, second ball is 15 m below the top. Hence, height of the top is $15 + 5 = 20 \text{ m}$.

Exercise:

Problem:

One ball is dropped from the top at a height 60 m, when another ball is projected up in the same line of motion. Two balls hit each other 20 m below the top. Compare the speeds of the ball when they strike.

Solution:

For the motion of first ball dropped from the top, let downward direction be positive :

$$v_1 = u + at$$

$$\Rightarrow v_1 = at = 10t$$

For the ball dropped from the top,

$$x = ut + \frac{1}{2}at^2$$

$$\Rightarrow -20 = 0Xt + \frac{1}{2}X - 10Xt^2$$

$$\Rightarrow 20 = 5t^2$$

$$\Rightarrow t = \pm 2 \text{ s}$$

Neglecting negative value, $t = 2\text{s}$. Hence, velocity of the ball dropped from the top is :

$$v_1 = 10t = 10 \times 2 = 20 \text{ m/s}$$

For the motion of second ball projected from the bottom, let upward direction be positive :

$$v_2 = u + at$$

$$\Rightarrow v_2 = u - 10 \times 2 = u - 20$$

Clearly, we need to know u . For upward motion,

$$x = ut + \frac{1}{2}at^2$$

$$\Rightarrow 40 = u \times 2 + \frac{1}{2}X - 10X2^2$$

$$\Rightarrow 40 = 2u - 20 = 30m/s$$

$$\Rightarrow v_2 = 30 - 20 = 10 \text{ m/s}$$

Thus,

$$\Rightarrow \frac{v_1}{v_2} = \frac{20}{10} = \frac{2}{1}$$

Note: See module titled “ [Vertical motion under gravity \(application\)](#) for more questions.

Vertical motion under gravity (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the accelerated motion under gravity. The questions are categorized in terms of the characterizing features of the subject matter :

- Motion plots
- Equal displacement
- Equal time
- Displacement in a particular second
- Twice in a position
- Collision in air

Motion plots

Example:

Problem : A ball is dropped from a height of 80 m. If the ball loses half its speed after each strike with the horizontal floor, draw (i) speed – time and (ii) velocity – time plots for two strikes with the floor. Consider vertical downward direction as positive and $g = 10 \text{ m/s}^2$.

Solution : In order to draw the plot, we need to know the values of speed and velocity against time. However, the ball moves under gravity with a constant acceleration 10 m/s^2 . As such, the speed and velocity between strikes are uniformly increasing or decreasing at constant rate. It means that we need to know end values, when the ball strikes the floor or when it reaches the maximum height.

In the beginning when the ball is released, the initial speed and velocity both are equal to zero. Its velocity, at the time first strike, is obtained from the equation of motion as :

$$v^2 = 0 + 2gh$$

$$\Rightarrow v = \sqrt{(2gh)} = \sqrt{(2 \times 10 \times 80)} = 40 \text{ m/s}$$

Corresponding speed is :

$$|v| = 40 \text{ m/s}$$

The time to reach the floor is :

$$t = \frac{v-u}{a} = \frac{40-0}{10} = 4 \text{ s}$$

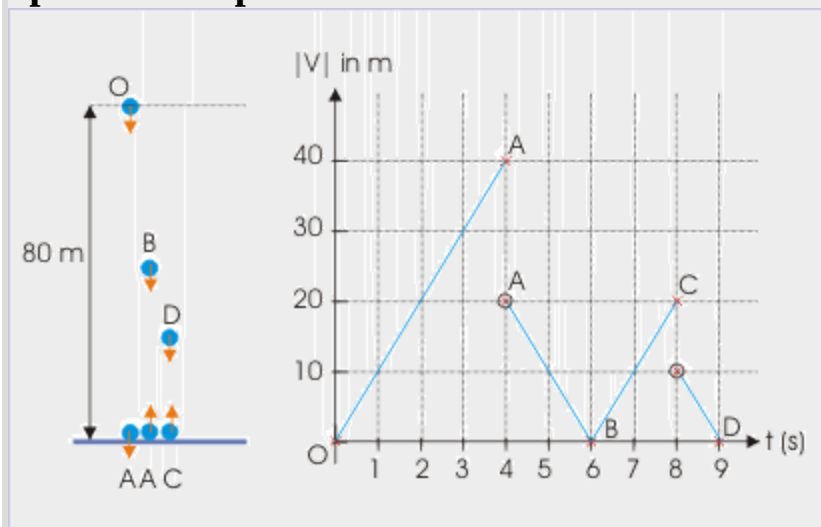
According to question, the ball moves up with half the speed. Hence, its speed after first strike is :

$$|v| = 20 \text{ m/s}$$

The corresponding upward velocity after first strike is :

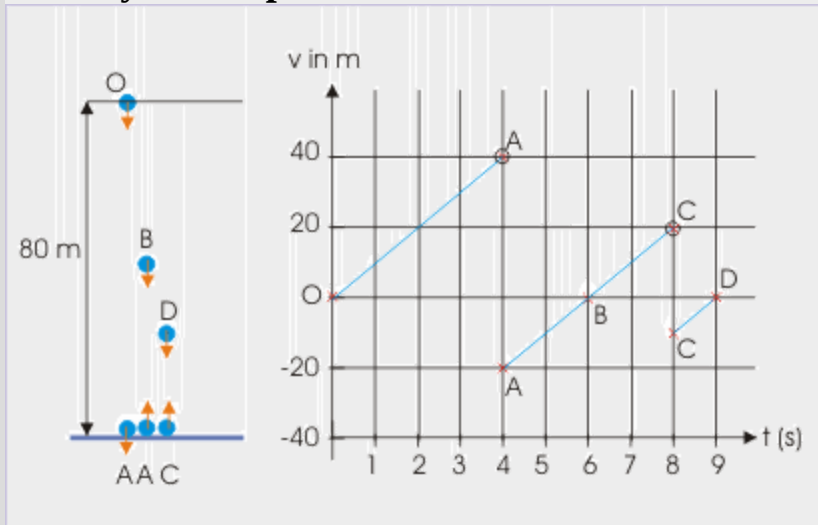
$$v = -20 \text{ m/s}$$

Speed – time plot



The ball dropped from a height loses half its height on each strike with the horizontal surface.

Velocity – time plot



The ball dropped from a height loses half its height on each strike with the horizontal surface.

It reaches a maximum height, when its speed and velocity both are equal to zero. The time to reach maximum height is :

$$t = \frac{v-u}{a} = \frac{0+20}{10} = 2 \text{ s}$$

After reaching the maximum height, the ball returns towards floor and hits it with the same speed with which it was projected up,

$$v = 20 \text{ m/s}$$

Corresponding speed is :

$$|v| = 20 \text{ m/s}$$

The time to reach the floor is :

$$t = 2 \text{ s}$$

Again, the ball moves up with half the speed with which ball strikes the floor. Hence, its speed after second strike is :

$$|v| = 10 \text{ m/s}$$

The corresponding upward velocity after first strike is :

$$v = -10 \text{ m/s}$$

It reaches a maximum height, when its speed and velocity both are equal to zero. The time to reach maximum height is :

$$t = \frac{v-u}{a} = \frac{0+10}{10} = 1 \text{ s}$$

Example:

Problem : A ball is dropped vertically from a height “h” above the ground. It hits the ground and bounces up vertically to a height “h/3”. Neglecting subsequent motion and air resistance, plot its velocity “v” with the height qualitatively.

Solution : We can proceed to plot first by fixing the origin of coordinate system. Let this be the ground. Let us also assume that vertical upward direction is the positive y-direction of the coordinate system. The ball remains above ground during the motion. Hence, height (y) is always positive. Since we are required to plot velocity .vs. height (displacement), we can use the equation of motion that relates these two quantities :

$$v^2 = u^2 + 2ay$$

During the downward motion, $u = 0$, $a = -g$ and displacement (y) is positive. Hence,

$$v^2 = -2gy$$

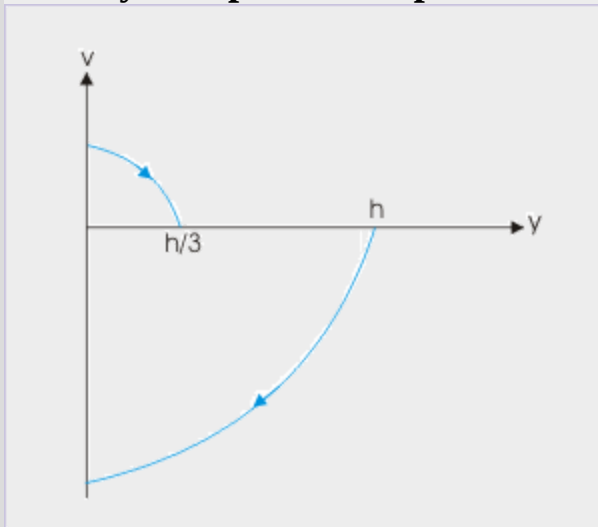
This is a quadratic equation. As such the plot is a parabola as motion progresses i.e. as y decreases till it becomes equal to zero (see plot below y -axis).

Similarly, during the upward motion, the ball has certain velocity so that it reaches $1/3$ rd of the height. It means that ball has a velocity less than that with which it strikes the ground. However, this velocity of rebound is in the upwards direction i.e positive direction of the coordinate system. In the nutshell, we should start drawing upward motion with a smaller positive velocity. The equation of motion, now, is :

$$v^2 = 2gy$$

Again, the nature of the plot is a parabola the relation being a quadratic equation. The plot progresses till displacement becomes equal to $y/3$ (see plot above y -axis). The two plots should look like as given here :

Velocity – displacement plot



The nature of plot is parabola.

Equal displacement

Example:

Problem : A particle is released from a height of “3h”. Find the ratios of time taken to fall through equal heights “h”.

Solution : Let t_1 , t_2 and t_3 be the time taken to fall through successive heights “h”. Then, the vertical linear distances traveled are :

$$h = \frac{1}{2}gt_1^2$$

$$2h = \frac{1}{2}g(t_1 + t_2)^2$$

$$3h = \frac{1}{2}g(t_1 + t_2 + t_3)^2$$

$$\Rightarrow t_1^2 : (t_1 + t_2)^2 : (t_1 + t_2 + t_3)^2 :: 1 : 2 : 3$$

$$\Rightarrow t_1 : t_1 + t_2 : t_1 + t_2 + t_3 :: 1 : \sqrt{2} : \sqrt{3}$$

In order to reduce this proportion into the one as required in the question, we carry out a general reduction of the similar type. In the end, we shall apply the result to the above proportion. Now, a general reduction of the proportion of this type is carried out as below. Let :

$$t_1 : t_1 + t_2 : t_1 + t_2 + t_3 :: a : b : c$$

$$\Rightarrow \frac{t_1}{a} = \frac{t_1+t_2}{b} = \frac{t_1+t_2+t_3}{c}$$

From the first two ratios,

$$\Rightarrow \frac{t_1}{a} = \frac{t_1+t_2-t_1}{b-a} = \frac{t_2}{b-a}$$

Similarly, from the last two ratios,

$$\Rightarrow \frac{t_1+t_2}{b} = \frac{t_1+t_2+t_3-t_1-t_2}{c-b} = \frac{t_3}{c-b}$$

Now, combining the reduced ratios, we have :

$$\Rightarrow \frac{t_1}{a} = \frac{t_2}{b-a} = \frac{t_3}{c-b}$$

$$\Rightarrow t_1 : t_2 : t_3 :: a : b - a : c - b$$

In the nutshell, we conclude that if

$$t_1 : t_1 + t_2 : t_1 + t_2 + t_3 :: a : b : c$$

Then,

$$t_1 : t_2 : t_3 :: a : b - a : c - b$$

Applying this result as obtained to the question in hand,

$$t_1 : t_1 + t_2 : t_1 + t_2 + t_3 :: 1 : \sqrt{2} : \sqrt{3}$$

We have :

$$\Rightarrow t_1 : t_2 : t_3 :: 1 : \sqrt{2} - 1 : \sqrt{3} - \sqrt{2}$$

Equal time

Example:

Problem : Balls are successively dropped one after another at an equal interval from a tower. At the instant, 9th ball is released, the first ball hits the ground. Which of the ball in series is at $\frac{3}{4}$ of the height of the tower?

Solution : Let “t” be the equal time interval. The first ball hits the ground when 9th ball is dropped. It means that the first ball has fallen for a total time $(9-1)t = 8t$. Let n^{th} ball is at $\frac{3}{4}$ th of the height of tower. Then,
For the first ball,

$$\Rightarrow h = \frac{1}{2}g \times 8t^2$$

For the nth ball,

$$\Rightarrow h - \frac{3h}{4} = \frac{h}{4} = \frac{1}{2}g \times (9 - n)^2 t^2$$

Combining two equations, we have :

$$\Rightarrow h = 2g \times (9 - n)^2 t^2 = \frac{1}{2}g \times 8t^2 = 32gt^2$$

$$\Rightarrow (9 - t)^2 = 16$$

$$\Rightarrow 9 - n = 4$$

$$\Rightarrow n = 5$$

Displacement in a particular second

Example:

Problem : A ball is dropped vertically from a tower. If the vertical distance covered in the last second is equal to the distance covered in first 3 seconds, then find the height of the tower. Consider $g = 10 \text{ m/s}^2$.

Solution : Let us consider that the ball covers the height y_n in n^{th} second. Then, the distance covered in the n^{th} second is given as :

$$y_n = u + \frac{g}{2} \times (2n - 1) = 0 + \frac{10}{2} \times (2n - 1) = 10n - 5$$

On the other hand, the distance covered in first 3 seconds is :

$$y_3 = ut + \frac{1}{2}gt^2 = 0 + \frac{1}{2} \times 10 \times 3^2 = 45 \text{ m}$$

According to question,

$$y_n = y_3$$

$$\Rightarrow 10n - 5 = 45$$

$$\Rightarrow n = 5 \text{ s}$$

Therefore, the height of the tower is :

$$\Rightarrow y = \frac{1}{2} \times 10 \times 5^2 = 125 \text{ m}$$

Twice in a position

Example:

Problem : A ball, thrown vertically upward from the ground, crosses a point “A” in time “ t_1 ”. If the ball continues to move up and then return to the ground in additional time “ t_2 ”, then find the height of “A” from the ground. .

Solution : Let the upward direction be the positive reference direction. The displacement equation for vertical projection from the ground to the point “A”, is :

$$h = ut_1 + \frac{1}{2}gt_1^2$$

In this equation, the only variable that we do not know is initial velocity. In order to determine initial velocity, we consider the complete upward motion till the ball reaches the maximum height. We know that the ball takes half of the total time to reach maximum height. It means that time for upward motion till maximum height is :

$$t_{\frac{1}{2}} = \frac{t_1 + t_2}{2}$$

As we know the time of flight, final velocity and acceleration, we can know initial velocity :

$$\begin{aligned} 0 &= u - gt_{\frac{1}{2}} \\ \Rightarrow u &= gt_{\frac{1}{2}} = \frac{(t_1 + t_2)g}{2} \end{aligned}$$

Substituting this value, the height of point “A” is :

$$\begin{aligned} h &= ut_1 + \frac{1}{2}gt_1^2 \\ \Rightarrow h &= \frac{(t_1 + t_2)g}{2} \times t_1 - \frac{1}{2}gt_1^2 \\ \Rightarrow h &= \frac{gt_1^2 + gt_1t_2 - gt_1^2}{2} \\ \Rightarrow h &= \frac{gt_1t_2}{2} \end{aligned}$$

Example:

Problem : A ball, thrown vertically upward from the ground, crosses a point “A” at a height 80 meters from the ground. If the ball returns to the same position after 6 second, then find the velocity of projection.

(Consider $g = 10 \text{ m/s}^2$).

Solution : Since two time instants for the same position is given, it is indicative that we may use the displacement equation as it may turn out to be quadratic equation in time “t”. The displacement, “y”, is (considering upward direction as positive y-direction) :

$$y = ut + \frac{1}{2}gt^2$$

$$80 = ut + \frac{1}{2} \times -10t^2 = ut - 5t^2$$

$$5t^2 - ut + 80 = 0$$

We can express time “t”, using quadratic formulae,

$$\Rightarrow t_2 = \frac{-(-u) + \sqrt{\{(-u)^2 - 4 \times 5 \times 80\}}}{2 \times 5} = \frac{u + \sqrt{(u^2 - 1600)}}{10}$$

Similarly,

$$\Rightarrow t_1 = \frac{u - \sqrt{(u^2 - 1600)}}{10}$$

According to question,

$$\Rightarrow t_2 - t_1 = \frac{\sqrt{(u^2 - 1600)}}{5}$$

$$\Rightarrow 6 = \frac{\sqrt{(u^2 - 1600)}}{5}$$

$$\Rightarrow \sqrt{(u^2 - 1600)} = 30$$

Squaring both sides we have,

$$\Rightarrow u^2 - 1600 = 900$$

$$\Rightarrow u = 50 \text{ m/s}$$

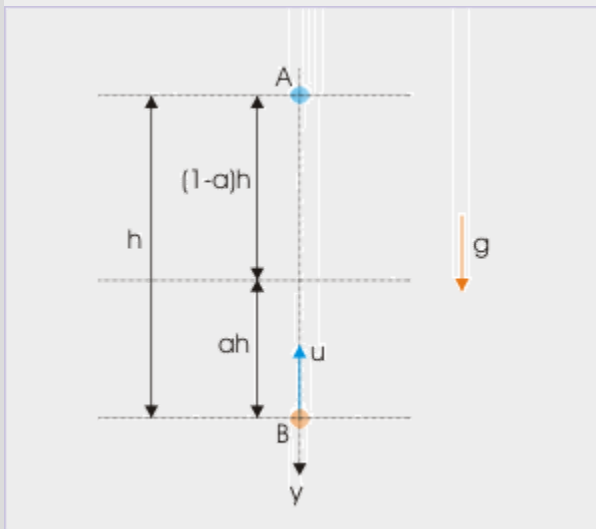
Collision in air

Example:

Problem : A ball “A” is dropped from a height, “h”, when another ball, “B” is thrown upward in the same vertical line from the ground. At the instant the balls collide in the air, the speed of “A” is twice that of “B”. Find the height at which the balls collide.

Solution : Let the velocity of projection of “B” be “u” and let the collision occurs at a fraction “a” of the height “h” from the ground. We should pause and make mental note of this new approach to express an intermediate height in terms of fraction of total height (take the help of figure).

Collision in the mid air



The balls collide as they move
in same vertical line.

Adhering to the conventional approach, we could have denoted the height at which collision takes place. For example, we could have considered collision at a vertical displacement y from the ground. Then displacements of two balls would have been :

“ y ” and “ $h - y$ ”

Obviously, the second expression of displacement is polynomial of two terms because of minus sign involved. On the other hand, the displacements of two balls in terms of fraction, are :

“ ah ” and “ $(1-a)h$ ”

The advantage of the second approach (using fraction) is that displacements are stated in terms of the product of two quantities. The variable “ h ” appearing in each of the terms cancel out if they appear on either side of the equation and we ultimately get equation in one variable i.e. “ a ”. In the nutshell, the approach using fraction reduces variables in the resulting equation. This point will be highlighted at the appropriate point in the solution to appreciate why we should use fraction?

Now proceeding with the question, the vertical displacement of “B” is ah and that of “A” is $(1-a)h$. We observe here that each of the balls takes the same time to cover respective displacements. Using equation of constant acceleration in one dimension (considering downward vertical direction positive),

For ball “A”,

$$(1 - a)h = \frac{1}{2}gt^2$$

For ball “B”,

$$-ah = -ut + \frac{1}{2}gt^2$$

The value of “ t ” obtained from the equation of ball “A” is :

$$\Rightarrow t = \sqrt{\left\{ \frac{2(1-a)h}{g} \right\}}$$

Substituting in the equation of ball “B”, we have :

$$\begin{aligned}
&\Rightarrow -ah = -u\sqrt{\left\{\frac{2(1-a)h}{g}\right\}} + (1-a)h \\
&\Rightarrow u\sqrt{\left\{\frac{2(1-a)h}{g}\right\}} = h \\
&\Rightarrow u = \sqrt{\left\{\frac{gh}{2(1-a)}\right\}}
\end{aligned}$$

Now, according to the condition given in the question,

$$\begin{aligned}
v_A &= 2v_B \\
\Rightarrow v_A^2 &= 4v_B^2
\end{aligned}$$

Using equations of motion for constant acceleration, we have :

$$\begin{aligned}
&\Rightarrow 2g(1-a)h = 4(u^2 - 2gah) \\
&\Rightarrow 2g(1-a)h = 4\left\{\frac{gh}{2(1-a)} - 2gah\right\} \\
&\Rightarrow (1-a) = \left\{\frac{1}{(1-a)} - 4a\right\}
\end{aligned}$$

Recall our discussion on the use of fraction. We can write down the corresponding steps for conventional method to express displacements side by side and find that the conventional approach will lead us to an equation as :

$$(h - y) = \left\{\frac{h}{(h-y)} - 4y\right\}$$

Evidently, this is an equation in two variables and hence can not be solved. It is this precise limitation of conventional approach that we switched to fraction approach to get an equation in one variable,

$$\Rightarrow (1-a) = \left\{\frac{1}{(1-a)} - 4a\right\}$$

$$\Rightarrow (1 - a)^2 = 1 - 4a(1 - a)$$

$$\Rightarrow 1 + a^2 - 2a = 1 - 4a + a^2$$

$$\Rightarrow 3a^2 - 2a = 0$$

$$\Rightarrow a(3a - 2) = 0$$

$$\Rightarrow a = 0, \frac{2}{3}$$

Neglecting, $a = 0$,

$$\Rightarrow a = \frac{2}{3}$$

The height from the ground at which collision takes place is :

$$\Rightarrow y = ah = \frac{2h}{3}$$

Non-uniform acceleration

Non-uniform acceleration constitutes the most general description of motion. It refers to variation in the rate of change in velocity. Simply put, it means that acceleration changes during motion. This variation can be expressed either in terms of position (x) or time (t). We understand that if we can describe non-uniform acceleration in one dimension, we can easily extend the analysis to two or three dimensions using composition of motions in component directions. For this reason, we shall confine ourselves to the consideration of non-uniform i.e. variable acceleration in one dimension.

In this module, we shall describe non-uniform acceleration using expressions of velocity or acceleration in terms of either of time, “ t ”, or position, “ x ”. We shall also consider description of non-uniform acceleration by expressing acceleration in terms of velocity. As a matter of fact, there can be various possibilities. Besides, non-uniform acceleration may involve interpretation acceleration - time or velocity - time graphs.

Accordingly, analysis of non-uniform acceleration motion is carried out in two ways :

- Using calculus
- Using graphs

Analysis using calculus is generic and accurate, but is limited to the availability of expression of velocity and acceleration. It is not always possible to obtain an expression of motional attributes in terms of “ x ” or “ t ”. On the other hand, graphical method lacks accuracy, but this method can be used with precision if the graphs are composed of regular shapes.

Using calculus involves differentiation and integration. The integration allows us to evaluate expression of acceleration for velocity and evaluate expression of velocity for displacement. Similarly, differentiation allows us to evaluate expression of position for velocity and evaluate expression of velocity for acceleration. We have already worked with expression of position in time. We shall work here with other expressions. Clearly, we need to know a bit about differentiation and integration before we proceed to analyze non-uniform motion.

Important calculus results

Integration is anti-differentiation i.e. an inverse process. We can compare differentiation and integration of basic algebraic, trigonometric, exponential and logarithmic functions to understand the inverse relation between processes. In the next section, we list few important differentiation and integration results for reference.

Differentiation

$$\frac{d}{dx} x^n = nx^{n-1}; \quad \frac{d}{dx} (ax + b)^n = na(ax + b)^{n-1}$$

$$\frac{d}{dx} \sin ax = a \cos ax; \quad \frac{d}{dx} \cos ax = -a \sin ax$$

$$\frac{d}{dx} e^x = e^x; \quad \frac{d}{dx} \log_e x = \frac{1}{x}$$

Integration

$$\int x^n dx = \frac{x^{n+1}}{n+1}; \quad \int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)}$$

$$\int \sin ax dx = -\frac{\cos ax}{a}; \quad \int \cos ax dx = \frac{\sin ax}{a}$$

$$\int e^x dx = e^x; \quad \int \frac{dx}{x} = \log_e x$$

Velocity and acceleration is expressed in terms of time “t

Let the expression of acceleration in x is given as function a(t). Now, acceleration is related to velocity as :

$$a(t) = \frac{dv}{dt}$$

We obtain expression for velocity by rearranging and integrating :

$$\Rightarrow dv = a(t)dt$$

$$\Rightarrow \Delta v = \int a(t)dt$$

This relation yields an expression of velocity in "t" after using initial conditions of motion. We obtain expression for position/ displacement by using defining equation, rearranging and integrating :

$$v(t) = \frac{dx}{dt}$$

$$\Rightarrow dx = v(t)dt$$

$$\Rightarrow \Delta x = \int v(t)dt$$

This relation yields an expression of position in "t" after using initial conditions of motion.

Example:

Problem : The acceleration of a particle along x-axis varies with time as :

$$a = t - 1$$

Velocity and position both are zero at t= 0. Find displacement of the particle in the interval from t=1 to t = 3 s. Consider all measurements in SI unit.

Solution : Using integration result obtained earlier for expression of acceleration :

$$\Delta v = v_2 - v_1 = \int a(t)dt$$

Here, $v_1 = 0$. Let $v_2 = v$, then :

$$v = \int_0^t a(t) dt = \int_0^t (t - 1) dt$$

Note that we have integrated RHS between time 0 and t seconds in order to get an expression of velocity in “t”.

$$\Rightarrow v = \frac{t^2}{2} - t$$

Using integration result obtained earlier for expression of velocity :

$$\Delta x = \int v(t) dt = \int \left(\frac{t^2}{2} - t \right) dt$$

Integrating between t=1 and t=3, we have :

$$\Delta x = \int_1^3 \left(\frac{t^2}{2} - t \right) dt = \left[\frac{t^3}{6} - \frac{t^2}{2} \right]_1^3 = \frac{27}{6} - \frac{9}{2} - \frac{1}{6} + \frac{1}{2} = 5 - 4 = 1m$$

Example:

Problem : If the velocity of particle moving along a straight line is proportional to $\frac{3}{4}$ th power of time, then how do its displacement and acceleration vary with time?

Solution : Here, we need to find a higher order attribute (acceleration) and a lower order attribute displacement. We can find high order attribute by differentiation, whereas we can find lower order attribute by integration.

$$v \propto t^{\frac{3}{4}}$$

Let,

$$v = kt^{\frac{3}{4}}$$

where k is a constant. The acceleration of the particle is :

$$a = \frac{dv(t)}{dt} = \frac{3kt^{\frac{3}{4}-1}}{4} = \frac{3kt^{-\frac{1}{4}}}{4}$$

Hence,

$$v \propto t^{-\frac{1}{4}}$$

For lower order attribute “displacement”, we integrate the function :

$$dx = vdt$$

$$\Rightarrow \Delta x = \int vdt = \int kt^{\frac{3}{4}}dt = \frac{4kt^{\frac{7}{4}}}{7}$$

$$\Delta x \propto t^{\frac{7}{4}}$$

Velocity and acceleration is expressed in terms of time “x”

Let the expression of acceleration in x given as function a(x). Now, the defining equation of instantaneous acceleration is:

$$a(x) = \frac{dv}{dt}$$

In order to incorporate differentiation with position, “x”, we rearrange the equation as :

$$a(x) = \frac{dv}{dx} \times \frac{dx}{dt} = \frac{v dv}{dx}$$

We obtain expression for velocity by rearranging and integrating :

$$\int v dv = \int a(x) dx$$

This relation yields an expression of velocity in x. We obtain expression for position/ displacement by using defining equation, rearranging and integrating

:

$$v(x) = \frac{dx}{dt}$$

$$\Rightarrow dt = \frac{dx}{v(x)}$$

$$\Rightarrow \Delta t = \int \frac{dx}{v(x)}$$

This relation yields an expression of position in t.

Example:

Problem : The acceleration of a particle along x-axis varies with position as :

$$a = 9x$$

Velocity is zero at t = 0 and x=2m. Find speed at x=4. Consider all measurements in SI unit.

Solution : Using integration,

$$\int v dv = \int a(x) dx$$

$$\int_0^v v dv = \int_2^x 9x dx$$

$$\Rightarrow \left[\frac{v^2}{2} \right]_0^v = 9 \left[\frac{x^2}{2} \right]_2^x$$

$$\Rightarrow \frac{v^2}{2} = \frac{9x^2}{2} - \frac{36}{2}$$

$$\Rightarrow v^2 = 9x^2 - 36 = 9(x^2 - 4)$$

$$\Rightarrow v = \pm 3\sqrt{(x^2 - 4)}$$

We neglect negative sign as particle is moving in x-direction with positive acceleration. At $x = 4$ m,

$$\Rightarrow v = 3\sqrt{(4^2 - 4)} = 3\sqrt{12} = 6\sqrt{3} \text{ m/s}$$

Acceleration in terms of velocity

Let acceleration is expressed as function of velocity as :

$$a = a(v)$$

Obtaining expression of velocity in time, “t”

The acceleration is defined as :

Arranging terms with same variable on one side of the equation, we have :

$$dt = \frac{dv}{a(v)}$$

Integrating, we have :

$$\Rightarrow \Delta t = \int \frac{dv}{a(v)}$$

Evaluation of this relation results an expression of velocity in t. Clearly, we can proceed as before to obtain expression of position in t.

Obtaining expression of velocity in time, “x”

In order to incorporate differentiation with position, “x”, we rearrange the defining equation of acceleration as :

$$a(v) = \frac{dv}{dx} \times \frac{dx}{dt} = \frac{v dv}{dx}$$

Arranging terms with same variable on one side of the equation, we have :

$$dx = \frac{v dv}{a(v)}$$

Integrating, we have :

$$\Rightarrow \Delta x = \int \frac{v dv}{a(v)}$$

Evaluation of this relation results an expression of velocity in x. Clearly, we can proceed as before to obtain expression of position in t.

Example:

Problem : The motion of a body in one dimension is given by the equation $dv(t)/dt = 6.0 - 3v(t)$, where $v(t)$ is the velocity in m/s and “t” in seconds. If the body was at rest at $t = 0$, then find the expression of velocity.

Solution : Here, acceleration is given as a function in velocity as independent variable. We are required to find expression of velocity in time. Using integration result obtained earlier for expression of time :

$$\Rightarrow \Delta t = \int \frac{dv}{a(v)}$$

Substituting expression of acceleration and integrating between $t=0$ and $t=t$, we have :

$$\Rightarrow \Delta t = t = \int \frac{dv(t)}{6-3v(t)}$$

$$\Rightarrow t = \frac{\ln \left[\frac{6-3v(t)}{6} \right]}{-3}$$

$$\Rightarrow 6 - 3v(t) = 6e^{-3t}$$

$$\Rightarrow 3v(t) = 6(1 - e^{-3t})$$

$$\Rightarrow v(t) = 2(1 - e^{-3t})$$

Graphical method

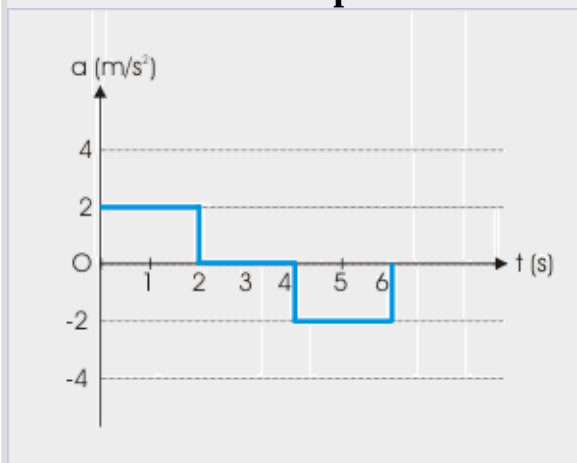
We analyze graphs of motion in parts keeping in mind regular geometric shapes involved in the graphical representation. Here, we shall work with two examples. One depicts variation of velocity with respect to time. Other depicts variation of acceleration with respect to time.

Velocity .vs. time

Example:

Problem : A particle moving in a straight line is subjected to accelerations as given in the figure below :

Acceleration – time plot



Acceleration – time plot

If $v = 0$ and $t = 0$, then draw velocity – time plot for the same time interval.

Solution : In the time interval between $t = 0$ and $t = 2$ s, acceleration is constant and is equal to 2 m/s^2 . Hence, applying equation of motion for final velocity, we have :

$$v = u + at$$

But, $u = 0$ and $a = 2 \text{ m/s}^2$.

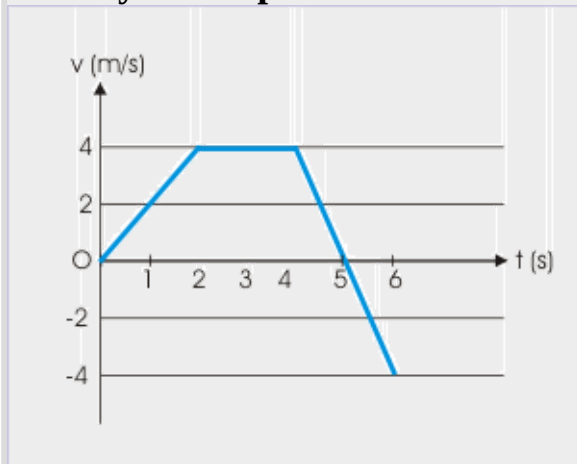
$$\Rightarrow v = 2t$$

The velocity at the end of 2 seconds is 4 m/s. Since acceleration is constant, the velocity – time plot for the interval is a straight line. On the other hand, the acceleration is zero in the time interval between $t = 2$ and $t = 4$ s. Hence, velocity remains constant in this time interval. For time interval $t = 4$ and 6 s, acceleration is -4 m/s^2 .

From the plot it is clear that the velocity at $t = 4$ s is equal to the velocity at $t = 2$ s, which is given by :

$$\Rightarrow u = 2 \times 2 = 4 \text{ m/s}$$

Velocity – time plot



Corresponding velocity – time plot.

Putting this value as initial velocity in the equation of velocity, we have :

$$\Rightarrow v = 4 - 4t$$

We should, however, be careful in drawing velocity plot for $t = 4 \text{ s}$ to $t = 6 \text{ s}$. We should realize that the equation above is valid in the time interval considered. The time $t = 4 \text{ s}$ from the beginning corresponds to $t = 0 \text{ s}$ and $t = 6 \text{ s}$ corresponds to $t = 2 \text{ s}$ for this equation.

Velocity at $t = 5 \text{ s}$ is obtained by putting $t = 1 \text{ s}$ in the equation,

$$\Rightarrow v = 4 - 4 \times 1 = 0$$

Velocity at $t = 6 \text{ s}$ is obtained by putting $t = 2 \text{ s}$ in the equation,

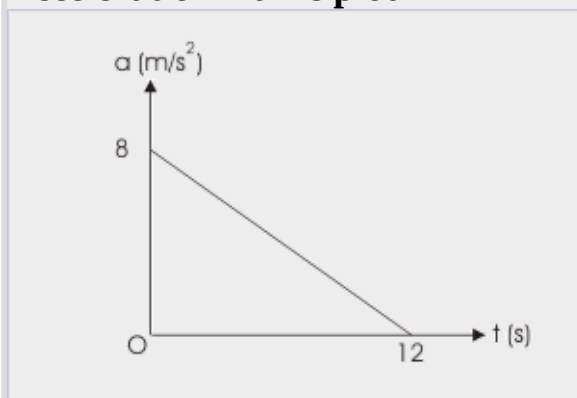
$$\Rightarrow v = 4 - 4 \times 2 = -4 \text{ m/s}$$

Acceleration .vs. time

Example:

Problem : A particle starting from rest and undergoes a rectilinear motion with acceleration “a”. The variation of “a” with time “t” is shown in the figure. Find the maximum velocity attained by the particle during the motion.

Acceleration – time plot



The area under plot gives change in velocity.

Solution : We see here that the particle begins from rest and is continuously accelerated in one direction (acceleration is always positive through out the motion) at a diminishing rate. Note that though acceleration is decreasing, but remains positive for the motion. It means that the particle attains maximum velocity at the end of motion i.e. at $t = 12$ s.

The area under the plot on acceleration – time graph gives change in velocity. Since the plot start at $t = 0$ i.e. the beginning of the motion, the areas under the plot gives the velocity at the end of motion,

$$\Delta v = v_2 - v_1 = v - 0 = v = \frac{1}{2} \times 8 \times 12 = 48 \text{ m/s}$$

$$v_{\max} = 48 \text{ m/s}$$

Exercises

- First identify : what is given and what is required. Establish relative order between given and required attribute.
- Use differentiation method to get a higher order attribute in the following order : displacement (position vector) $\rightarrow$ velocity $\rightarrow$ acceleration.
- Use integration method to get a lower order attribute in the following order : acceleration $\rightarrow$ velocity $\rightarrow$ displacement (position vector).
- Since we are considering accelerated motion in one dimension, graphical representation of motion is valid. The interpretation of plot in terms of slope of the curve gives higher order attribute, whereas interpretation of plot in terms of area under the plot gives lower order attribute.

Exercise:

Problem:

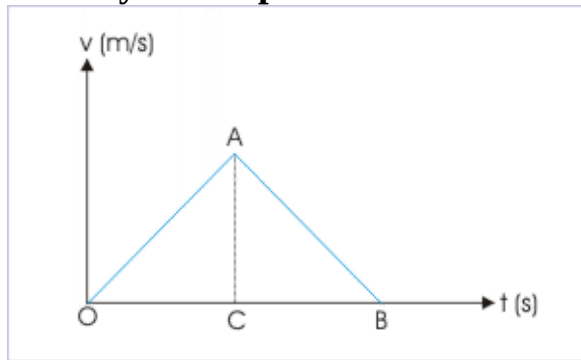
A particle of mass m moves on the x -axis. It starts from rest $t = 0$ from the point $x = 0$, and comes to rest at $t = 1$ at the point $x = 1$. No other information is available about its motion at intermediate time ($0 < t < 1$). Prove that magnitude of instantaneous acceleration during the motion can not be less than 4 m/s^2 .

Solution:

In between the starting and end point, the particle may undergo any combination of acceleration and deceleration. According to question, we are required to know the minimum value of acceleration for any combination possible.

Let us check the magnitude of acceleration for the simplest combination and then we evaluate other complex possible scenarios. Now, the simplest scenario would be that the particle first accelerates and then decelerates for equal time and at the same rate to complete the motion.

Velocity – time plot



Velocity – time plot

In order to assess the acceleration for a linear motion , we would make use of the fact that area under v-t plot gives the displacement (= 1 m). Hence,

$$\Delta OAB = \frac{1}{2} \times vt = \frac{1}{2} \times v \times 1 = 1 \text{ m}$$

$$\Rightarrow v_{\max} = 2 \text{ m/s}$$

Thus, the maximum velocity during motion under this condition is 2 m/s. On the other hand, the acceleration for this motion is equal to the slope of the line,

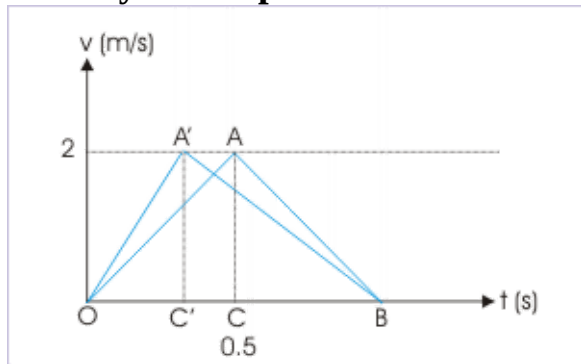
$$a = \tan \theta = \frac{2}{0.5} = 4 \text{ m/s}^2$$

Now let us complicate the situation, in which particle accelerates for shorter period and decelerates gently for a longer period (see figure

below). This situation would result velocity being equal to 2 m/s and acceleration greater than 4 m/s^2 .

It is so because time period is fixed (1 s) and hence base of the triangle is fixed. Also as displacement in given time is 1 m. It means that area under the triangle is also fixed (1 m). As area is half of the product of base and height, it follows that the height of the triangle should remain same i.e 2 m/s. For this reason the graphical representations of possible variation of acceleration and deceleration may look like as shown in the figure here.

Velocity – time plot



Area under velocity – time plot gives displacement, whereas slope of the plot gives the acceleration.

Clearly, the slope of OA' is greater than 4 m/s^2 .

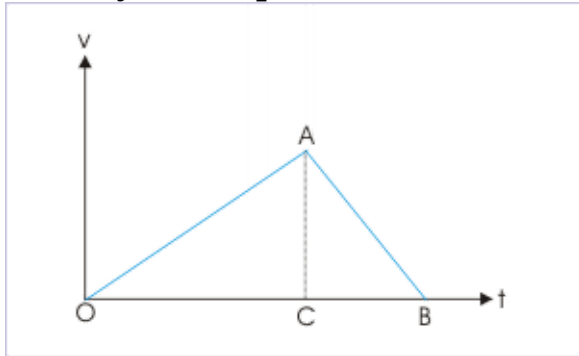
We may argue that why to have a single combination of acceleration and deceleration? What if the particle undergoes two cycles of acceleration and deceleration? It is obvious that such consideration will again lead to similar analysis for symmetric and non-symmetric acceleration, which would be bounded by the value of time and area of the triangle.

Thus we conclude that the minimum value of acceleration is 4 m/s^2 .

Exercise:

Problem:

A particle accelerates at constant rate “a” from rest at and then decelerates at constant rate “b” to come to rest. If the total time elapsed is “t” during this motion, then find (i) maximum velocity achieved and (ii) total displacement during the motion.

Velocity – time plot

Area under velocity – time plot gives displacement , whereas slope of the plot gives the acceleration.

Solution:

The particle initially accelerates at a constant rate. It means that its velocity increases with time. The particle, therefore, gains maximum velocity say, $v_{\max}$, before it begins to decelerate. Let t_1 and t_2 be the time intervals, for which particle is in acceleration and deceleration respectively. Then,

$$t = t_1 + t_2$$

From the triangle OAC, the magnitude of acceleration is equal to the slope of straight line OA. Hence,

$$a = \frac{v_{\max}}{t_1}$$

Similarly from the triangle CAB, the magnitude of acceleration is equal to the slope of straight line AB. Hence,

$$b = \frac{v_{\max}}{t_2}$$

Substituting values of time, we have :

$$t = \frac{v_{\max}}{a} + \frac{v_{\max}}{b} = v_{\max} \left(\frac{a+b}{ab} \right)$$

$$v_{\max} = \frac{abt}{(a+b)}$$

In order to find the displacement, we shall use the fact that area under velocity – time plot is equal to displacement. Hence,

$$x = \frac{1}{2} \times v_{\max} t$$

Putting the value of maximum velocity, we have :

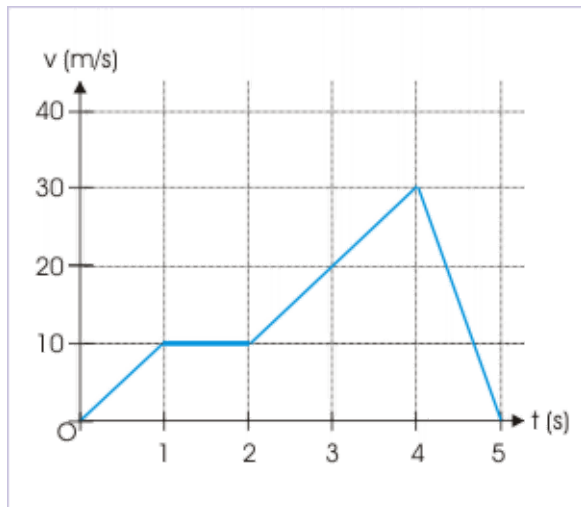
$$\Rightarrow x = \frac{abt^2}{2(a+b)}$$

Exercise:

Problem:

The velocity – time plot of the motion of a particle along x - axis is as shown in the figure. If $x = 0$ at $t = 0$, then (i) analyze acceleration of particle during the motion (ii) draw corresponding displacement – time plot (showing nature of the plot) and (iii) find total displacement.

Velocity – time plot



Slope of the plot gives
acceleration; area under plot
gives displacement.

Solution:

Graphically, slope of the plot yields acceleration and area under the plot yields displacement.

Since the plot involves regular geometric shapes, we shall find areas of these geometric shapes individually and then add them up to get various points for displacement – time plot.

The displacement between times 0 and 1 s is :

$$\Delta x = \frac{1}{2} \times 1 \times 10 = 5m$$

The velocity between 0 and 1 s is increasing at constant rate. The acceleration is constant and its magnitude is equal to the slope, which is $10/1 = 10 \text{ m/s}^2$. Since acceleration is in the direction of motion, the particle is accelerated. The resulting displacement – time curve is a parabola with an increasing slope.

The displacement between times 1 and 2 s is :

$$\Delta x = 1 \times 10 = 10m$$

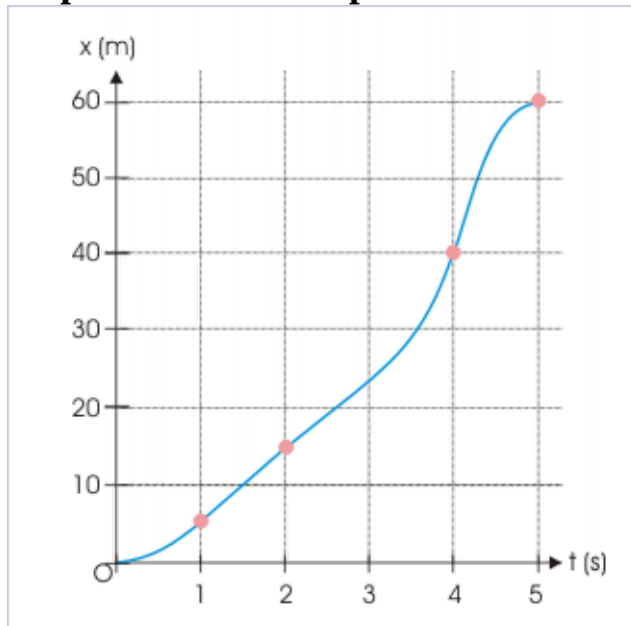
The velocity between 1 and 2 s is constant and as such acceleration is zero. The resulting displacement – time curve for this time interval is a straight line having slope equal to that of the magnitude of velocity.

The displacement at the end of 2 s is $5 + 10 = 15$ m

Now, the displacement between times 2 and 4 s is :

$$\Delta x = \frac{1}{2} \times (20 + 30) \times 1 = 25m$$

Displacement – time plot



Corresponding displacement – time plot of the given velocity- - time plot.

The velocity between 2 and 4 s is increasing at constant rate. The acceleration, therefore, is constant and its magnitude is equal to the slope of the plot, $\frac{20}{2} = 10 \text{ m/s}^2$. Since acceleration is in the direction of motion, the particle is accelerated and the resulting displacement – time curve is a parabola with increasing slope.

The displacement at the end of 4 s is $5 + 10 + 25 = 40$ m

Now, the displacement between times 4 and 5 s is :

$$\Delta x = \frac{1}{2} \times 30 \times 1 = 15m$$

The velocity between 2 and 4 s is decreasing at constant rate, but always remains positive. As such, the acceleration is constant, but negative and its magnitude is equal to the slope, which is $^{30}_1 = 30 \text{ m/s}^2$. Since acceleration is in the opposite direction of motion, the particle is decelerated. The resulting displacement – time curve is an inverted parabola with decreasing slope.

The final displacement at the end of 5 s is $5 + 10 + 25 + 15 = 55$ m

Characteristics of motion : One dimensional, Unidirectional, variable acceleration in one dimension (magnitude)

Exercise:

Problem:

The motion of a body in one dimension is given by the equation $dv(t)/dt = 6.0 - 3v(t)$, where $v(t)$ is the velocity in m/s and “t” in seconds. If the body was at rest at $t = 0$, then find the terminal speed.

Solution:

In order to answer this question, we need to know the meaning of terminal speed. The terminal speed is the speed when the body begins to move with constant speed. Now, the first time derivate of velocity gives the acceleration of the body :

$$\frac{dv(t)}{dt} = a = 6 - 3v(t)$$

For terminal motion, $a = 0$. Hence,

$$a = 6 - 3v(t) = 0$$

$$v(t) = \frac{6}{3} = 2 \text{ m/s}$$

Exercise:**Problem:**

The motion of a body in one dimension is given by the equation $dv(t)/dt = 6.0 - 3v(t)$, where $v(t)$ is the velocity in m/s and “t” in seconds. If the body was at rest at $t = 0$, then find the magnitude of the initial acceleration and also find the velocity, when the acceleration is half the initial value.

Solution:

We have to find the acceleration at $t = 0$ from the equation given as :

$$\frac{dv(t)}{dt} = a = 6 - 3v(t)$$

Generally, we would have been required to know the velocity function so that we can evaluate the function for time $t = 0$. In this instant case, though the function of velocity is not known, but we know the value of velocity at time $t = 0$. Thus, we can know acceleration at $t = 0$ by just substituting the value for velocity function at that time instant as :

$$a = 6 - 3v(t) = 6 - 3 \times 0 = 6 \text{ m/s}^2$$

Now when the acceleration is half the initial acceleration,

$$\Rightarrow a' = \frac{a}{2} = \frac{6}{2} = 3 \text{ m/s}^2$$

Hence,

$$\Rightarrow 6 - 3v(t) = 3$$

$$\Rightarrow v(t) = \frac{3}{3} = 1 \text{ m/s}$$

Relative velocity in one dimension

The measurements, describing motion, are subject to the state of motion of the frame of reference with respect to which measurements are made. Our day to day perception of motion is generally earth's view – a view common to all bodies at rest with respect to earth. However, we encounter occasions when there is perceptible change to our earth's view. One such occasion is traveling on the city trains. We find that it takes lot longer to overtake another train on a parallel track. Also, we see two people talking while driving separate cars in the parallel lane, as if they were stationary to each other!. In terms of kinematics, as a matter of fact, they are actually stationary to each other - even though each of them are in motion with respect to ground.

In this module, we set ourselves to study motion from a perspective other than that of earth. Only condition we subject ourselves is that two references or two observers making the measurements of motion of an object, are moving at constant velocity (We shall learn afterward that two such reference systems moving with constant velocity is known as inertial frames, where Newton's laws of motion are valid.).

The observers themselves are not accelerated. There is, however, no restriction on the motion of the object itself, which the observers are going to observe from different reference systems. The motion of the object can very well be accelerated. Further, we shall study relative motion for two categories of motion : (i) one dimension (in this module) and (ii) two dimensions (in another module). We shall skip three dimensional motion – though two dimensional study can easily be extended to three dimensional motion as well.

Relative motion in one dimension

We start here with relative motion in one dimension. It means that the individual motions of the object and observers are along a straight line with only two possible directions of motion.

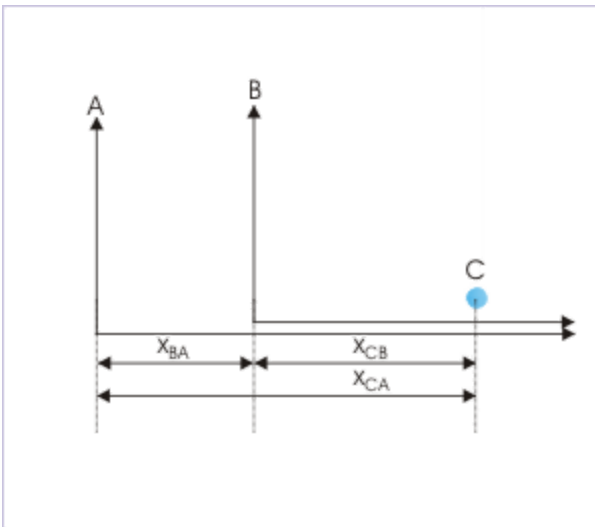
Position of the point object

We consider two observers “A” and “B”. The observer “A” is at rest with earth, whereas observer “B” moves with a velocity v_{BA} with respect to the observer “A”. The two observers watch the motion of the point like object “C”. The motions of “B” and “C” are along the same straight line.

Note: It helps to have a convention about writing subscripted symbol such as v_{BA} . The first subscript indicates the entity possessing the attribute (here velocity) and second subscript indicates the entity with respect to which measurement is made. A velocity like v_{BA} shall, therefore, mean velocity of “B” with respect to “A”.

The position of the object “C” as measured by the two observers “A” and “B” are x_{CA} and x_{CB} as shown in the figure. The observers are represented by their respective frame of reference in the figure.

Position



Here,

$$x_{CA} = x_{BA} + x_{CB}$$

Velocity of the point object

We can obtain velocity of the object by differentiating its position with respect to time. As the measurements of position in two references are different, it is expected that velocities in two references are different, because one observer is at rest, whereas other observer is moving with constant velocity.

$$v_{CA} = \frac{dx_{CA}}{dt}$$

and

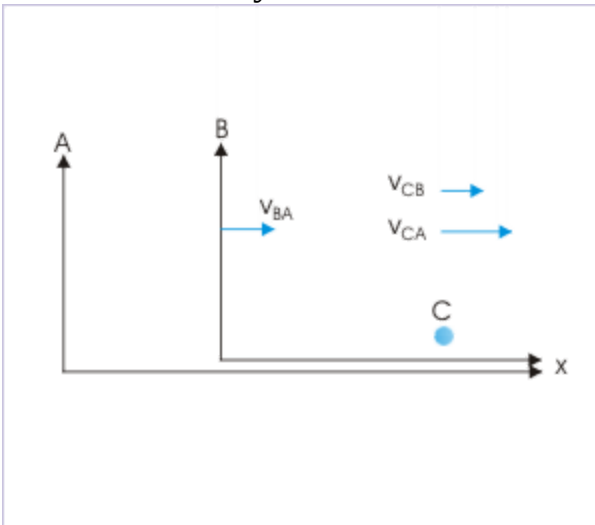
$$v_{CB} = \frac{dx_{CB}}{dt}$$

Now, we can obtain relation between these two velocities, using the relation $x_{CA} = x_{BA} + x_{CB}$ and differentiating the terms of the equation with respect to time :

$$\frac{dx_{CA}}{dt} = \frac{dx_{BA}}{dt} + \frac{dx_{CB}}{dt}$$

$$\Rightarrow v_{CA} = v_{BA} + v_{CB}$$

Relative velocity



The meaning of the subscripted velocities are :

- v_{CA} : velocity of object "C" with respect to "A"
- v_{CB} : velocity of object "C" with respect to "B"
- v_{BA} : velocity of object "B" with respect to "A"

Example:

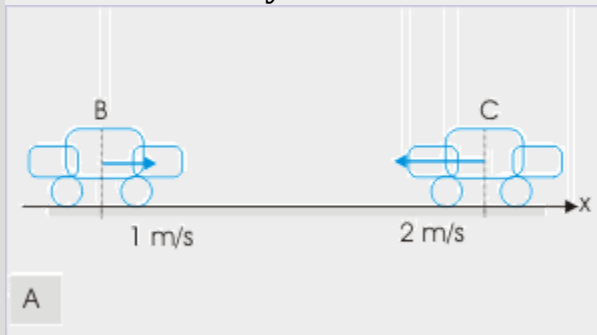
Problem : Two cars, standing a distance apart, start moving towards each other with speeds 1 m/s and 2 m/s along a straight road. What is the speed with which they approach each other ?

Solution : Let us consider that "A" denotes Earth, "B" denotes first car and "C" denotes second car. The equation of relative velocity for this case is :

$$\Rightarrow v_{CA} = v_{BA} + v_{CB}$$

Here, we need to fix a reference direction to assign sign to the velocities as they are moving opposite to each other and should have opposite signs. Let us consider that the direction of the velocity of B is in the reference direction, then

Relative velocity



$$v_{BA} = 1 \text{ m/s} \text{ and } v_{CA} = -2 \text{ m/s}.$$

Now :

$$v_{CA} = v_{BA} + v_{CB}$$

$$\Rightarrow -2 = 1 + v_{CB}$$

$$\Rightarrow v_{CB} = -2 - 1 = -3 \text{ m/s}$$

This means that the car "C" is approaching "B" with a speed of -3 m/s along the straight road. Equivalently, it means that the car "B" is approaching "C" with a speed of 3 m/s along the straight road. We, therefore, say that the two cars approach each other with a relative speed of 3 m/s.

Acceleration of the point object

If the object being observed is accelerated, then its acceleration is obtained by the time derivative of velocity. Differentiating equation of relative velocity, we have :

$$\begin{aligned}v_{CA} &= v_{BA} + v_{CB} \\ \Rightarrow \frac{dv_{CA}}{dt} &= \frac{dv_{BA}}{dt} + \frac{dv_{CB}}{dt} \\ \Rightarrow a_{CA} &= a_{BA} + a_{CB}\end{aligned}$$

The meaning of the subscripted accelerations are :

- a_{CA} : acceleration of object "C" with respect to "A"
- a_{CB} : acceleration of object "C" with respect to "B"
- a_{BA} : acceleration of object "B" with respect to "A"

But we have restricted ourselves to reference systems which are moving at constant velocity. This means that relative velocity of "B" with respect to "A" is a constant. In other words, the acceleration of "B" with respect to "A" is zero i.e. $a_{BA} = 0$. Hence,

$$\Rightarrow a_{CA} = a_{CB}$$

The observers moving at constant velocity, therefore, measure same acceleration of the object. As a matter of fact, this result is characteristics of inertial frame of reference. The reference frames, which measure same acceleration of an object, are inertial frames of reference.

Interpretation of the equation of relative velocity

The important aspect of relative velocity in one dimension is that velocity has only two possible directions. We need not use vector notation to write or evaluate equation of relative velocities in one dimension. The velocity, therefore, can be treated as signed scalar variable; plus sign indicating velocity in the reference direction and minus sign indicating velocity in opposite to the reference direction.

Equation with reference to earth

The equation of relative velocities refers velocities in relation to different reference system.

$$v_{CA} = v_{BA} + v_{CB}$$

We note that two of the velocities are referred to A. In case, “A” denotes Earth’s reference, then we can conveniently drop the reference. A velocity without reference to any frame shall then mean Earth’s frame of reference.

$$\Rightarrow v_C = v_B + v_{CB}$$

$$\Rightarrow v_{CB} = v_C - v_B$$

This is an important relation. This is the working relation for relative motion in one dimension. We shall be using this form of equation most of the time, while working with problems in relative motion. This equation can be used effectively to determine relative velocity of two moving objects with uniform velocities (C and B), when their velocities in Earth’s reference are known. Let us work out an exercise, using new notation and see the ease of working.

Example:

Problem : Two cars, initially 100 m distant apart, start moving towards each other with speeds 1 m/s and 2 m/s along a straight road. When would

they meet ?

Solution : The relative velocity of two cars (say 1 and 2) is :

$$v_{21} = v_2 - v_1$$

Let us consider that the direction v_1 is the positive reference direction. Here, $v_1 = 1$ m/s and $v_2 = -2$ m/s. Thus, relative velocity of two cars (of 2 w.r.t 1) is :

$$\Rightarrow v_{21} = -2 - 1 = -3 \text{ m/s}$$

This means that car "2" is approaching car "1" with a speed of -3 m/s along the straight road. Similarly, car "1" is approaching car "2" with a speed of 3 m/s along the straight road. Therefore, we can say that two cars are approaching at a speed of 3 m/s. Now, let the two cars meet after time "t" :

$$t = \frac{\text{Displacement}}{\text{Relative velocity}} = \frac{100}{3} = 33.3 \text{ s}$$

Order of subscript

There is slight possibility of misunderstanding or confusion as a result of the order of subscript in the equation. However, if we observe the subscript in the equation, it is easy to formulate a rule as far as writing subscript in the equation for relative motion is concerned. For any two subscripts say "A" and "B", the relative velocity of "A" (first subscript) with respect to "B" (second subscript) is equal to velocity of "A" (first subscript) subtracted by the velocity of "B" (second subscript) :

$$v_{AB} = v_A - v_B$$

and the relative velocity of B (first subscript) with respect to A (second subscript) is equal to velocity of B (first subscript) subtracted by the velocity of A (second subscript):

$$v_{BA} = v_B - v_A$$

Evaluating relative velocity by making reference object stationary

An inspection of the equation of relative velocity points to an interesting feature of the equation. We need to emphasize that the equation of relative velocity is essentially a vector equation. In one dimensional motion, we have taken the liberty to write them as scalar equation :

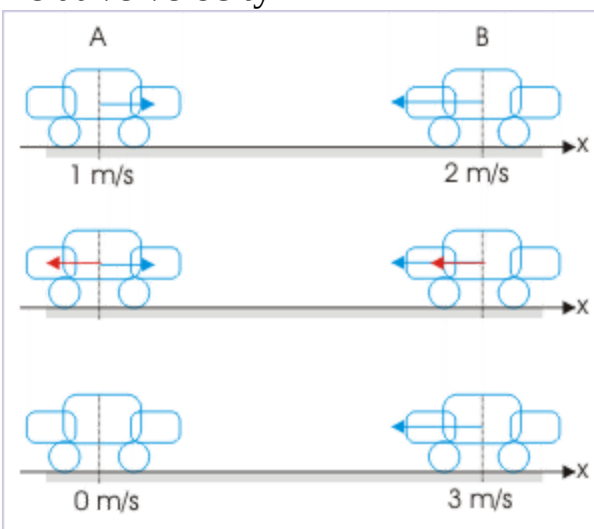
$$v_{BA} = v_B - v_A$$

Now, the equation comprises of two vector quantities (v_B and $-v_A$) on the right hand side of the equation. The vector “ $-v_A$ ” is actually the negative vector i.e. a vector equal in magnitude, but opposite in direction to “ v_A ”. Thus, we can evaluate relative velocity as following :

1. Apply velocity of the reference object (say object "A") to both objects and render the reference object at rest.
2. The resultant velocity of the other object ("B") is equal to relative velocity of "B" with respect to "A".

This concept of rendering the reference object stationary is explained in the figure below. In order to determine relative velocity of car "B" with reference to car "A", we apply velocity vector of car "A" to both cars. The relative velocity of car "B" with respect to car "A" is equal to the resultant velocity of car "B".

Relative velocity



This technique is a very useful tool for consideration of relative motion in two dimensions.

Direction of relative velocities

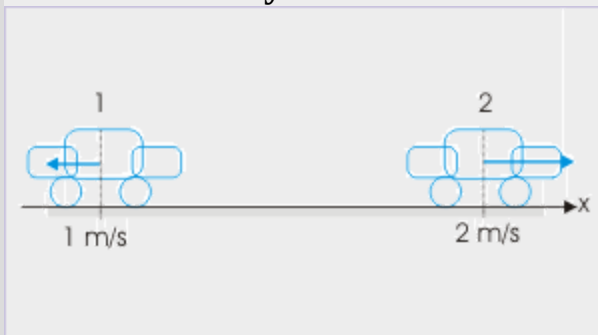
For a pair of two moving objects moving uniformly, there are two values of relative velocity corresponding to two reference frames. The values differ only in sign – not in magnitude. This is clear from the example here.

Example:

Problem : Two cars start moving away from each other with speeds 1 m/s and 2 m/s along a straight road. What are relative velocities ? Discuss the significance of their sign.

Solution : Let the cars be denoted by subscripts “1” and “2”. Let us also consider that the direction v_2 is the positive reference direction, then relative velocities are :

Relative velocity



$$\Rightarrow v_{12} = v_1 - v_2 = -1 - 2 = -3 \text{ m/s}$$

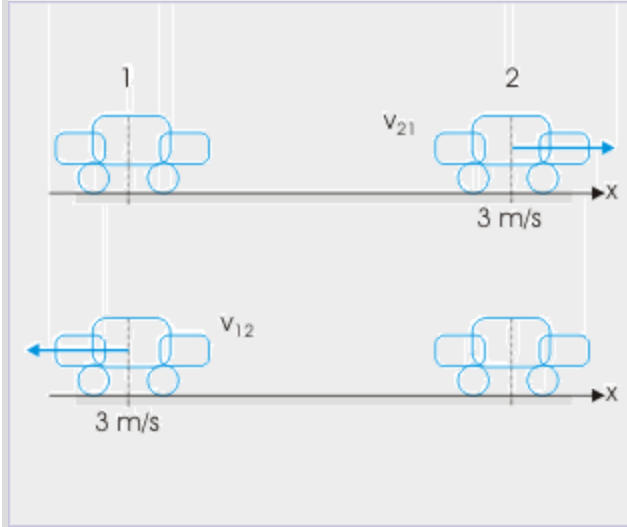
$$\Rightarrow v_{21} = v_2 - v_1 = 2 - (-1) = 3 \text{ m/s}$$

The sign attached to relative velocity indicates the direction of relative velocity with respect to reference direction. The directions of relative velocity are different, depending on the reference object.

However, two relative velocities with different directions mean same physical situation. Let us read the negative value first. It means that car 1

moves away from car 2 at a speed of 3 m/s in the direction opposite to that of car 2. This is exactly the physical situation. Now for positive value of relative velocity, the value reads as car 2 moves from car 1 in the direction of its own velocity. This also is exactly the physical situation. There is no contradiction as far as physical interpretation is concerned. Importantly, the magnitude of approach – whatever be the sign of relative velocity – is same.

Relative velocity



Relative velocity .vs. difference in velocities

It is very important to understand that relative velocity refers to two moving bodies – not a single body. Also that relative velocity is a different concept than the concept of "difference of two velocities", which may pertain to the same or different objects. The difference in velocities represents difference of “final” velocity and “initial” velocity and is independent of any order of subscript. In the case of relative velocity, the order of subscripts are important. The expression for two concepts viz relative velocity and difference in velocities may look similar, but they are different concepts.

Relative acceleration

We had restricted our discussion up to this point for objects, which moved with constant velocity. The question, now, is whether we can extend the concept of relative velocity to acceleration as well. The answer is yes. We can attach similar meaning to most of the quantities - scalar and vector both. It all depends on attaching physical meaning to the relative concept with respect to a particular quantity. For example, we measure potential energy (a scalar quantity) with respect to an assumed datum.

Extending concept of relative velocity to acceleration is done with the restriction that measurements of individual accelerations are made from the same reference.

If two objects are moving with different accelerations in one dimension, then the relative acceleration is equal to the net acceleration following the same working relation as that for relative velocity. For example, let us consider that an object designated as "1" moves with acceleration " a_1 " and the other object designated as "2" moves with acceleration " a_2 " along a straight line. Then, relative acceleration of "1" with respect to "2" is given by :

$$\Rightarrow a_{12} = a_1 - a_2$$

Similarly, relative acceleration of "2" with respect to "1" is given by :

$$\Rightarrow a_{21} = a_2 - a_1$$

Worked out problems

Example:

Relative motion

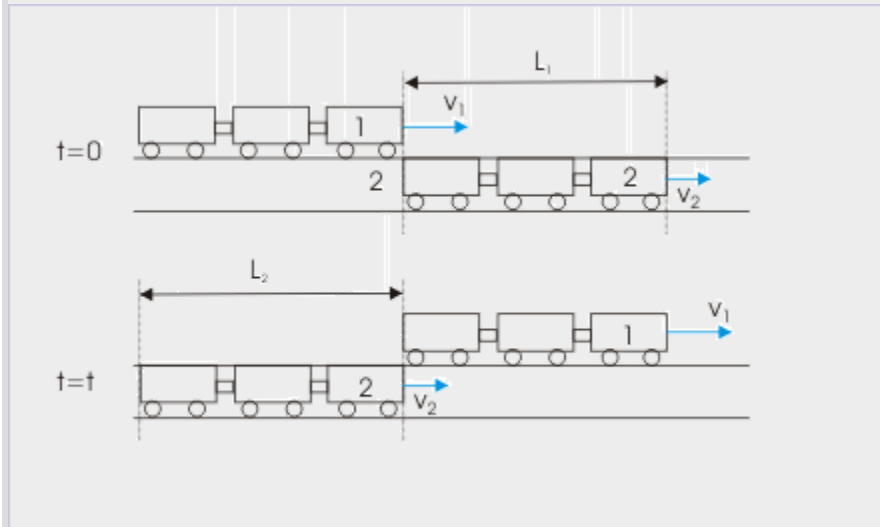
Problem : Two trains are running on parallel straight tracks in the same direction. The train, moving with the speed of 30 m/s overtakes the train ahead, which is moving with the speed of 20 m/s. If the train lengths are 200 m each, then find the time elapsed and the ground distance covered by the trains during overtake.

Solution : First train, moving with the speed of 30 m/s overtakes the second train, moving with the speed of 20 m/s. The relative speed with which first train overtakes the second train,

$$v_{12} = v_1 - v_2 = 30 - 20 = 10 \text{ m/s.}$$

The figure here shows the initial situation, when faster train begins to overtake and the final situation, when faster train goes past the slower train. The total distance to be covered is equal to the sum of each length of the trains ($L_1 + L_2$) i.e. $200 + 200 = 400 \text{ m}$. Thus, time taken to overtake is :

The total relative distance



The total relative distance to cover during overtake is equal to the sum of lengths of each train.

$$t = \frac{400}{10} = 40 \text{ s.}$$

In this time interval, the two trains cover the ground distance given by:

$$s = 30 \times 40 + 20 \times 40 = 1200 + 800 = 2000 \text{ m.}$$

Exercise:

Problem:

In the question given in the example, if the trains travel in the opposite direction, then find the time elapsed and the ground distance covered by the trains during the period in which they cross each other.

Solution:

$$v_{12} = v_1 - v_2 = 30 - (-20) = 50 \text{ m/s.}$$

The total distance to be covered is equal to the sum of each length of the trains i.e. $200 + 200 = 400 \text{ m}$. Thus, time taken to overtake is :

$$t = \frac{400}{50} = 8 \text{ s.}$$

Now, in this time interval, the two trains cover the ground distance given by:

$$s = 30 \times 8 + 20 \times 8 = 240 + 160 = 400 \text{ m.}$$

In this case, we find that the sum of the lengths of the trains is equal to the ground distance covered by the trains, while crossing each other.

Check your understanding

Check the module titled [Relative velocity in one dimension \(Check your understanding\)](#) to test your understanding of the topics covered in this module.

Relative velocity in one dimension(application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1. Foremost thing in solving problems of relative motion is about visualizing measurement. If we say a body "A" has relative velocity " v " with respect to another moving body "B", then we simply mean that we are making measurement from the moving frame (reference) of "B".
2. The concept of relative velocity applies to two objects. It is always intuitive to designate one of the objects as moving and other as reference object.
3. It is helpful in solving problem to make reference object stationary by applying negative of its velocity to both objects. The resultant velocity of the moving object is equal to the relative velocity of the moving object with respect to reference object. If we interpret relative velocity in this manner, it gives easy visualization as we are accustomed to observing motion from stationary state.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the relative velocity in one dimension.

Example:

Problem : A jet cruising at a speed of 1000 km/hr ejects hot air in the opposite direction. If the speed of hot air with respect to Jet is 800 km/hr, then find its speed with respect to ground.

Solution : Let the direction of Jet be x – direction. Also, let us denote jet with “A” and hot air with “B”. Here,

$$v_A = 1000 \text{ km / hr}$$

$$v_B = ?$$

$$v_{BA} = -800 \text{ km / hr}$$

Now,

$$v_{BA} = v_B - v_A$$

$$v_B = v_A + v_{AB}$$

$$= 1000 + (-800) = 200 \text{ km / hr}$$

The speed of the hot air with respect to ground is 200 km/hr.

Example:

Problem : If two bodies, at constant speeds, move towards each other, then the linear distance between them decreases at 6 km/hr. If the bodies move in the same direction with same speeds, then the linear distance between them increases at 2 km/hr. Find the speeds of two bodies (in km/hr).

Solution : Let the speeds of the bodies are “u” and “v” respectively. When they move towards each other, the relative velocity between them is :

$$v_{AB} = v_A - v_B = u - (-v) = u + v = 6 \text{ km / hr}$$

When they are moving in the same direction, the relative velocity between them is :

$$v_{AB} = v_A - v_B = u - v = 2 \text{ km / hr}$$

Solving two linear equations, we have :

$$u = 4 \text{ km/hr}$$

$$v = 2 \text{ km/hr}$$

Example:

Problem : Two cars A and B move along parallel paths from a common point in a given direction. If “u” and “v” be their speeds ($u > v$), then find the separation between them after time “t”.

Solution : The relative velocity of the cars is :

$$v_{12} = v_1 - v_2 = u - v$$

The separation between the cars is :

$$x = v_{12}t = (u - v)t$$

Example:

Problem : Two trains of length 100 m each, running on parallel track, take 20 seconds to overtake and 10 seconds to cross each other. Find their speeds (in m/s).

Solution : The distance traveled in the two cases is $100 + 100 = 200\text{m}$. Let their speeds be “u” and “v”. Now, relative velocity for the overtake is :

$$v_{12} = v_1 - v_2 = u - v$$

$$200 = (u - v) \times 20$$

$$\Rightarrow u - v = 10$$

The relative velocity to cross each other is :

$$v_{12}' = v_1 - v_2 = u - (-v) = u + v$$

$$200 = (u + v) \times 10$$

$$\Rightarrow u + v = 20$$

Solving two linear equations, we have :

$u = 15 \text{ m/s}$ and $v = 5 \text{ m/s}$.

Example:

Problem : Two cars are moving in the same direction at the same speed, "u". The cars maintain a linear distance "x" between them. An another car (third car) coming from opposite direction meets the two cars at an interval of "t", then find the speed of the third car.

Solution : Let "u" be the speed of either of the two cars and "v" be the speed of the third car. The relative velocity of third car with respect to either of the two cars is :

$$v_{\text{rel}} = u + v$$

The distance between the two cars remains same as they are moving at equal speeds in the same direction. The events of meeting two cars separated by a distance "x" is described in the context of relative velocity. We say equivalently that the third car is moving with relative velocity "u+v", whereas the first two cars are stationary. Thus, distance covered by the third car with relative velocity is given by :

$$x = (u + v)t$$

$$v = \frac{(x-ut)}{t} = \frac{x}{t} - u$$

Example:

Problem : Two cars, initially at a separation of 12 m, start simultaneously. First car "A", starting from rest, moves with an acceleration 2 m/s^2 , whereas the car "B", which is ahead, moves with a constant velocity 1 m/s along the same direction. Find the time when car "A" overtakes car "B".

Solution : We shall attempt this question, using concept of relative velocity. First we need to understand what does the event "overtaking" mean? Simply said, it is the event when both cars are at the same position at a particular time instant. Subsequently, the car coming from behind moves ahead.

Now relative velocity has an appropriate interpretation suiting to this situation. The relative velocity of "A" with respect to "B" means that the

car “B” is stationary, while car “A” moves with relative velocity. Thus, problem reduces to a simple motion of a single body moving with a certain velocity (relative velocity) and covering a certain linear distance (here separation of 12 m) with certain acceleration. Note that car “B” has no acceleration.

Relative velocity of “A” with respect to “B”, i.e. v_{AB} , at the start of motion is :

$$v_{AB} = v_A - v_B = 0 - 1 = 1 \text{ m/s}$$

Relative acceleration of “A” with respect to “B”, i.e. a_{AB} is :

$$a_{AB} = a_A - a_B = 2 - 0 = 2 \text{ m/s}^2$$

Applying equation of motion for a single body (remember the second body is rendered stationary),

$$\begin{aligned}x &= ut + \frac{1}{2}at^2 \\ \Rightarrow 12 &= -1 \times t + \frac{1}{2} \times 2 \times t^2 \\ \Rightarrow t^2 - t - 12 &= 0 \\ \Rightarrow t^2 + 3t - 4t - 12 &= 0 \\ \Rightarrow (t + 3)(t - 4) &= 0\end{aligned}$$

Neglecting negative value,

$$\Rightarrow t = 4 \text{ s}$$

Note : We can solve this problem without using the concept of relative velocity as well. The car “A” moves 12 m more than the distance covered by “B” at the time of overtake. If car “B” moves by “x” meters, then car “A” travels “x+12” meters. It means that :

$$\text{Displacement of “A”} = \text{Displacement of “B”} + 12$$

$$\frac{1}{2}at^2 = vt + 12$$

Putting values,

$$\Rightarrow \frac{1}{2} \times 2 \times t^2 = 1 \times t + 12$$
$$\Rightarrow t^2 - t - 12 = 0$$

This is the same equation as obtained earlier. Hence,

$$\Rightarrow t = 4 \text{ s}$$

Evidently, it appears to be much easier when we do not use the concept of relative velocity. This is the case with most situations in one dimensional motion of two bodies. It is easier not to use concept of relative velocity in one dimensional motion to the extent possible. We shall learn subsequently, however, that the concept of relative velocity has an edge in analyzing two dimensional motion, involving two bodies.

Relative velocity in two dimensions

The concept of relative motion in two or three dimensions is exactly same as discussed for the case of one dimension. The motion of an object is observed in two reference systems as before – the earth and a reference system, which moves with constant velocity with respect to earth. The only difference here is that the motion of the reference system and the object ,being observed, can take place in two dimensions. The condition that observations be carried out in inertial frames is still a requirement to the scope of our study of relative motion in two dimensions.

As a matter of fact, theoretical development of the equation of relative velocity is so much alike with one dimensional case that the treatment in this module may appear repetition of the text of earlier module. However, application of relative velocity concept in two dimensions is different in content and details, requiring a separate module to study the topic.

Relative motion in two dimensions

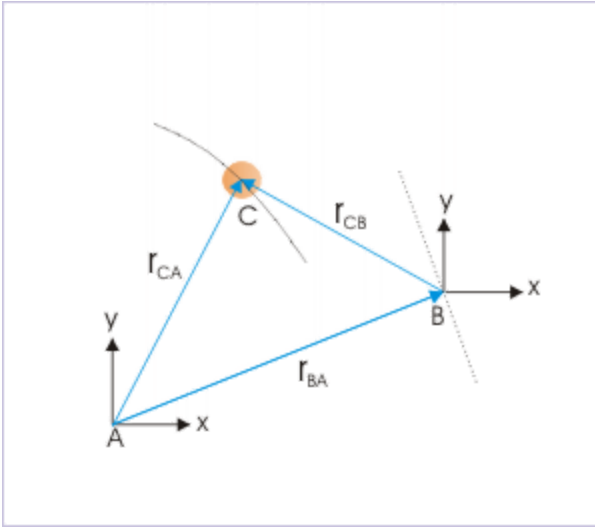
The important aspect of relative motion in two dimensions is that we can not denote vector attributes of motion like position, velocity and acceleration as signed scalars as in the case of one dimension. These attributes can now have any direction in two dimensional plane (say “xy” plane) and as such they should be denoted with either vector notations or component scalars with unit vectors.

Position of the point object

We consider two observers A and B. The observer “A” is at rest with respect to earth, whereas observer “B” moves with a constant velocity with respect to the observer on earth i.e. “A”. The two observers watch the motion of the point like object “C”. The motions of “B” and “C” are as shown along dotted curves in the figure below. Note that the path of observer "B" is a straight line as it is moving with constant velocity. However, there is no such restriction on the motion of object C, which can be accelerated as well.

The position of the object “C” as measured by the two observers “A” and “B” are $\mathbf{r}_{CA}$ and $\mathbf{r}_{CB}$. The observers are represented by their respective frame of reference in the figure.

Positions



The observers are represented by their respective frame of reference.

Here,

$$\mathbf{r}_{CA} = \mathbf{r}_{BA} + \mathbf{r}_{CB}$$

Velocity of the point object

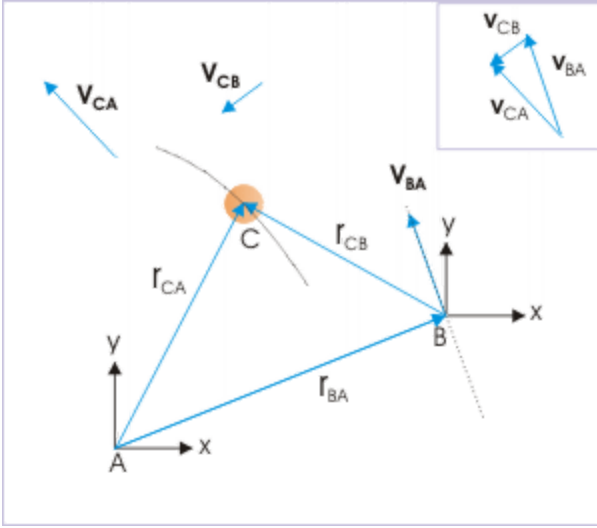
We can obtain velocity of the object by differentiating its position with respect to time. As the measurements of position in two references are different, it is expected that velocities in two references are different,

$$\mathbf{v}_{CA} = \frac{d\mathbf{r}_{CA}}{dt}$$

and

$$\mathbf{v}_{CB} = \frac{d\mathbf{r}_{CB}}{dt}$$

The velocities of the moving object “C” ($\mathbf{v}_{CA}$ and $\mathbf{v}_{CB}$) as measured in two reference systems are shown in the figure. Since the figure is drawn from the perspective of “A” i.e. the observer on the ground, the velocity $\mathbf{v}_{CA}$ of the object "C" with respect to "A" is tangent to the curved path. Velocity



The observers measure different velocities.

Now, we can obtain relation between these two velocities, using the relation $\mathbf{r}_{CA} = \mathbf{r}_{BA} + \mathbf{r}_{CB}$ and differentiating the terms of the equation with respect to time as :

$$\frac{d\mathbf{r}_{CA}}{dt} = \frac{d\mathbf{r}_{BA}}{dt} + \frac{d\mathbf{r}_{CB}}{dt}$$

$$\Rightarrow \mathbf{v}_{CA} = \mathbf{v}_{BA} + \mathbf{v}_{CB}$$

The meaning of the subscripted velocities are :

- $\mathbf{v}_{CA}$: velocity of object "C" with respect to "A"
- $\mathbf{v}_{CB}$: velocity of object "C" with respect to "B"

- $\mathbf{v}_{BA}$: velocity of object "B" with respect to "A"

Example:

Problem : A boy is riding a cycle with a speed of $5\sqrt{3}$ m/s towards east along a straight line. It starts raining with a speed of 15 m/s in the vertical direction. What is the direction of rainfall as seen by the boy.

Solution : Let us denote Earth, boy and rain with symbols A, B and C respectively. The question here provides the velocity of B and C with respect to A (Earth).

$$v_{BA} = 5\sqrt{3} \text{ m/s}$$

$$v_{CA} = 15 \text{ m/s}$$

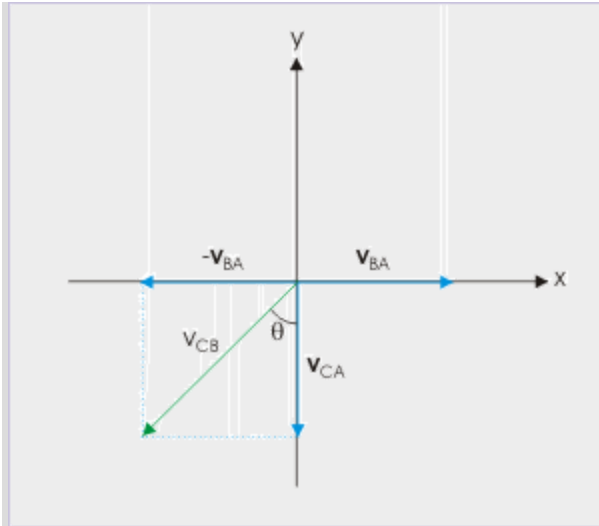
We need to determine the direction of rain (C) with respect to boy (B) i.e. $\mathbf{v}_{CB}$.

$$\mathbf{v}_{CA} = \mathbf{v}_{BA} + \mathbf{v}_{CB}$$

$$\Rightarrow \mathbf{v}_{CB} = \mathbf{v}_{CA} - \mathbf{v}_{BA}$$

We now draw the vector diagram to evaluate the terms on the right side of the equation. Here, we need to evaluate “ $\mathbf{v}_{CA} - \mathbf{v}_{BA}$ ”, which is equivalent to “ $\mathbf{v}_{CA} + (-\mathbf{v}_{BA})$ ”. We apply parallelogram theorem to obtain vector sum as represented in the figure.

Relative velocity



For the boy (B), the rain appears to fall, making an angle “ θ ” with the vertical (-y direction).

$$\Rightarrow \tan \theta = \frac{v_{BA}}{v_{CA}} = \frac{5\sqrt{3}}{15} = \frac{1}{\sqrt{3}} = \tan 30^\circ$$

$$\Rightarrow \theta = 30^\circ$$

Acceleration of the point object

If the object being observed is accelerated, then its acceleration is obtained by the time derivative of velocity. Differentiating equation of relative velocity, we have :

$$\Rightarrow \frac{d\mathbf{v}_{CA}}{dt} = \frac{d\mathbf{v}_{BA}}{dt} + \frac{d\mathbf{v}_{CB}}{dt}$$

$$\Rightarrow \mathbf{a}_{CA} = \mathbf{a}_{BA} + \mathbf{a}_{CB}$$

The meaning of the subscripted accelerations are :

- $\mathbf{a}_{CA}$: acceleration of object "C" with respect to "A"
- $\mathbf{a}_{CB}$: acceleration of object "C" with respect to "B"
- $\mathbf{a}_{BA}$: acceleration of object "B" with respect to "A"

But we have restricted ourselves to reference systems which are moving at constant velocity. This means that relative velocity of "B" with respect to "A" is a constant. In other words, the acceleration of "B" with respect to "A" is zero i.e. $\mathbf{a}_{BA} = 0$. Hence,

$$\Rightarrow \mathbf{a}_{CA} = \mathbf{a}_{CB}$$

The observers moving at constant velocities, therefore, measure same acceleration of the object (C).

Interpretation of the equation of relative velocity

The interpretation of the equation of relative motion in two dimensional motion is slightly tricky. The trick is entirely about the ability to analyze vector quantities as against scalar quantities. There are few alternatives at our disposal about the way we handle the vector equation.

In broad terms, we can either use graphical techniques or vector algebraic techniques. In graphical method, we can analyze the equation with graphical representation along with analytical tools like Pythagoras or Parallelogram theorem of vector addition as the case may be. Alternatively, we can use vector algebra based on components of vectors.

Our approach shall largely be determined by the nature of inputs available for interpreting the equation.

Equation with reference to earth

The equation of relative velocities refers velocities in relation to different reference system.

$$\mathbf{v}_{CA} = \mathbf{v}_{BA} + \mathbf{v}_{CB}$$

We note that two of the velocities are referred to A. In case, "A" denotes Earth's reference, then we can conveniently drop the reference. A velocity without reference to any frame shall then mean Earth's frame of reference.

$$\Rightarrow \mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$$

$$\Rightarrow \mathbf{v}_{CB} = \mathbf{v}_C - \mathbf{v}_B$$

This is an important relation. As a matter of fact, we shall require this form of equation most of the time, while working with problems in relative motion. This equation can be used effectively to determine relative velocity of two moving objects with uniform velocity (C and B), when their velocities in Earth's reference are known.

Note : As in the case of one dimensional case, we can have a working methodology to find the relative velocity in two dimensions. In brief, we drop the reference to ground all together. We simply draw two velocities as given $\mathbf{v}_A$, $\mathbf{v}_B$. Then, we reverse the direction of reference velocity $\mathbf{v}_B$ and find the resultant relative velocity, $\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$, applying parallelogram theorem or using algebraic method involving unit vectors.

In general, for any two objects "A" and "B", moving with constant velocities,

$$\Rightarrow \mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

Example:

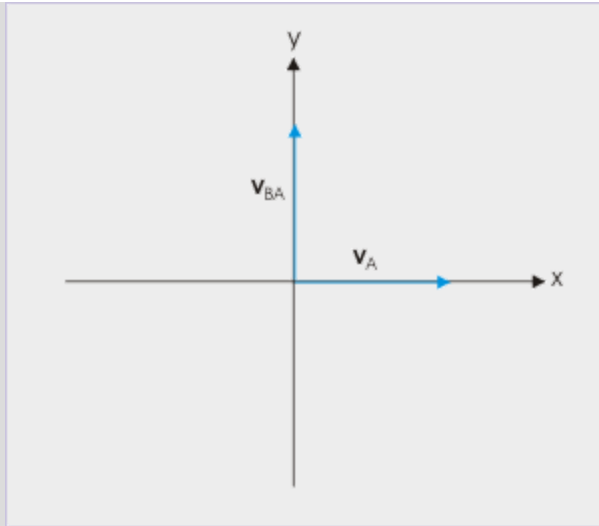
Problem : A person is driving a car towards east at a speed of 80 km/hr. A train appears to move towards north with a velocity of $80\sqrt{3}$ km/hr to the person driving the car. Find the speed of the train as measured with respect to earth.

Solution : Let us first identify the car and train as "A" and "B". Here, we are provided with the speed of car ("A") with respect to Earth i.e. " v_A " and speed of train ("B") with respect to "A" i.e v_{BA} .

$$v_A = 80 \text{ km / hr}$$

$$v_{BA} = 80\sqrt{3} \text{ km / hr}$$

Relative velocity



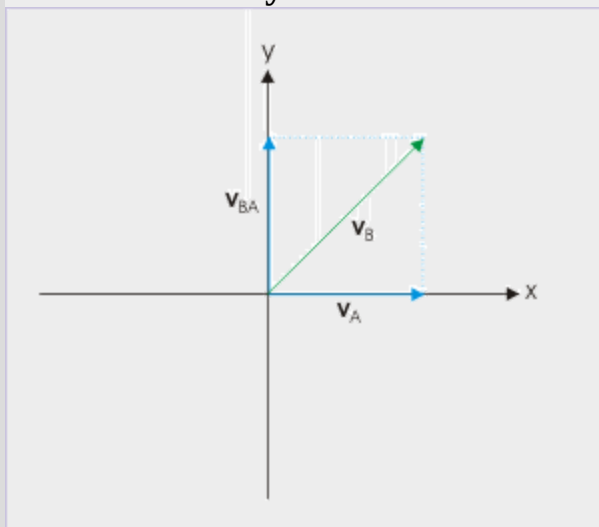
We are required to find the speed of train (“B”) with respect to Earth i.e. $\mathbf{v}_B$, . From equation of relative motion, we have :

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$$

$$\Rightarrow \mathbf{v}_B = \mathbf{v}_{BA} + \mathbf{v}_A$$

To evaluate the right hand side of the equation, we draw vectors “ $\mathbf{v}_{BA}$ ” and “ $\mathbf{v}_A$ ” and use parallelogram law to find the actual speed of the train.

Relative velocity



$$\Rightarrow v_B = \sqrt{v_{BA}^2 + v_A^2} = \sqrt{(80\sqrt{3})^2 + 80^2} = 160 \text{ km / hr}$$

Evaluation of equation using analytical technique

We have already used analytical method to evaluate vector equation of relative velocity. Analytical method makes use of Pythagoras or Parallelogram theorem to determine velocities.

Analytical method, however, is not limited to making use of Pythagoras or Parallelogram theorem. Depending on situation, we may use simple trigonometric relation as well to evaluate equation of relative motion in two dimensions. Let us work out an exercise to emphasize application of such geometric (trigonometric) analytical technique.

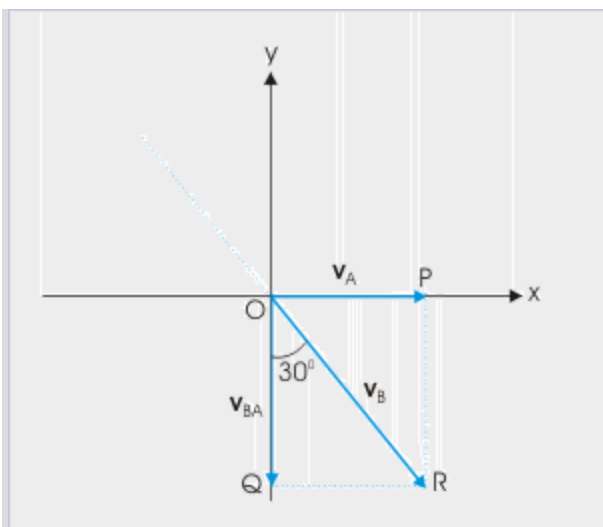
Example:

Problem : A person, standing on the road, holds his umbrella to his back at an angle 30° with the vertical to protect himself from rain. He starts running at a speed of 10 m/s along a straight line. He finds that he now has to hold his umbrella vertically to protect himself from the rain. Find the speed of raindrops as measured with respect to (i) ground and (ii) the moving person.

Solution : Let us first examine the inputs available in this problem. To do this let us first identify different entities with symbols. Let A and B denote the person and the rain respectively. The initial condition of the person gives the information about the direction of rain with respect to ground - notably not the speed with which rain falls. It means that we know the direction of velocity $\mathbf{v}_B$. The subsequent condition, when person starts moving, tells us the velocity of the person "A" with respect to ground i.e $\mathbf{v}_A$. Also, it is given that the direction of relative velocity of rain "B" with respect to the moving person "A" is vertical i.e. we know the direction of relative velocity $\mathbf{v}_{BA}$.

We draw three vectors involved in the problem as shown in the figure. OP represents $\mathbf{v}_A$; OQ represents $\mathbf{v}_B$; OR represents $\mathbf{v}_{BA}$.

Relative velocity



In $\triangle OCB$,

$$v_B = OR = \frac{QR}{\sin 30^\circ}$$

$$\Rightarrow v_B = \frac{10}{\frac{1}{2}} = 20 \text{ m/s}$$

and

$$v_{BA} = OQ = \frac{QR}{\tan 30^\circ}$$

$$\Rightarrow v_{BA} = \frac{10}{\frac{1}{\sqrt{3}}} = 10\sqrt{3} \text{ m/s}$$

Equation in component form

So far we have used analytical method to evaluate vector equation of relative velocity. It is evident that vector equation also renders to component form – particularly when inputs are given in component form along with unit vectors.

Here, we shall highlight one very important aspect of component analysis, which helps us to analyze complex problems. The underlying concept is that consideration of motion in mutually perpendicular direction is independent of each other. This aspect of independence is emphasized in

analyzing projectile motion, where motions in vertical and horizontal directions are found to be independent of each other (it is an experimental fact).

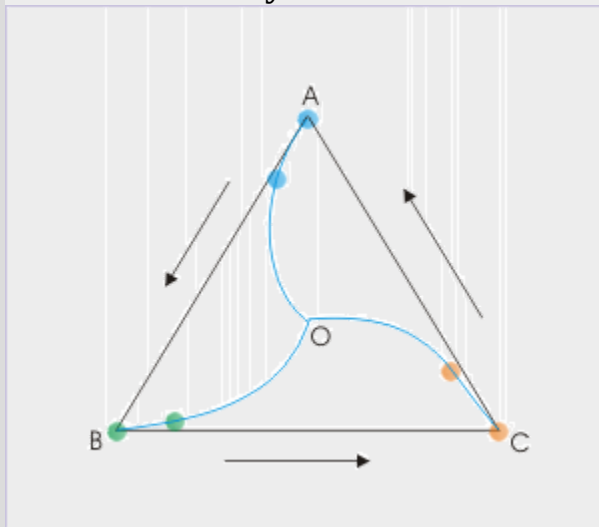
We work out the exercise to illustrate the application of the technique, involving component analysis.

Example:

Problem : Three particles A,B and C situated at the vertices of an equilateral triangle starts moving simultaneously at a constant speed “ v ” in the direction of adjacent particle, which falls ahead in the anti-clockwise direction. If “ a ” be the side of the triangle, then find the time when they meet.

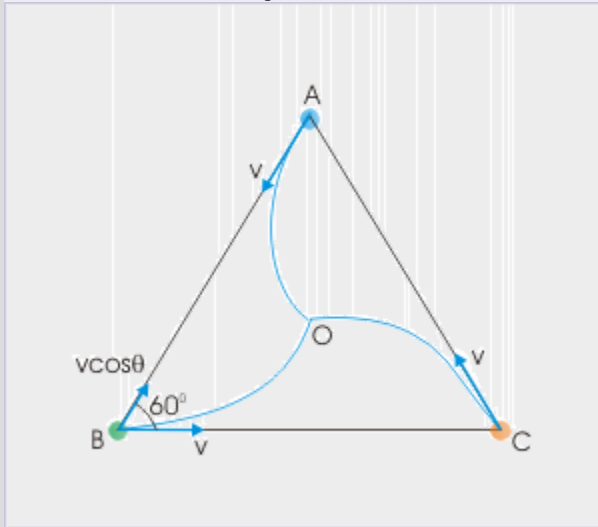
Solution : Here, particle “A” follows “B”, “B” follows “C” and “C” follows “A”. The direction of motion of each particle keeps changing as motion of each particle is always directed towards other particle. The situation after a time “ t ” is shown in the figure with a possible outline of path followed by the particles before they meet.

Relative velocity



This problem appears to be complex as the path of motion is difficult to be defined. But, it has a simple solution in component analysis. Let us consider the pair “A” and “B”. The initial component of velocities in the direction of line joining the initial position of the two particles is “ v ” and “ $v\cos\theta$ ” as shown in the figure here :

Relative velocity



The component velocities are directed towards each other. Now, considering the linear (one dimensional) motion in the direction of AB, the relative velocity of “A” with respect to “B” is :

$$v_{AB} = v_A - v_B$$

$$v_{AB} = v - (-v \cos \theta) = v + v \cos \theta$$

In equilateral triangle, $\theta = 60^\circ$,

$$v_{AB} = v + v \cos 60^\circ = v + \frac{v}{2} = \frac{3v}{2}$$

The time taken to cover the displacement “a” i.e. the side of the triangle,

$$t = \frac{2a}{3v}$$

Check your understanding

Check the module titled [Relative velocity in two dimensions \(application\)](#) to test your understanding of the topics covered in this module.

Relative velocity in two dimensions (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1. Solution of problems involving relative motion in two dimensions involves evaluation of vector equation. The evaluation or analysis of vector equation is not limited to the use of pythagoras theorem, but significantly makes use of goemtric consideration like evaluating trigonometirc ratios.
2. Generally, we attempt graphical solution. This is so because graphical solution is intuitive and indicative of actual physical phenomenon. However, most of the problem can equally be handled with the help of algebraic vector analysis, involving unit vectors.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the relative velocity in two dimensions. The questions are categorized in terms of the characterizing features of the subject matter :

- Velocity of an individual object
- Relative velocity
- Closest approach

Velocity of an individual object

Example:

Problem : A man, moving at 3 km/hr along a straight line, finds that the rain drops are falling at 4 km/hr in vertical direction. Find the angle with which rain drop hits the ground.

Solution : Let the man be moving in x-direction. Let us also denote man with “A” and rain drop with “B”. Here, we need to know the direction of rain drop with respect to ground i.e. the direction of $\mathbf{v}_B$.

Here,

$$v_A = 3 \text{ km / hr}$$

$$v_B = ?$$

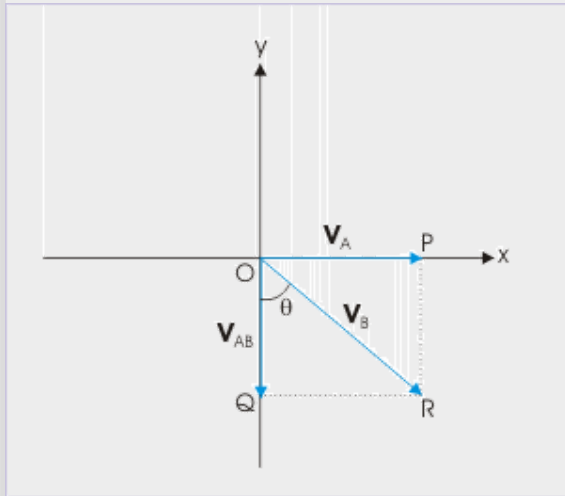
$$v_{BA} = 4 \text{ km / hr : in the vertical direction}$$

Using equation, $\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$,

$$\Rightarrow \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}$$

In order to evaluate the right hand side of the equation, we construct the vector diagram as shown in the figure.

Relative motion in two dimensions



From inspection of given data and using appropriate trigonometric function in ΔOBR , we have :

$$\tan \theta = \frac{3}{4} = \tan 37^\circ$$

$$\Rightarrow \theta = 37^\circ$$

Example:

Problem : A person, moving at a speed of 1 m/s, finds rain drops falling (from back) at 2 m/s at an angle 30° with the vertical. Find the speed of raindrop (m/s)

with which it hits the ground.

Solution : Let the person be moving in x-direction. Let us also denote man with “A” and rain drop with “B”. Here we need to know the speed of the rain drops with respect to ground i.e. $\mathbf{v}_B$.

Here,

$$v_A = 1 \text{ m/s}$$

$$v_B = ?$$

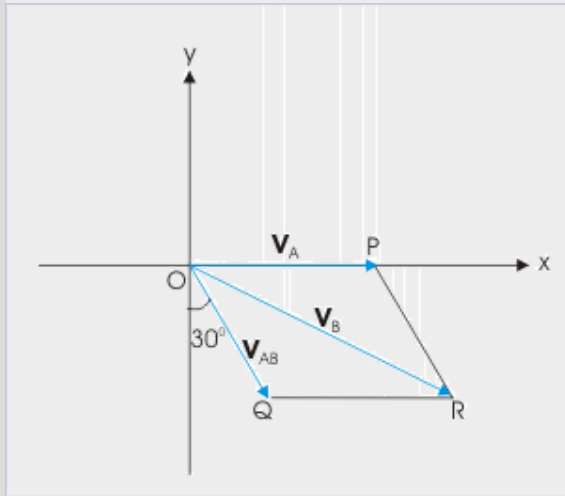
$$v_{BA} = 2 \text{ m/s}$$

Using equation, $\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$,

$$\Rightarrow \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}$$

In order to evaluate the right hand side of the equation, we construct the vector diagram as shown in the figure.

Relative motion in two dimensions



From parallelogram theorem,

$$v_B = \sqrt{(v_A^2 + v_{BA}^2 + 2v_A v_{BA} \cos 60^\circ)}$$

$$\Rightarrow v_B = \sqrt{(1^2 + 2^2 + 2 \times 1 \times 2 \times \frac{1}{2})}$$

$$\Rightarrow v_B = \sqrt{(1 + 4 + 2)} = \sqrt{7} \text{ m/s}$$

Example:

Problem : A boy moves with a velocity $0.5\mathbf{i} - \mathbf{j}$ in m/s. He receives rains at a velocity $0.5\mathbf{i} - 2\mathbf{j}$ in m/s. Find the speed at which rain drops meet the ground.

Solution : Let the person be moving along OA. Let us also denote man with “A” and rain drop with “B”. Here we need to know the speed at which rain drops fall on the ground (v_B).

Here,

$$\mathbf{v}_A = 0.5\mathbf{i} - \mathbf{j}$$

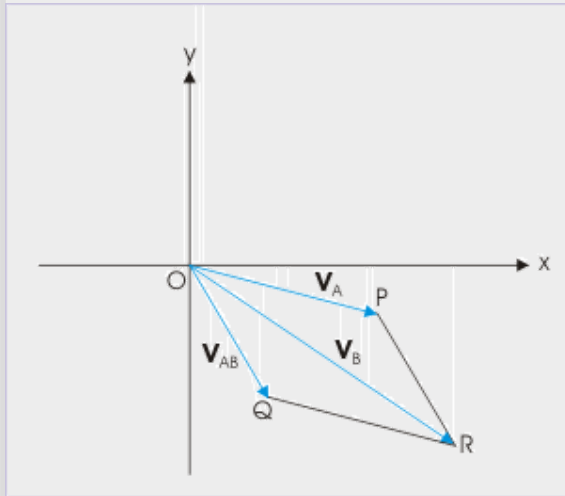
$$v_B = ?$$

$$\mathbf{v}_{BA} = 0.5\mathbf{i} - 2\mathbf{j}$$

Using equation, $\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$,

$$\Rightarrow \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}$$

Relative motion in two dimensions



$$\Rightarrow \mathbf{v}_B = 0.5\mathbf{i} - \mathbf{j} + 0.5\mathbf{i} - 2\mathbf{j} = \mathbf{i} - 3\mathbf{j}$$

$$\Rightarrow v_B = \sqrt{(1 + 9)} = \sqrt{(10)} \text{ m/s}$$

Relative velocity

Example:

Problem : Rain drop appears to fall in vertical direction to a person, who is walking at a velocity 3 m/s in a given direction. When the person doubles his velocity in the same direction, the rain drop appears to come to make an angle 45° from the vertical. Find the speed of the rain drop.

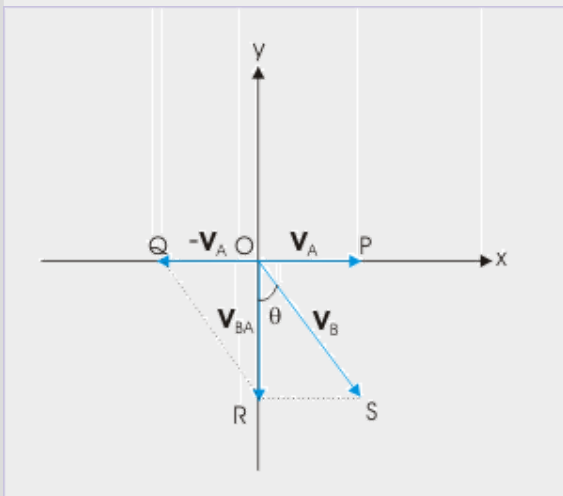
Solution : This is a slightly tricky question. Readers may like to visualize the problem and solve on their own before going through the solution given here. Let us draw the situations under two cases. Here, only the directions of relative velocities in two conditions are given. The figure on left represents initial situation. Here, the vector OP represents velocity of the person ($\mathbf{v}_A$); OR represents relative velocity of rain drop with respect to person ($\mathbf{v}_{BA}$); OS represents velocity of rain drop.

The figure on right represents situation when person starts moving with double velocity. Here, the vector OT represents velocity of the person ($\mathbf{v}_{A1}$); OW represents relative velocity of rain drop with respect to person ($\mathbf{v}_{BA1}$). We should note that velocity of rain ($\mathbf{v}_B$) drop remains same and as such, it is represented by OS represents as before.

Relative velocity

Raindrop appears to fall vertically.

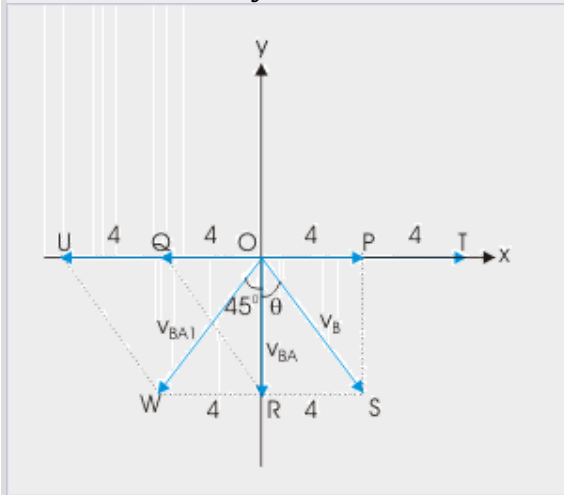
Raindrop appears to fall at angle of 45° .



[missing_resource: rmt6.gif]

According to question, we are required to know the speed of raindrop. It means that we need to know the angle " θ " and the side OS, which is the magnitude of velocity of raindrop. It is intuitive from the situation that it would help if consider the vector diagram and carry out geometric analysis to find these quantities. For this, we substitute the vector notations with known magnitudes as shown here.

Relative velocity



Values of different segments are shown.

We note here that

$$WR = UQ = 4 \text{ m/s}$$

Clearly, triangles ORS and ORW are congruent triangles as two sides and one enclosed angle are equal.

$$WR = RS = 4 \text{ m/s}$$

$$OR = OR$$

$$\angle ORW = \angle ORS = 90^\circ$$

Hence,

$$\angle WOR = \angle SOR = 45^\circ$$

In triangle ORS,

$$\sin 45^\circ = \frac{RS}{OS}$$

$$\Rightarrow OS = \frac{RS}{\sin 45^\circ} = 4\sqrt{2} \text{ m/s}$$

Example:

Problem : Rain drop appears to fall in vertical direction to a person, who is walking at a velocity 3 m/s in a given direction. When the person doubles his velocity in the same direction, the rain drop appears to come to make an angle 45° from the vertical. Find the speed of the rain drop, using unit vectors.

Solution : It is the same question as earlier one, but is required to be solved using unit vectors. The solution of the problem in terms of unit vectors gives us an insight into the working of algebraic analysis and also let us appreciate the power and elegance of using unit vectors to find solution of the problem, which otherwise appears to be difficult.

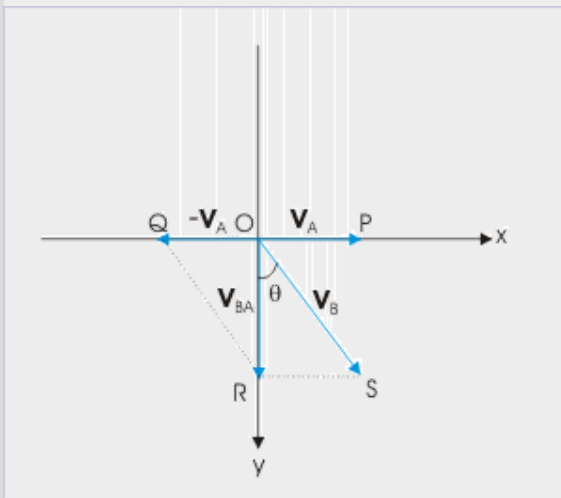
Let the velocity of raindrop be :

$$\mathbf{v}_B = a\mathbf{i} + b\mathbf{j}$$

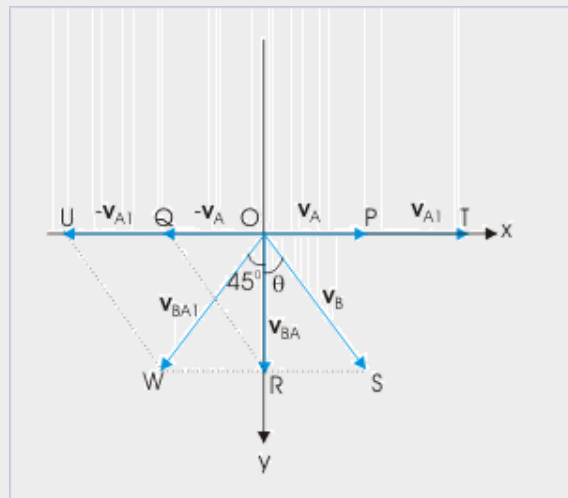
where “a” and “b” are constants. Note here that we have considered vertically downward direction as positive.

Relative velocity

Raindrop appears to fall vertically.



Raindrop appears to fall at angle of 45° .



According to question,

$$\mathbf{v}_A = 4\mathbf{i}$$

The relative velocity of raindrop with respect to person is :

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A = a\mathbf{i} + b\mathbf{j} - 4\mathbf{i} = (a - 4)\mathbf{i} + b\mathbf{j}$$

However, it is given that the raindrop appears to fall in vertical direction. It means that relative velocity has no component in x-direction. Hence,

$$a - 4 = 0$$

$$a = 4 \quad m/s$$

and

$$\mathbf{v}_{BA} = b\mathbf{j}$$

In the second case,

$$\mathbf{v}_{A_1} = 8\mathbf{i}$$

$$\mathbf{v}_{BA_1} = \mathbf{v}_B - \mathbf{v}_A = a\mathbf{i} + b\mathbf{j} - 8\mathbf{i} = (a - 8)\mathbf{i} + b\mathbf{j} = (4 - 8)\mathbf{i} + b\mathbf{j} = -4\mathbf{i} + b\mathbf{j}$$

It is given that the raindrop appears to fall in the direction, making an angle 45° with the vertical.

$$\tan 135^\circ = -1 = -\frac{4}{b}$$

Note that tangent of the angle is measured from positive x – direction ($90^\circ + 45^\circ = 135^\circ$) of the coordinate system.

$$b = 4$$

Thus, velocity of the raindrop is :

$$\mathbf{v}_B = a\mathbf{i} + b\mathbf{j} = -4\mathbf{i} + b\mathbf{j}$$

The speed of raindrop is :

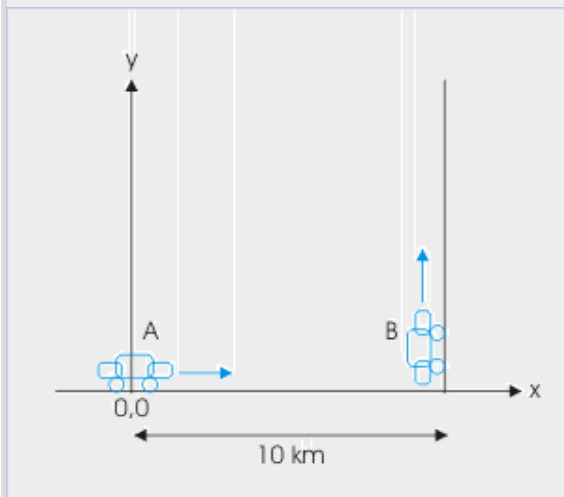
$$\Rightarrow \mathbf{v}_B = \sqrt{(4^2 + 4^2)} = 4\sqrt{2} \quad m/s$$

Closest approach

Example:

Problem : The car “B” is ahead of car “A” by 10 km on a straight road at a given time. At this instant, car “B” turns left on a road, which makes right angle to the original direction. If both cars are moving at a speed of 40 km/hr, then find the closest approach between the cars and the time taken to reach closest approach.

Motion of two cars



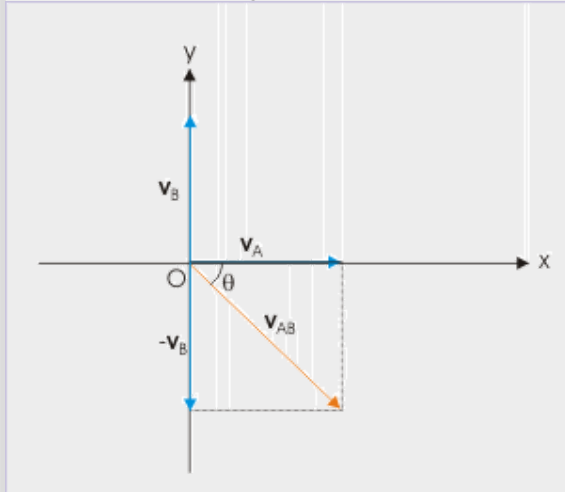
Motion of two cars in
perpendicular directions

Solution : Closest approach means that the linear distance between the cars is minimum. There are two ways to handle this question. We can proceed in the conventional manner, considering individual velocity as observed from ground reference. We use calculus to find the closest approach. Alternatively, we can use the concept of relative velocity and determine the closest approach. Here, we shall first use the relative velocity technique. By solving in two ways, we shall reinforce the conceptual meaning of relative velocity and see that its interpretation in terms of a stationary reference is a valid conception. The car “A” moves with certain relative velocity with respect to “B”, which is given by :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

Since they are moving with same speed and at right angle to each other, relative velocity is as shown in the figure. Using pythagoras theorem, the magnitude of relative velocity is :

Relative velocity



Relative velocity is obtained as sum of vectors.

$$v_{AB} = \sqrt{(v^2 + v^2)} = \sqrt{(40^2 + 40^2)} = 40\sqrt{2} \text{ m/s}$$

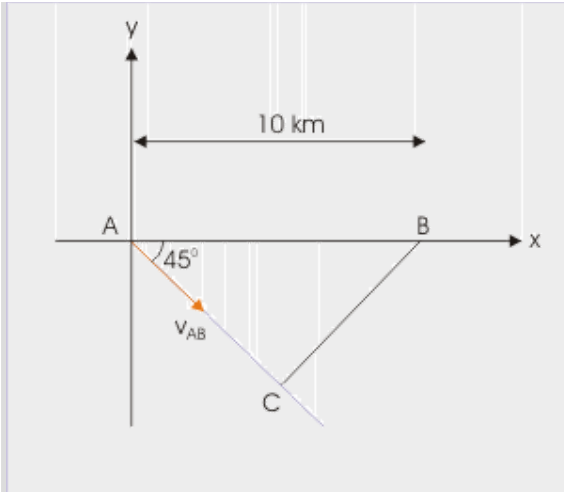
Its direction with respect to original direction is :

$$\tan \theta = \frac{40}{40} = 1$$

$$\theta = 45^\circ$$

As we have discussed, relative velocity can be conceptually interpreted as if car “B” is stationary and car “A” is moving with a velocity $40\sqrt{2}$ making an angle 45° . Since perpendicular drawn from the position of “B” (which is stationary) to the path of motion of “A” is the shortest linear distance and hence the closest approach. Therefore, closest approach by trigonometry of ΔABC ,

Relative velocity



Closest approach

$$r_{\min} = AB \sin 45^\circ = 10 \times \frac{1}{\sqrt{2}} = 5\sqrt{2} \text{ km}$$

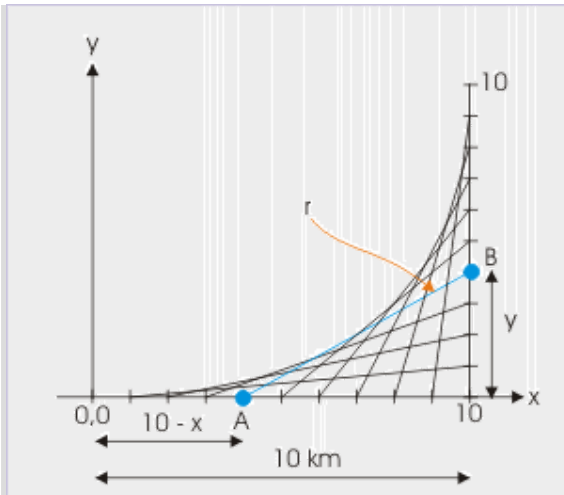
We should understand here that car “A” is moving in the direction as shown from the perspective of car “B”.

The time taken by car “A” to travel $5\sqrt{2}$ km with a velocity of $40\sqrt{2}$ m/s along the direction is :

$$t = \frac{5\sqrt{2}}{40\sqrt{2}} = \frac{1}{8} \text{ hr} = 7.5 \text{ min}$$

Now, let us now attempt to analyze motion from the ground’s perspective. The figure here shows the positions of the car and linear distance between them at a displacement of 1 km. The linear distance first decreases and then increases. At a given time, the linear distance between the cars is :

Motion of two cars



Closest approach

$$r = \sqrt{\{(10 - x)^2 + y^2\}}$$

For minimum distance, first time derivative is equal to zero. Hence,

$$\Rightarrow \frac{dr}{dt} = \frac{-2(10 - x) \frac{dx}{dt} + 2y \frac{dy}{dt}}{2\sqrt{\{(10 - x)^2 + y^2\}}} = 0$$

$$\Rightarrow -2(10 - x) \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\Rightarrow -10 + x + y = 0$$

$$\Rightarrow x + y = 10$$

Since cars are moving with same speed, $x = y$. Hence,

$$x = y = 5 \text{ km}$$

The closest approach, therefore, is :

$$r_{\min} = \sqrt{\{(10 - x)^2 + y^2\}} = \sqrt{\{(10 - 5)^2 + 5^2\}} = 5\sqrt{2}$$

Analysing motion in a medium

Resultant motion follows the principle of independence of motion in mutually perpendicular directions.

The motion of an object body in a medium is composed of two motions : (i) motion of the object and (ii) motion of the medium. The resulting motion as observed by an observer in the reference frame is the resultant of two motions. The basic equation that governs the context of study is the equation of relative motion in two dimensions :

Equation:

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

As discussed in the previous module, the motion of the body can alternatively be considered as the resultant of two motions. This concept of resultant motion is essentially an equivalent way of stating the concept of relative motion. Rearranging the equation, we have :

Equation:

$$\mathbf{v}_A = \mathbf{v}_{AB} + \mathbf{v}_B$$

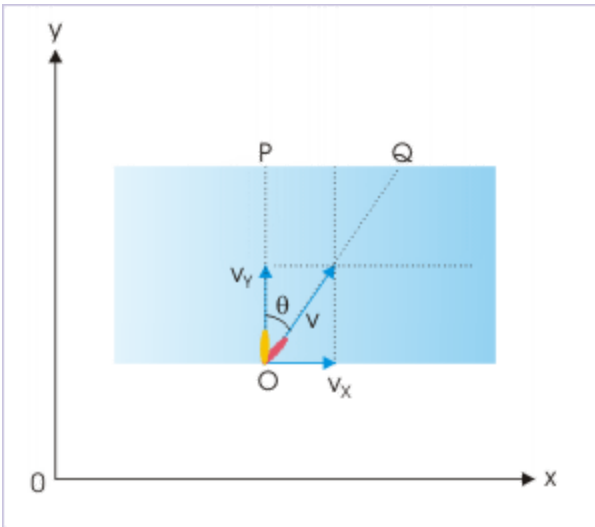
Both these equations are a vector equations, which can be broadly dealt in two ways (i) analytically (using graphics) and (ii) algebraically (using unit vectors). The use of a particular method depends on the inputs available and the context of motion.

Concept of independence of motion

Analysis of the motion of a body in a medium, specifically, makes use of independence of motions in two perpendicular directions. On a rectilinear coordinate system, the same principle can be stated in terms of component velocities. For example, let us consider the motion of a boat in a river, which tries to reach a point on the opposite bank of the river. The boat sails in the direction perpendicular to the direction of stream. Had the water been still, the boat would have reached the point exactly across the river with a velocity (v_y). But, water body is not still. It has a velocity in x-direction.

The boat, therefore, drifts in the direction of the motion of the water stream (v_x).

Motion of a Boat



Motions in two mutually perpendicular directions are independent of each other.

The important aspect of this motion is that the drift (x) depends on the component of velocity in x -direction. This drift (x) is independent of the component of velocity in y -direction. Why? Simply because, it is an experimental fact, which is fundamental to natural phenomena. We shall expand on this aspect while studying projectile motion also, where motions in vertical and horizontal directions are independent of each other.

In the case of boat, the displacements in the mutually perpendicular coordinate directions are :

Equation:

$$x = v_x t$$

Equation:

$$y = v_y t$$

Motion of boat in a stream

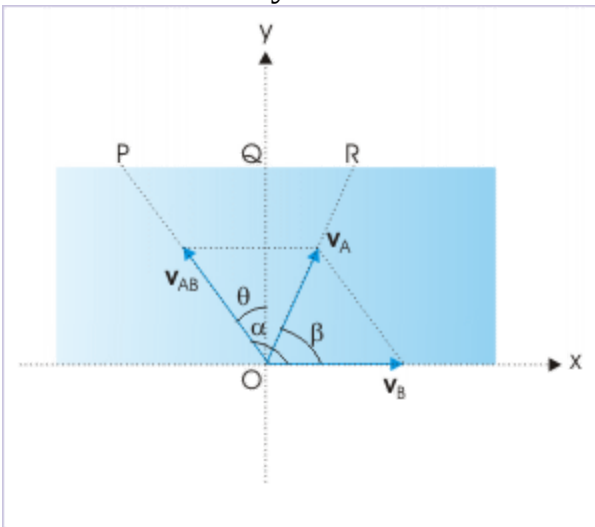
In this section, we shall consider a general situation of sailing of a boat in a moving stream of water. In order to keep our context simplified, we consider that stream is unidirectional in x-direction and the width of stream, “d”, is constant.

Let the velocities of boat (A) and stream (B) be “ $\mathbf{v}_A$ ” and “ $\mathbf{v}_B$ ” respectively with respect to ground. The velocity of boat (A) with respect to stream (B), therefore, is :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

$$\Rightarrow \mathbf{v}_A = \mathbf{v}_{AB} + \mathbf{v}_B$$

Resultant velocity



The boat moves with the resultant velocity.

These velocities are drawn as shown in the figure. This is clear from the figure that boat sails in the direction, making an angle “ θ ” with y-direction, but reaches destination in different direction. The boat obviously is carried

along in the stream in x-direction. The displacement in x-direction ($x = QR$) from the directly opposite position to actual position on the other side of the stream is called the drift of the boat.

Resultant velocity

The magnitude of resultant velocity is obtained, using parallelogram theorem,

Equation:

$$v_A = \sqrt{(v_{AB}^2 + v_B^2 + 2v_{AB}v_B \cos \alpha)}$$

where “ α ” is the angle between $\mathbf{v}_B$ and $\mathbf{v}_{AB}$ vectors. The angle “ β ” formed by the resultant velocity with x-direction is given as :

Equation:

$$\tan \beta = \frac{v_{AB} \sin \alpha}{v_B + v_{AB} \cos \alpha}$$

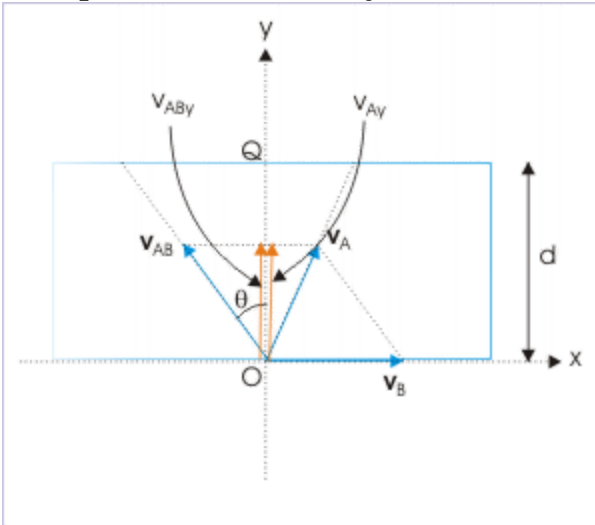
Time to cross the stream

The boat covers a linear distance equal to the width of stream “ d ” in time “ t ” in y-direction. Applying the concept of independence of motions in perpendicular direction, we can say that boat covers a linear distance “ $OQ = d$ ” with a speed equal to the component of resultant velocity in y-direction.

Now, resultant velocity is composed of (i) velocity of boat with respect to stream and (ii) velocity of stream. We observe here that velocity of stream is perpendicular to y – direction. As such, it does not have any component in y – direction. We, therefore, conclude that the component of resultant velocity is equal to the component of the velocity of boat with respect to stream in y -direction,. Note the two equal components shown in the figure.

They are geometrically equal as they are altitudes of same parallelogram.
Hence,

Components of velocity



Components of velocity of boat
w.r.t stream and velocity of
stream in y-direction are equal.

$$v_{Ay} = v_{AB_y} = v_{AB} \cos \theta$$

where “ θ ” is the angle that relative velocity of boat w.r.t stream makes with the vertical.

Equation:

$$t = \frac{d}{v_{Ay}} = \frac{d}{v_{AB} \cos \theta}$$

We can use either of the two expressions to calculate time to cross the river, depending on the inputs available.

Drift of the boat

The displacement of the boat in x-direction is independent of motion in perpendicular direction. Hence, displacement in x-direction is achieved with the component of resultant velocity in x-direction,

$$x = (v_{Ax})t = (v_B - v_{ABx})t = (v_B - v_{AB} \sin \theta)t$$

Substituting for time “t”, we have :

Equation:

$$x = (v_B - v_{AB} \sin \theta) \frac{d}{v_{AB} \cos \theta}$$

Special cases

Shortest interval of time to cross the stream

The time to cross the river is given by :

$$t = \frac{d}{v_{Ay}} = \frac{d}{v_{AB} \cos \theta}$$

Clearly, time is minimum for greatest value of denominator. The denominator is maximum for $\theta = 0^\circ$. For this value,

Equation:

$$t_{\min} = \frac{d}{v_{AB}}$$

This means that the boat needs to sail in the perpendicular direction to the stream to reach the opposite side in minimum time. The drift of the boat for this condition is :

Equation:

$$x = \frac{v_B d}{v_{AB}}$$

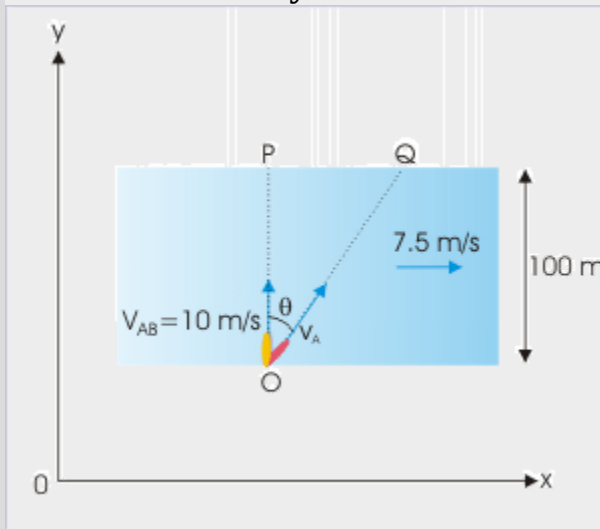
Example:

Problem : A boat, which has a speed of 10 m/s in still water, points directly across the river of width 100 m. If the stream flows with the velocity 7.5 m/s in a linear direction, then how far downstream does the boat touch on the opposite bank.

Solution : Let the direction of stream be in x-direction and the direction across stream is y-direction. We further denote boat with “A” and stream with “B”. Now, from the question, we have :

$$v_{AB} = 10 \text{ m/s}$$

$$v_B = 7.5 \text{ m/s}$$

Resultant velocity

The boat moves at an angle with vertical.

The motions in two mutually perpendicular directions are independent of each other. In order to determine time (t), we consider motion in y – direction,

$$\Rightarrow t = \frac{OP}{v_{AB}} = \frac{100}{10} = 10 \text{ s}$$

The displacement in x-direction is :

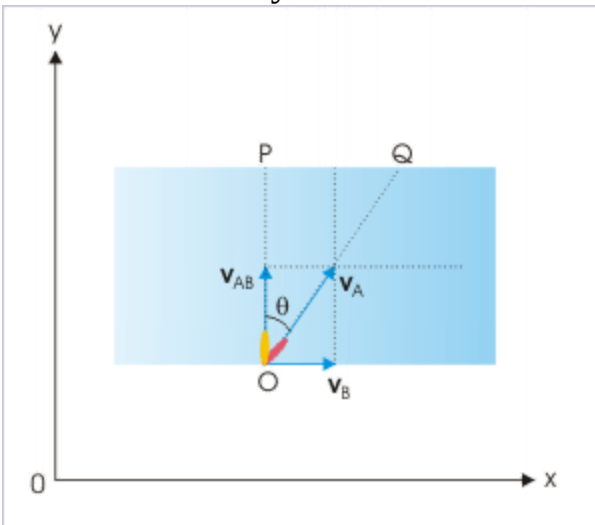
$$PQ = v_B \times t$$

Putting this value, we have :

$$x = PQ = v_B \times t = 7.5 \times 10 = 75 \text{ m}$$

The velocity of the boat w.r.t stream and the stream velocity are perpendicular to each other in this situation of shortest time as shown here in the figure. Magnitude of resultant velocity in this condition, therefore, is given as :

Resultant velocity



The boat moves at an angle with y-direction.

Equation:

$$v_A = \sqrt{(v_{AB}^2 + v_B^2)}$$

The angle that the resultant makes with y-direction (perpendicular to stream direction) is :

$$\tan \theta = \frac{v_B}{v_{AB}}$$

Time to cross the river, in terms of linear distance covered during the motion, is :

Equation:

$$t = \frac{OQ}{\sqrt{(v_{AB}^2 + v_B^2)}}$$

Direction to reach opposite point of the stream

If the boat is required to reach a point directly opposite, then it should sail upstream. In this case, the resultant velocity of the boat should be directed in y –direction. The drift of the boat is zero here. Hence,

$$x = \frac{(v_B - v_{AB} \sin \theta)d}{v_{AB} \cos \theta} = 0$$

$$\Rightarrow v_B - v_{AB} \sin \theta = 0$$

Equation:

$$\Rightarrow \sin \theta = \frac{v_B}{v_{AB}}$$

Thus. the boat should sail upstream at an angle given by above expression to reach a point exactly opposite to the point of sailing.

The velocity to reach opposite point of the stream

The angle at which boat sails to reach the opposite point is :

$$\sin \theta = \frac{v_B}{v_{AB}}$$

This expression points to a certain limitation with respect to velocities of boat and stream. If velocity of boat in still water is equal to the velocity of stream, then

$$\begin{aligned}\sin \theta &= \frac{v_B}{v_{AB}} = 1 = \sin 90^\circ \\ \Rightarrow \theta &= 90^\circ\end{aligned}$$

It means that boat has to sail in the direction opposite to the stream to reach opposite point. This is an impossibility from the point of physical reality. Hence, we can say that velocity of boat in still water should be greater than the velocity of stream ($v_{AB} > v_B$) in order to reach a point opposite to the point of sailing.

In any case, if $v_{AB} < v_B$, then the boat can not reach the opposite point as sine function can not be greater than 1.

Shortest path

The magnitude of linear distance covered by the boat is given by :

Equation:

$$s = \sqrt{(d^2 + x^2)}$$

It is evident from the equation that linear distance depends on the drift of the boat, “x”. Thus, shortest path corresponds to shortest drift. Now, there are two situations depending on the relative magnitudes of velocities of boat and stream.

Note: We should be aware that though the perpendicular distance to stream (width of the river) is the shortest path, but boat may not be capable to follow this shortest path in the first place.

1: $v_{AB} > v_B$

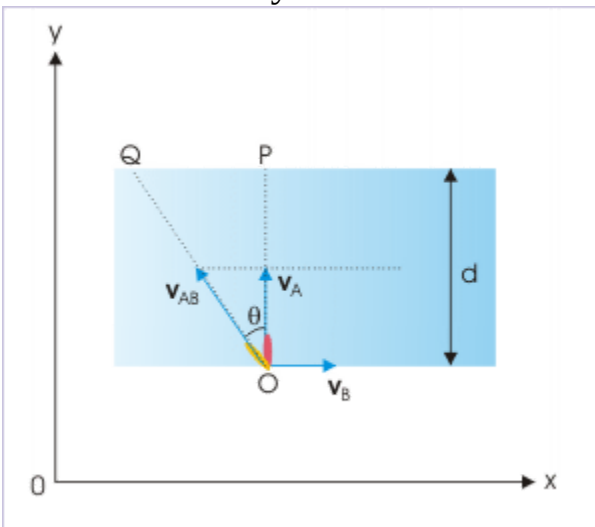
We have seen that when stream velocity (v_B) is less than the velocity of boat in still water, the boat is capable to reach the opposite point across the stream. For this condition, drift (x) is zero and represents the minimum value. Accordingly, the shortest path is :

Equation:

$$s_{\min} = d$$

The boat needs to sail upstream at the specified angle. In this case, the resultant velocity is directed across the river in perpendicular direction and its magnitude is given by :

Resultant velocity



The boat moves perpendicular to stream.

Equation:

$$v_A = \sqrt{(v_{AB}^2 - v_B^2)}$$

The time taken to cross the river is :

Equation:

$$t = \frac{d}{\sqrt{(v_{AB}^2 - v_B^2)}}$$

2: $v_{AB} < v_B$

In this case, the boat is carried away from the opposite point in the direction of stream. Now, the drift “x” is given as :

$$x = \frac{(v_B - v_{AB} \sin \theta)d}{v_{AB} \cos \theta}$$

For minimum value of “x”, first time derivative of “x” is equal to zero,

Equation:

$$\frac{dx}{dt} = 0$$

We need to find minimum drift and corresponding minimum length of path, subject to this condition.

Motion of an object in a medium

We have discussed the motion in the specific reference of boat in water stream. However, the consideration is general and is applicable to the motion of a body in a medium. For example, the discussion and analysis

can be extended to the motion of an aircraft, whose velocity is modified by the motion of the wind.

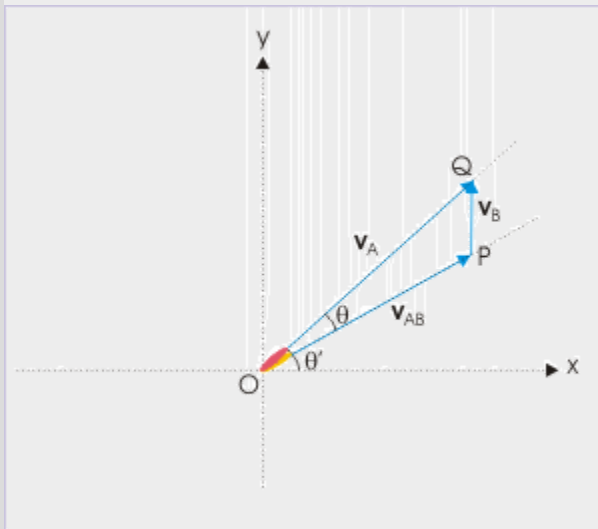
Example:

Problem : An aircraft flies with a wind velocity of $200\sqrt{2}$ km/hr blowing from south. If the relative velocity of aircraft with respect to wind is 1000 km/hr, then find the direction in which aircraft should fly such that it reaches a destination in north – east direction.

Solution : The figure here shows the velocities. OP denotes the velocity of the aircraft in the still air or equivalently it represents the relative velocity of aircraft with respect to air in motion; PQ denotes the velocity of the wind and OQ denotes the resultant velocity of the aircraft. It is clear that the aircraft should fly in the direction OP so that it is ultimately led to follow the north-east direction.

We should understand here that one of the velocities is resultant velocity of the remaining two velocities. It follows then that three velocity vectors are represented by the sides of a closed triangle.

Motion of an aircraft



The aircraft flies such a way that it keeps a north – east course.

We can get the direction of OP, if we can find the angle “ θ ”. The easiest technique to determine the angle between vectors composing a triangle is to apply sine law,

$$\frac{OP}{\sin 45^0} = \frac{PQ}{\sin \theta}$$

Putting values, we have :

$$\sin \theta = \frac{PQ \sin 45^0}{OP} = \frac{200\sqrt{2}}{1000x\sqrt{2}} = \frac{1}{5} = 0.2$$

$$\theta = \sin^{-1}(0.2)$$

Hence the aircraft should steer in the direction, making an angle with east as given by :

$$\theta = 45^0 - \sin^{-1}(0.2)$$

Resultant motion (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the resultant velocity. The questions are categorized in terms of the characterizing features of the subject matter :

- Velocity of the object
- Time to cross the stream
- Multiple references
- Minimum time, distance and speed

Velocity of the object

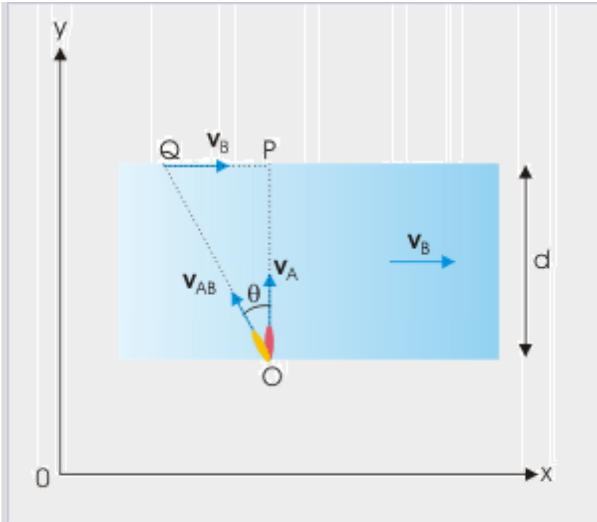
Example:

Problem : A person can swim at 1 m/s in still water. He swims to cross a river of width 200 m to a point exactly opposite to his/her initial position. If the water stream in river flows at 2 m/s in a linear direction, then find the time taken (in seconds) to reach the opposite point.

Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. Let us also denote person with “A” and water stream with “B”.

To reach the point across, the person has to swim upstream at an angle such that the velocity of the person with respect to ground($\mathbf{v}_A$) is across the direction of water stream. The situation is shown in the figure.

Relative velocity



Here,

Speed of the person (A) with respect to stream (B) : $v_{AB} = 1 \text{ m/s}$

Speed of water stream (B) with respect to ground : $v_B = 2 \text{ m/s}$

Speed of the person (A) with respect to ground : $v_A = ?$

$d = 200 \text{ m}$

From the ΔOAB ,

$$\begin{aligned} OQ^2 &= OP^2 + PQ^2 \\ \Rightarrow OP &= \sqrt{\{OQ^2 - PQ^2\}} \\ \Rightarrow t &= \frac{d}{\sqrt{\{OB^2 - AB^2\}}} \end{aligned}$$

It is clear from the denominator of the expression that for finite time, $OB > AB$. From the values as given in the question, $OB < AB$ and the denominator becomes square root of negative number. The result is interpreted to mean that the physical event associated with the expression is not possible. The swimmer, therefore, can not reach the point, which is exactly opposite to his position. The speed of the swimmer should be greater than that of the stream to reach the point lying exactly opposite. Note that we had explained the same situation in the module on the subject with the help of the value of " $\sin\theta$ ", which can not be greater than 1. We have taken a different approach here to illustrate the same limitation of the

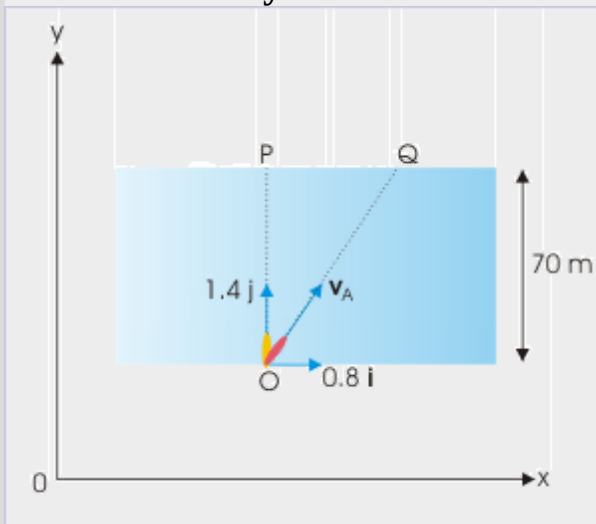
swimmer's ability to cross the water stream, showing that the interpretation is consistent and correct.

Example:

Problem : The direction of water stream in a river is along x – direction of the coordinate system attached to the ground. A swimmer swims across the river with a velocity $(0.8\mathbf{i} + 1.4\mathbf{j})$ m/s, as seen from the ground. If the river is 70 m wide, how long (in seconds) does he take to reach the river bank on the other side ?

Solution : We recognize here that the given velocity represents the resultant velocity $(\mathbf{v}_A)$ of the swimmer (A). The time to reach the river bank on the other side is a function of component velocity in y-direction.

Relative velocity



Here,

$$v_x = 0.8 \text{ m/s}$$

$$v_y = 1.4 \text{ m/s}$$

$$t = \frac{\text{Width of the river}}{v_y}$$

$$t = \frac{70}{1.4} = 50 \text{ s}$$

Example:

Problem : A person can swim at a speed “ u ” in still water. He points across the direction of water stream to cross a river. The water stream flows with a speed “ v ” in a linear direction. Find the direction in which he actually swims with respect to the direction of stream.

Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. Let us also denote person with "A" and water stream with “B”.

Here,

Speed of the person (A) with respect to stream (B) : $v_{AB} = u$

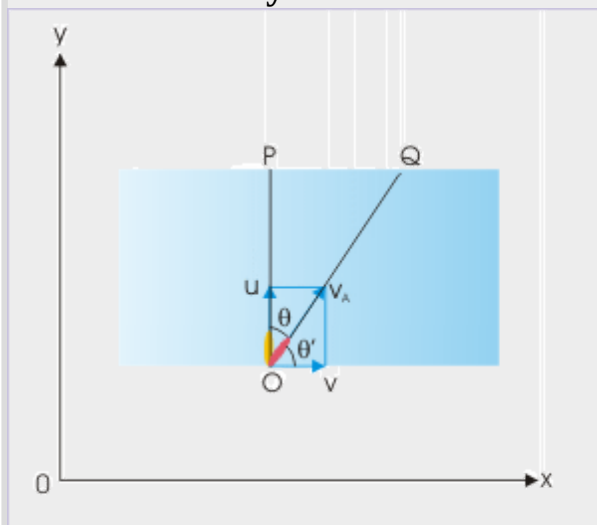
Speed of stream (B) with respect to ground : $v_B = v$

Speed of the person (A) with respect to ground : $v_C = ?$

Using equation, $\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$,

$$\Rightarrow \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}$$

From the figure,

Relative velocity

$$\tan \theta' = \frac{v_{AB}}{v_B} = \frac{u}{v}$$

The direction in which he actually swims with respect to the direction of stream is

$$\theta' = \tan^{-1}\left(\frac{u}{v}\right)$$

Time to cross the river

Example:

Problem : A person can swim at a speed 1 m/s in still water. He swims perpendicular to the direction of water stream, flowing at the speed 2 m/s. If the linear distance covered during the motion is 300 m, then find the time taken to cross the river.

Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. Let us also denote ground person with "A" and water stream with "B". This is clearly the situation corresponding to the least time for crossing the river.

Here,

Speed of the person (A) with respect to stream (B) : $v_{AB} = u = 1 \text{ m/s}$

Speed of water stream (B) with respect to ground : $v_B = v = 2 \text{ m/s}$

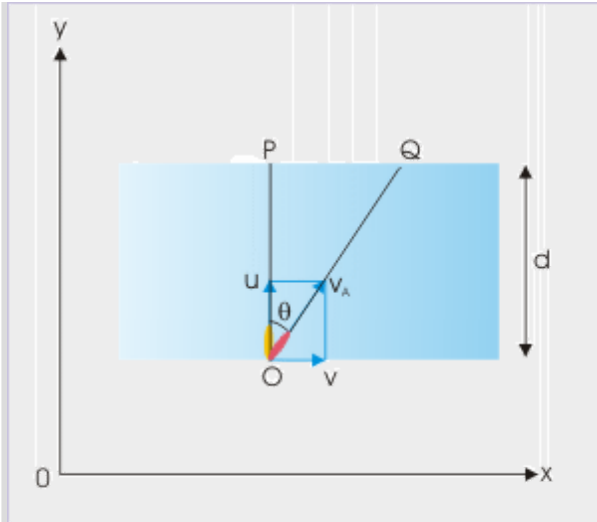
Speed of the person (A) with respect to ground : $v_A = ?$

We note here that the perpendicular linear distance i.e. the width of river is not given. Instead, the linear distance covered during the motion is given. Hence, we need to find the resultant speed in the direction of motion to find time. Using equation for the resultant velocity,

$$\Rightarrow \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}$$

From the figure, we have :

Relative velocity



$$v_A = \sqrt{\{v_{AB}^2 + v_B^2\}} = \sqrt{\{u^2 + v^2\}}$$

$$v_A = \sqrt{\{1^2 + 2^2\}} = \sqrt{5} \text{ m/s}$$

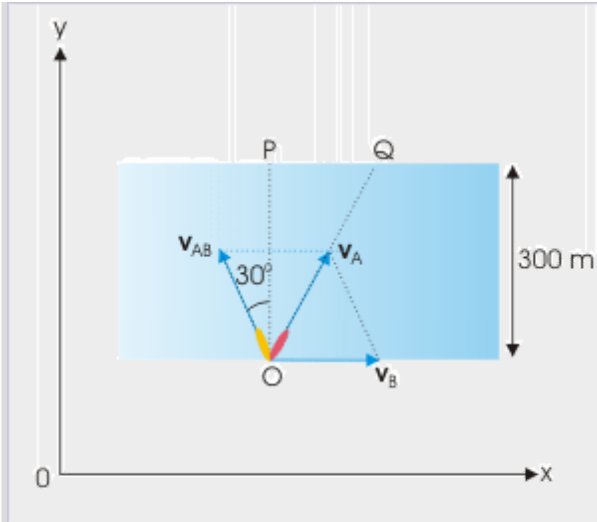
$$\Rightarrow t = \frac{500}{\sqrt{5}} = 100\sqrt{5} \text{ s}$$

Example:

Problem : A person can swim at a speed of $\sqrt{3}$ m/s in still water. He swims at an angle of 120° from the stream direction while crossing a river. The water stream flows with a speed of 1 m/s. If the river width is 300 m, how long (in seconds) does he take to reach the river bank on the other side ?

Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. Here, we need to know the component of the resultant velocity in the direction perpendicular to the stream.

Relative velocity



This approach, however, would be tedious. We shall use the fact that the component of " $\mathbf{v}_A$ " in any one of the two mutually perpendicular directions is equal to the sum of the components of $\mathbf{v}_{AB}$ and $\mathbf{v}_B$ in that direction.

$$\Rightarrow v_{Ay} = v_{AB} \cos 30^\circ$$

$$\Rightarrow v_{Ay} = \sqrt{3} \cos 30^\circ = \frac{3}{2} \text{ m/s}$$

Thus, time taken to cross the river is :

$$t = \frac{\text{Width of the river}}{v_{Ay}}$$

$$t = \frac{300 \times 2}{3} = 200 \text{ s}$$

Multiple references

Example:

Problem : A boat, capable of sailing at 2 m/s, moves upstream in a river. The water stream flows at 1 m/s. A person walks from the front end to the rear end of the boat at a speed of 1 m/s along the linear direction. What is the speed of the person (m/s) with respect to ground ?

Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. We further denote boat with “A”, stream with “B”, and person with “C”.

We shall work out this problem in two parts. In the first part, we shall find out the velocity of boat (A) with respect to ground and then we shall find out the velocity of person (C) with respect to ground.

Here,

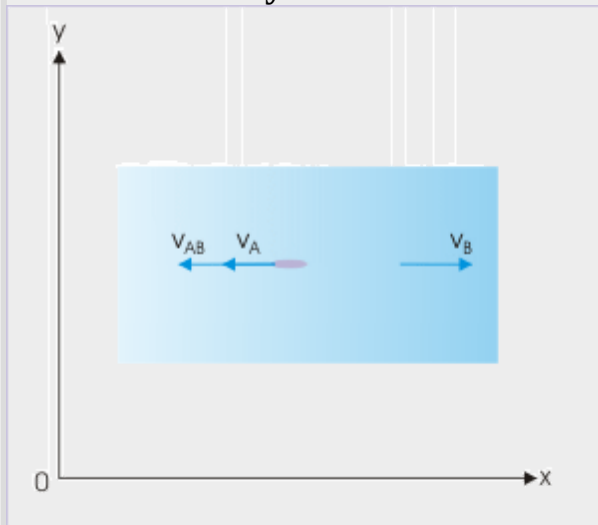
Velocity of boat (A) with respect to stream (B) : $v_{AB} = -2 \text{ m/s}$

Velocity of the stream (A) with respect to ground : $v_A = 1 \text{ m/s}$

Velocity of the person (C) with respect to boat (A) : $v_{CA} = 1 \text{ m/s}$

Velocity of the person (C) with respect to ground : $v_C = ?$

Relative velocity



The velocity of boat with respect to ground is equal to the resultant velocity of the boat as given by :

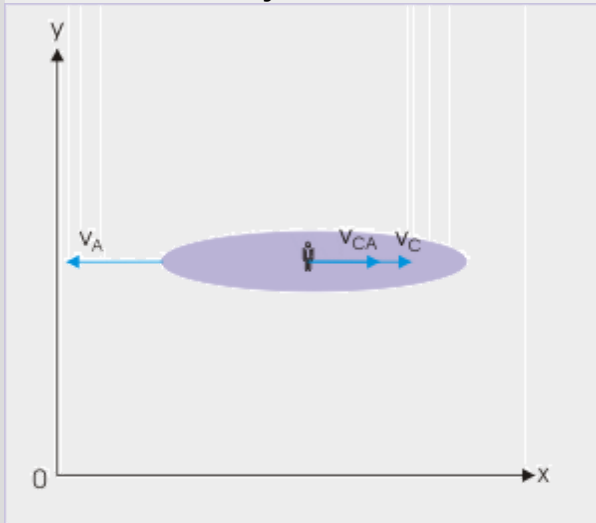
$$v_A = v_{AB} + v_B$$

$$\Rightarrow v_A = -2 + 1 = -1 \text{ m/s}$$

For the motion of person and boat, the velocity of the person with respect to ground is equal to the resultant velocity of (i) velocity of the person (C) with respect to boat (A) and (ii) velocity of the boat (A) with respect to ground. We note here that relative velocity of person with respect to boat is

given and that we have already determined the velocity of boat (A) with respect to ground in the earlier step. Hence,

Relative velocity



$$v_C = v_{CA} + v_A$$

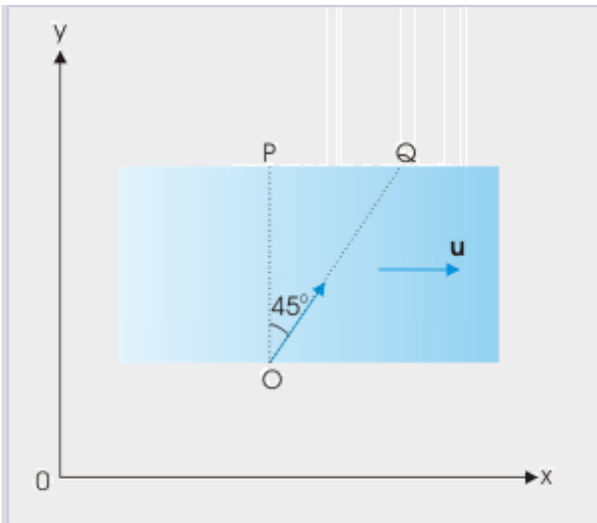
$$\Rightarrow v_C = 1 + (-1) = 0$$

Minimum time, distance and speed

Example:

Problem : A boy swims to reach a point “Q” on the opposite bank, such that line joining initial and final position makes an angle of 45 with the direction perpendicular to the stream of water. If the velocity of water stream is “u”, then find the minimum speed with which the boy should swim to reach his target.

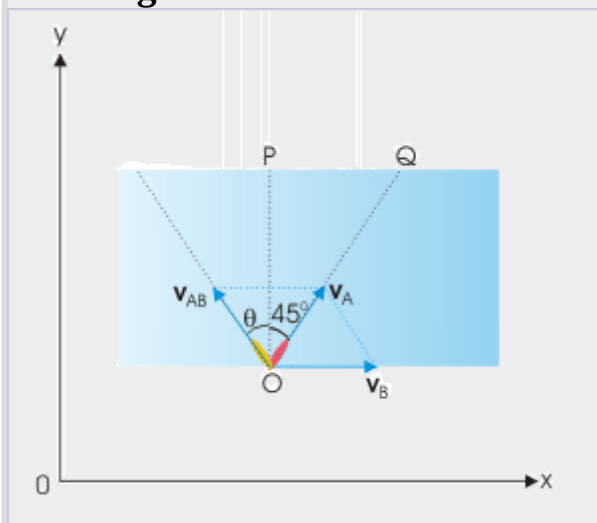
Crossing a river



The boy swims to reach point “Q”.

Solution : Let “A” and “B” denote the boy and the stream respectively. Here, we are required to know the minimum speed of boy, v_{AB} (say “v”) such that he reaches point “Q”. Now, he can adjust his speed with the direction he swims. Let the boy swims at an angle “ θ ” with a speed “v”.

Crossing a river



The boy swims to reach point “Q”.

Looking at the figure, it can be seen that we can make use of the given angle by taking trigonometric ratio such as tangent, which will involve speed of boy in still water (v) and the speed of water stream (u). This expression may then be used to get an expression for the minimum speed as required.

The slope of resultant velocity, v_A , is :

$$\tan 45^\circ = \frac{v_{Ax}}{v_{Ay}} = 1$$

$$\Rightarrow v_{Ax} = v_{Ay}$$

Now, the components of velocity in “x” and “y” directions are :

$$v_{Ax} = u - v \sin \theta$$

$$v_{Ay} = v \cos \theta$$

Putting in the equation we have :

$$u - v \sin \theta = v \cos \theta$$

Solving for “ v ”, we have :

$$\Rightarrow v = \frac{u}{(\sin \theta + \cos \theta)}$$

The velocity is minimum for a maximum value of denominator. The denominator is maximum for a particular value of the angle, θ ; for which :

$$\frac{d}{d\theta} (\sin \theta + \cos \theta) = 0$$

$$\Rightarrow \cos \theta - \sin \theta = 0$$

$$\Rightarrow \tan \theta = 1$$

$$\Rightarrow \theta = 45^\circ$$

It means that the boy swims with minimum speed if he swims in the direction making an angle of 45° with y-direction. His speed with this angle is :

$$v = \frac{u}{(\sin 45^\circ + \cos 45^\circ)} = \frac{\sqrt{2}u}{2} = \frac{u}{\sqrt{2}}$$

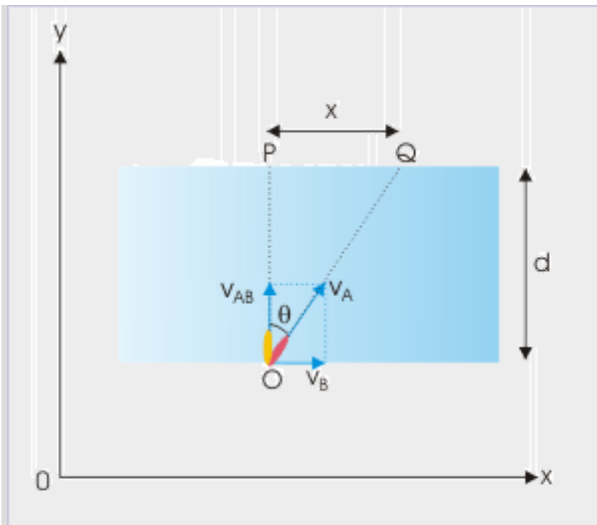
Example:

Problem : A boat crosses a river in minimum time, taking 10 minutes during which time it drifts by 120 m in the direction of stream. On the other hand, boat takes 12.5 minutes while moving across the river. Find (i) width of the river (ii) velocity of boat in still water and (iii) speed of the stream.

Solution : There are three pieces of information about "minimum time", "drift" and "time along shortest path". Individually each of these values translate into three separate equations, which can be solved to find the required values.

The boat takes minimum time, when it sails in the direction perpendicular to the stream (current). The time to cross the river is given by dividing width with component of resultant velocity (v_{Ay}). The boat, in this case, sails in the perpendicular direction. Hence, the component of resultant velocity is equal to the velocity of boat in still water (v_{AB}). The time to cross the river in this case is :

Crossing a river



The swims towards “P”.

$$t_{\min} = \frac{d}{v_{AB}} = \frac{d}{v_{AB}} = 10$$

$$d = 10v_{AB}$$

The drift in this time is given by :

$$x = v_B t_{\min}$$

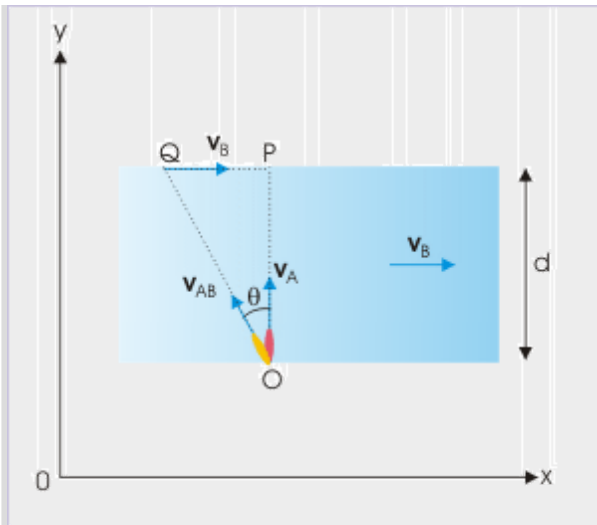
Putting values,

$$120 = v_B \quad x \quad 10$$

$$v_B = 12 \quad \text{meter / minute}$$

Now we need to use the information on shortest path. It is given that the boat moves across stream in 12.5 minutes. For this boat has to sail upstream at certain angle. The resultant speed is given by :

Crossing a river



The swimmer swims towards “P”.

$$v_A = \sqrt{v_{AB}^2 - v_B^2}$$

and the time taken is :

$$\frac{d}{\sqrt{v_{AB}^2 - v_B^2}} = 12.5$$

Substituting for “d” and “ v_B ” and squaring on both sides, we have :

$$(v_{AB}^2 - 12^2)12.5^2 = d^2 = 10^2 v_{AB}^2$$

$$v_{AB}^2(12.5^2 - 10^2) = 12^2 \times 12.5^2$$

$$v_{AB} = \frac{12 \times 12.5}{7.5} = 20 \text{ meter / minute}$$

and

$$d = 10v_{AB} = 10 \times 20 = 200 \text{ m}$$

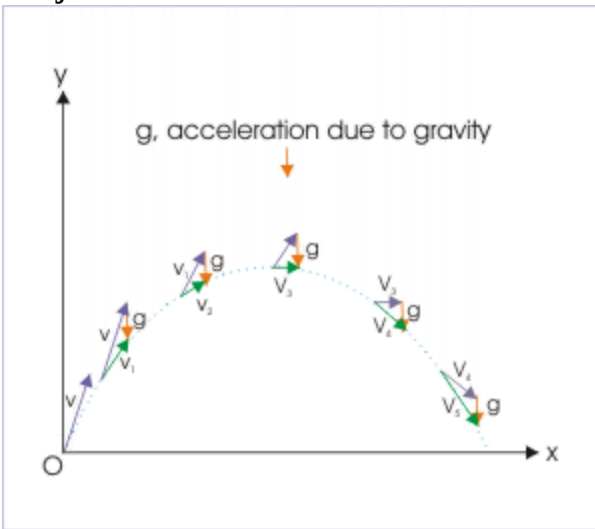
Projectile motion

Projectile motion is a special case of two dimensional motion with constant acceleration. Here, force due to gravity moderates linear motion of an object thrown at certain angle to the vertical direction. The resulting acceleration is a constant, which is always directed in vertically downward direction.

The projectile motion emphasizes one important aspect of constant acceleration that even constant acceleration, which is essentially unidirectional, is capable to produce two dimensional motion. The basic reason is that force and initial velocity of the object are not along the same direction. The linear motion of the projected object is continuously worked upon by the gravity, which results in the change of both magnitude and direction of the velocity. A change in direction of the velocity ensures that motion is not one dimensional.

The change in magnitude and direction of the velocity is beautifully managed so that time rate of change in velocity is always directed in vertically downward direction i.e. in the direction of gravity. This aspect is shown qualitatively for the motion in the figure below as velocity change successively at the end of every second from v_1 to v_2 to v_3 and so on..... by exactly a vector, whose magnitude is equal to acceleration due to gravity “g”.

Projectile motion



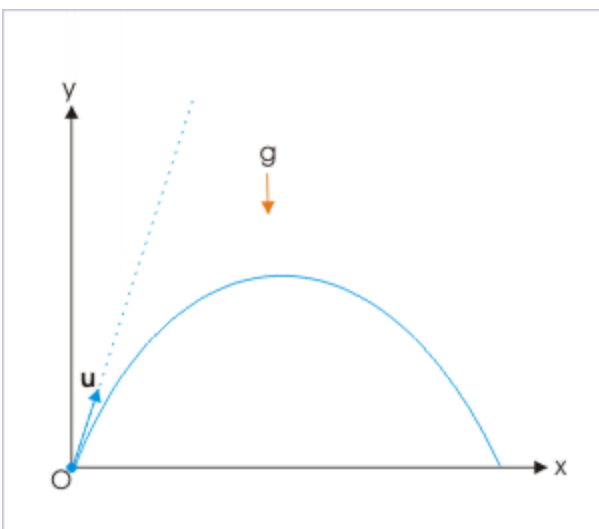
Velocity of the projectile
changes by acceleration vector
in unit time.

Force(s) in projectile motion

Flight of base ball, golf ball etc. are examples of projectile motion. In these cases, the projectile is projected with certain force at certain angle to vertical direction. The force that initiates motion is a contact force. Once the motion of the ball is initiated, the role of contact force is over. It does not subsequently affect or change the velocity of the ball as the contact is lost.

In order to emphasize, we restate three important facts about projectile motion. First, we need to apply force at the time of projection. This force as applied by hand or by any other mechanical device, accelerates projectile briefly till it is in contact with "thrower". The moment the projectile is physically disconnected with the throwing device, it moves with a velocity, which it gained during brief contact period. The role of force responsible for imparting motion is over. Second, motion of projectile is maintained if there is no net external force (Newton's laws of motion). This would be the case for projection in force free space. The projectile is initiated into the motion with certain initial velocity, say $\mathbf{u}$. Had there been no other force(s), then the ball would have moved along the dotted straight line (as shown in figure below) and might have been lost in to the space.

Projectile motion



Path of a projectile projected at an angle with horizontal direction.

Third, the projectile, once out in the space, is acted upon by the force due to gravity and air resistance. We, however, neglect the effect of air resistance for the time being and confine our study of the motion which is affected by force due to gravity acting downwards. The motion or velocity of projectile is then moderated i.e. accelerated (here, acceleration means change of speed or change of direction or both) by gravity. This is the only force. Hence, acceleration due to gravity is the only acceleration involved in the motion. This downward acceleration is a constant and is the acceleration in any projectile motion near Earth, which is not propelled or dragged.

Analysis of projectile motion

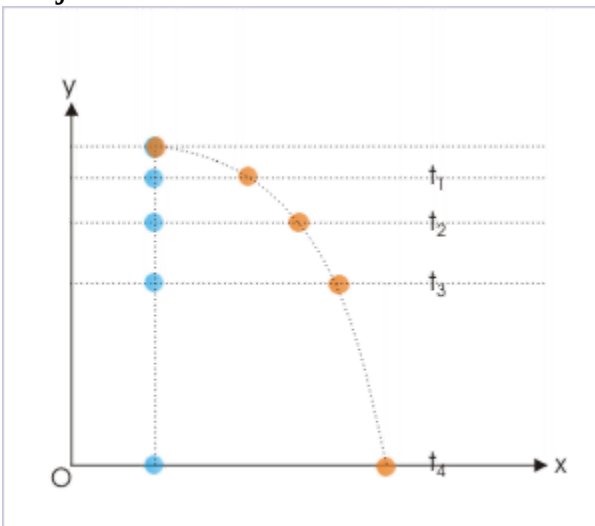
There is a very useful aspect of two dimensional motion that can be used with great effect. Two dimensional motion can be resolved in to two linear motions in two mutually perpendicular directions : (i) one along horizontal direction and (ii) the other along vertical direction. The linear motion in each direction can, then, be analyzed with the help of equivalent scalar system, in which signs of the attributes of the motion represent direction.

We can analyze the projectile motion with the help of equations of motion. As the motion occurs in two dimensions, we need to use vector equations and interpret them either graphically or algebraically as per the vector rules. We know that algebraic methods consisting of component vectors render vectors analysis in relatively simpler way. Still, vector algebra requires certain level of skills to manipulate vector components in two directions.

In the nutshell, the study of projectile motion is equivalent to two independent linear motions. This paradigm simplifies the analysis of projectile motion a great deal. Moreover, this equivalent construct is not merely a mathematical construct, but is a physically verifiable fact. The motions in vertical and horizontal directions are indeed independent of each other.

To illustrate this, let us consider the flight of two identical balls, which are initiated in motion at the same time. One ball is dropped vertically and another is projected in horizontal direction with some finite velocity from the same height. It is found that both balls reach the ground at the same time and also their elevations above the ground are same at all times during the motion.

Projectile motion



Comparing vertical and
projectile motion

The fact that two balls reach the ground simultaneously and that their elevations from the ground during the motion at all times are same, point to the important aspect of the motion that vertical motion in either of the two motions are identical. This implies that the horizontal motion of the second ball does not interfere with its vertical motion. By extension, we can also say that the vertical motion of the second ball does not interfere with its horizontal motion.

Projectile motion and equations of motion

Here, we describe the projectile motion with the help of a two dimensional rectangular coordinate system such that (This not not a requirement. One can choose reference coordinate system to one's convenience):

- Origin of the coordinate system coincides with point of projection.
- The x – axis represents horizontal direction.
- The y – axis represents vertical direction.

Initial velocity

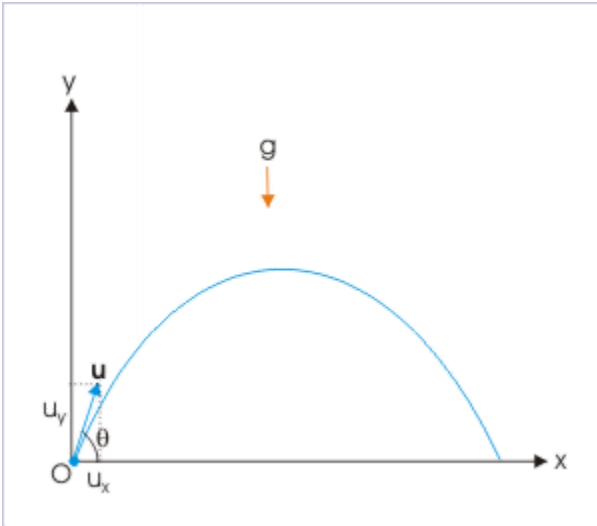
Let us consider that the projectile is thrown with a velocity “u” at an angle θ from the horizontal direction as shown in the figure. The component of initial velocity in the two directions are :

Equation:

$$u_x = u \cos \theta$$

$$u_y = u \sin \theta$$

Projectile motion



Component projection
velocities in x and y directions

Equations of motion in vertical direction

Motion in vertical direction is moderated by the constant force due to gravity. This motion, therefore, is described by one dimensional equations of motion for constant acceleration.

Velocity

The velocity in the vertical direction is given by :

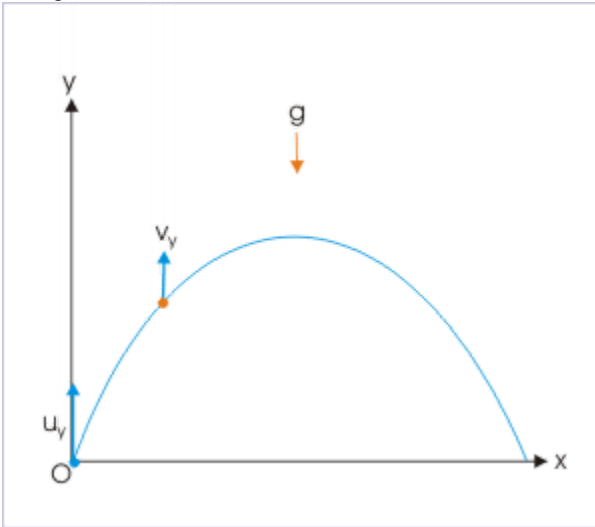
Equation:

$$v_y = u_y - gt$$

An inspection of equation - 2 reveals that this equation can be used to determine velocity in vertical direction at a given time “t” or to determine time of flight “t”, if final vertical velocity is given. This assumes

importance as we shall see that final vertical velocity at the maximum height becomes zero.

Projectile motion



Vertical component of velocity
during motion

The equation for velocity further reveals that the magnitude of velocity is reduced by an amount “ gt ” after a time interval of “ t ” during upward motion. The projectile is decelerated in this part of motion (velocity and acceleration are in opposite direction). The reduction in the magnitude of velocity with time means that it becomes zero corresponding to a particular value of “ t ”. The vertical elevation corresponding to the position, when projectile stops, is maximum height that projectile attains. For this situation ($v_y = 0$), the time of flight “ t ” is obtained as :

Equation:

$$\begin{aligned}v_y &= u_y - gt \\ \Rightarrow 0 &= u_y - gt \\ \Rightarrow t &= \frac{u_y}{g}\end{aligned}$$

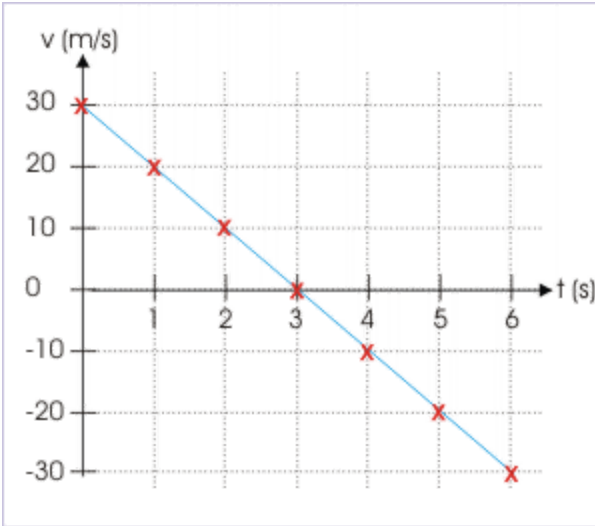
Immediately thereafter, projectile is accelerated in vertically downward direction with increasing speed. In order to appreciate variation of speed and velocity during projectile motion, we calculate the values of a projectile for successive seconds, which is projected with an initial velocity of 60 m/s making an angle of 30° with the horizontal. Here, vertical component of velocity is $60 \sin 30^\circ$ i.e. 30 m/s.

----- Time gt Velocity Magnitude of velocity (s)											
(m/s) (m/s) (m/s)			-----								
						0	0	30	30	1	10
										20	20
										2	20
10	10	3	30	0	0	4	40	-10	10	5	50
										-20	20
										6	60
										-30	30

Above table substantiate the observations made earlier. The magnitude of vertical velocity of the projectile first decreases during upward flight; becomes zero at maximum height; and, thereafter, picks up at the same rate during downward flight.

It is also seen from the data that each of the magnitude of vertical velocity during upward motion is regained during downward motion. In terms of velocity, for every vertical velocity there is a corresponding vertical velocity of equal magnitude, but opposite in direction.

The velocity – time plot of the motion is a straight line with negative slope. The negative slope here indicates that acceleration i.e acceleration due to gravity is directed in the opposite direction to that of positive y- direction.
Velocity – time plot



The velocity – time plot for constant acceleration in vertical direction

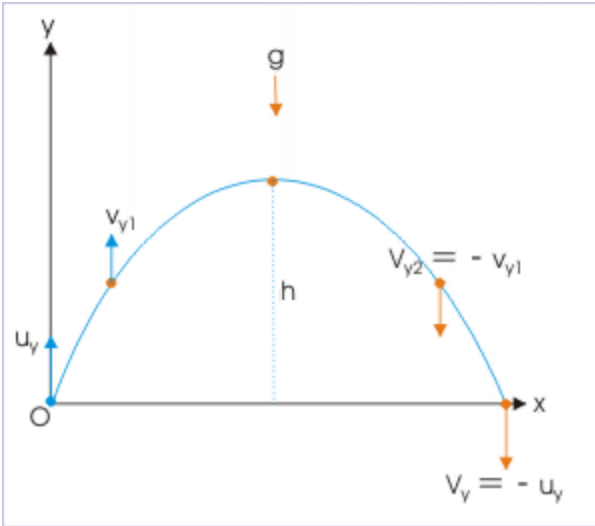
From the plot, we see that velocity is positive and acceleration is negative for upward journey, indicating deceleration i.e. decrease in speed.

The time to reach maximum height in this case is :

$$\Rightarrow t = \frac{u_y}{g} = \frac{30}{10} = 3 \text{ s}$$

The data in the table confirms this. Further, we know that vertical motion is independent of horizontal motion and time of flight for vertical motion is equal for upward and downward journey. This means that total time of flight is $2t$ i.e. $2 \times 3 = 6$ seconds. We must, however, be careful to emphasize that this result holds if the point of projection and point of return to the surface are on same horizontal level.

Projectile motion



Initial and final velocities are equal in magnitude but opposite in direction.

There is yet another interesting feature that can be drawn from the data set. The magnitude of vertical velocity of the projectile (30 m/s) at the time it hits the surface on return is equal to that at the time of the start of the motion. In terms of velocity, the final vertical velocity at the time of return is inverted initial velocity.

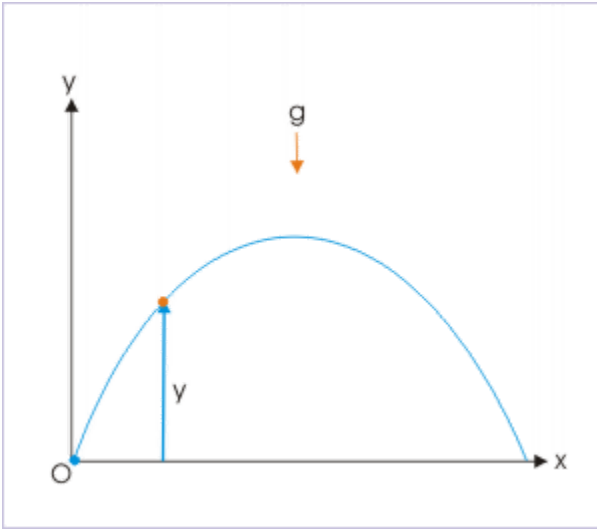
Displacement

The displacement in vertical direction is given by :

Equation:

$$y = u_y t - \frac{1}{2} g t^2$$

Projectile motion



Vertical displacement at a given time

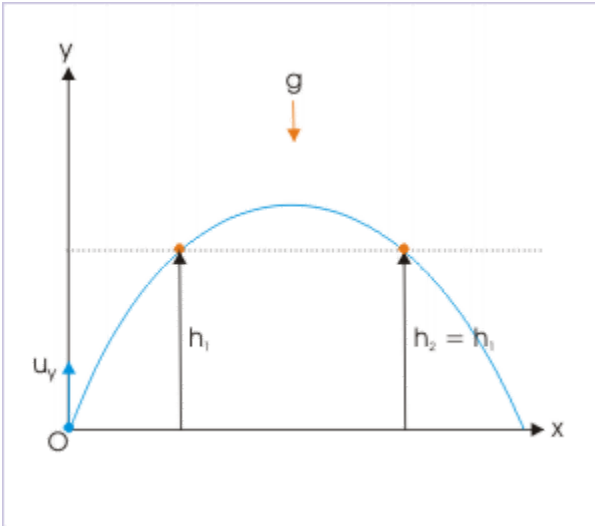
This equation gives vertical position or displacement at a given time. It is important to realize that we have simplified the equation $\Delta y = y_2 - y_1 = u_y t - \frac{1}{2}gt^2$ by selecting origin to coincide by the point of projection so that,

$$\Delta y = y_2 - y_1 = y \text{ (say)}$$

Thus, “y” with this simplification represents position or displacement.

The equation for position or displacement is a quadratic equation in time “t”. It means that solution of this equation yields two values of time for every value of vertical position or displacement. This interpretation is in fine agreement with the motion as projectile retraces all vertical displacement as shown in the figure.

Projectile motion



All vertical displacement is achieved twice by the projectile except the point of maximum height.

Time of flight

The time taken to complete the journey from the point of projection to the point of return is the time of the flight for the projectile. In case initial and final points of the journey are on the same horizontal level, then the net displacement in vertical direction is zero i.e. $y = 0$. This condition allows us to determine the total time of flight “T” as :

$$\begin{aligned}
 y &= u_y T - \frac{1}{2} g T^2 \\
 \Rightarrow 0 &= u_y T - \frac{1}{2} g T^2 \\
 \Rightarrow T(u_y - \frac{1}{2} g T) &= 0 \\
 \Rightarrow T = 0 \text{ or } T &= \frac{2u_y}{g}
 \end{aligned}$$

$T = 0$ corresponds to the time of projection. Hence neglecting the first value, the time of flight is :

Equation:

$$\Rightarrow T = \frac{2u_y}{g}$$

We see that total time of flight is twice the time projectile takes to reach the maximum height. It means that projectile takes equal times in "up" and "down" motion. In other words, time of ascent equals time of descent.

Example:

Problem : A ball is thrown upwards with a speed of 10 m/s making an angle 30° with horizontal and returning to ground on same horizontal level. Find (i) time of flight and (ii) time to reach the maximum height

Solution : Here, component of initial velocity in vertical direction is :

$$u_y = u \sin \theta = 10 \sin 30^\circ = 10 \times \frac{1}{2} = 5 \text{ m/s}$$

(i) The time of flight, T , is :

$$\Rightarrow T = \frac{2u_y}{g} = 2 \times \frac{5}{10} = 1 \text{ s}$$

(ii) Time to reach the maximum height is half of the total flight when starting and end points of the projectile motion are at same horizontal level. Hence, the time to reach the maximum height is 0.5 s.

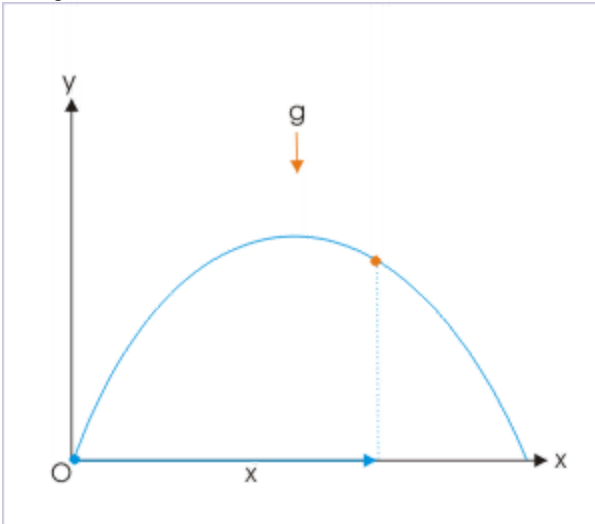
Equations of motion in horizontal direction

The force due to gravity has no component in horizontal direction. Since gravity is the only force acting on the projectile, this means that the motion in horizontal direction is not accelerated. Therefore, the motion in horizontal direction is an uniform motion. This implies that the component of velocity in x-direction is constant. As such, the position or displacement in x-direction at a given time "t" is :

Equation:

$$x = u_x t$$

Projectile motion



Horizontal displacement at a given time

This equation gives the value of horizontal position or displacement at any given instant.

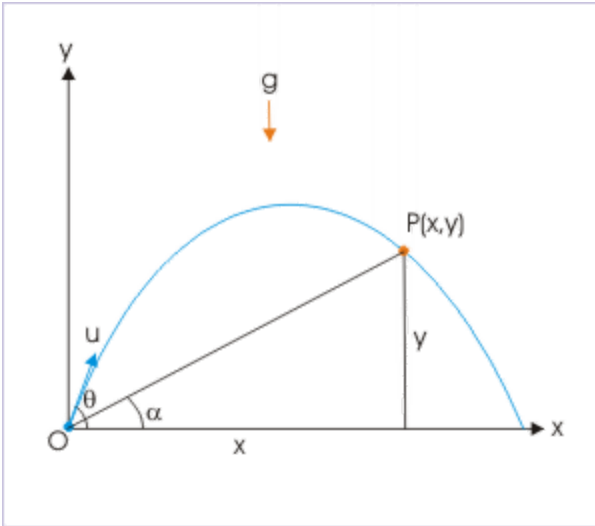
Displacement of projectile

The displacement of projectile is obtained by vector addition of displacements in x and y direction. The magnitude of displacement of the projectile from the origin at any given instant is :

Equation:

$$\text{Displacement, OP} = \sqrt{(x^2 + y^2)}$$

Displacement in projectile motion



The angle that displacement vector subtends on x-axis is :

Equation:

$$\tan \alpha = \frac{y}{x}$$

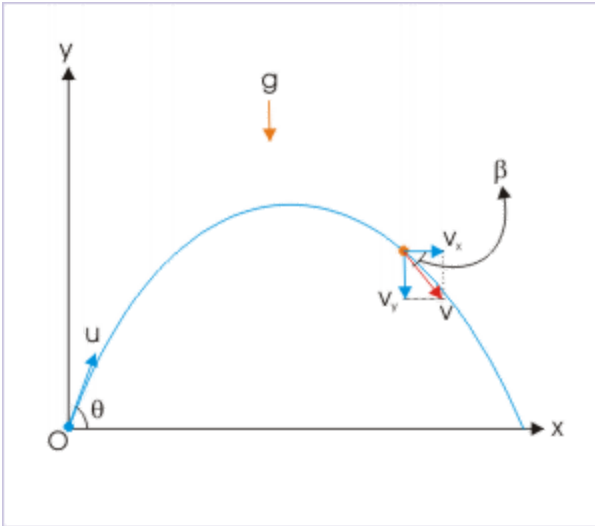
Velocity of projectile

The velocity of projectile is obtained by vector addition of velocities in x and y direction. Since component velocities are mutually perpendicular to each other, we can find magnitude of velocity of the projectile at any given instant, applying Pythagoras theorem :

Equation:

$$v = \sqrt{(v_x^2 + v_y^2)}$$

Velocity of a projectile



The angle that the resultant velocity subtends on x-axis is :

Equation:

$$\tan \beta = \frac{v_y}{v_x}$$

Example:

Problem : A ball is projected upwards with a velocity of 60 m/s at an angle 60° to the vertical. Find the velocity of the projectile after 1 second.

Solution : In order to find velocity of the projectile, we need to know the velocity in vertical and horizontal direction. Now, initial velocities in the two directions are (Note that the angle of projection is given in relation to vertical direction.):

$$u_x = u \sin \theta = 60 \sin 60^\circ = 60 \times \frac{\sqrt{3}}{2} = 30\sqrt{3} \text{ m/s}$$

$$u_y = u \cos \theta = 60 \cos 60^\circ = 60 \times \frac{1}{2} = 30 \text{ m/s}$$

Now, velocity in horizontal direction is constant as there is no component of acceleration in this direction. Hence, velocity after "1" second is :

$$v_x = u_x = 30\sqrt{3} \text{ m/s}$$

On the other hand, the velocity in vertical direction is obtained, using equation of motion as :

$$\begin{aligned}v_y &= u_y - gt \\ \Rightarrow v_y &= 30 - 10 \times 1 \\ \Rightarrow v_y &= 20 \text{ m/s}\end{aligned}$$

The resultant velocity, v , is given by :

$$\begin{aligned}v &= \sqrt{(v_x^2 + v_y^2)} \\ \Rightarrow v &= \sqrt{\left\{ (30\sqrt{3})^2 + (20)^2 \right\}} = \sqrt{(900 \times 3 + 400)} = 55.68 \text{ m/s}\end{aligned}$$

Equation of the path of projectile

Equation of projectile path is a relationship between “x” and “y”. The x and y – coordinates are given by equations,

$$\begin{aligned}y &= u_y t - \frac{1}{2}gt^2 \\ x &= u_x t\end{aligned}$$

Eliminating “t” from two equations, we have :

Equation:

$$y = \frac{u_y x}{u_x} - \frac{gx^2}{2u_x^2}$$

For a given initial velocity and angle of projection, the equation reduces to the form of $y = Ax + Bx^2$, where A and B are constants. The equation of “y” in “x” is the equation of parabola. Hence, path of the projectile motion is a parabola. Also, putting expressions for initial velocity components $u_x = u \cos \theta$ and $u_y = u \sin \theta$, we have :

Equation:

$$\Rightarrow y = \frac{(u \sin \theta)x}{u \cos \theta} - \frac{gx^2}{2u^2 \cos^2 \theta}$$

$$\Rightarrow y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$$

Some other forms of the equation of projectile are :

Equation:

$$\Rightarrow y = x \tan \theta - \frac{gx^2 \sec^2 \theta}{2u^2}$$

Equation:

$$\Rightarrow y = x \tan \theta - \frac{gx^2 (1 + \tan^2 \theta)}{2u^2}$$

Exercises

Exercise:

Problem:

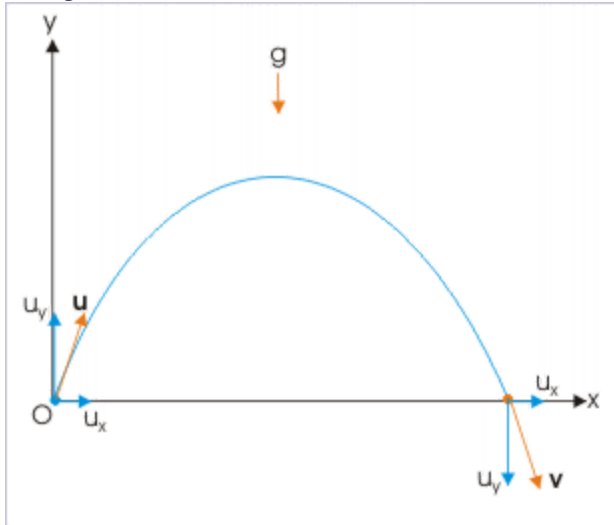
A projectile with initial velocity $2\mathbf{i} + \mathbf{j}$ is thrown in air (neglect air resistance). The velocity of the projectile before striking the ground is (consider $g = 10 \text{ m/s}^2$) :

(a) $\mathbf{i} + 2\mathbf{j}$ (b) $2\mathbf{i} - \mathbf{j}$ (c) $\mathbf{i} - 2\mathbf{j}$ (d) $2\mathbf{i} - 2\mathbf{j}$

Solution:

The vertical component of velocity of the projectile on return to the ground is equal in magnitude to the vertical component of velocity of projection, but opposite in direction. On the other hand, horizontal component of velocity remains unaltered. Hence, we can obtain velocity on the return to the ground by simply changing the sign of vertical component in the component expression of velocity of projection.

Projectile motion



Components of velocities

$$\mathbf{v} = 2\mathbf{i} - \mathbf{j}$$

Hence, option (b) is correct.

Exercise:

Problem:

Which of the following quantities remain unaltered during projectile motion :

- (a) vertical component of velocity and vertical component of acceleration
- (b) horizontal component of velocity and horizontal component of acceleration
- (c) vertical component of velocity and horizontal component of acceleration

(d) horizontal component of velocity and vertical component of acceleration

Solution:

The horizontal component of acceleration is zero. As such, horizontal component of velocity is constant. On the other hand, vertical component of acceleration is “g”, which is a constant. Clearly, vertical component of velocity is not constant.

Hence, options (b) and (d) are correct.

Exercise:

Problem:

Motion of a projectile is described in a coordinate system, where horizontal and vertical directions of the projectile correspond to x and y axes. At a given instant, the velocity of the projectile is $2\mathbf{i} + 3\mathbf{j}$ m/s. Then, we can conclude that :

- (a) the projectile has just started its motion
 - (b) the projectile is about to hit the ground
 - (c) the projectile is descending from the maximum height
 - (d) the projectile is ascending to the maximum height
-

Solution:

Here, the vertical component of the velocity (3 m/s) is positive. Therefore, projectile is moving in positive y-direction. It means that the projectile is still ascending to reach the maximum height (the vertical component of velocity at maximum height is zero). It is though possible that the given velocity is initial velocity of projection. However, the same can not be concluded.

Hence, option (d) is correct.

Exercise:**Problem:**

A projectile, thrown at angle " θ " with an initial velocity " u ", returns to the same horizontal ground level. If " x " and " y " coordinates are in horizontal and vertical directions, the equation of projectile in x and y coordinates has the form :

$$(a) \quad x = Ay - By^2$$

$$(b) \quad y = Ax - Bx^2$$

$$(c) \quad (1 - A)x = By^2$$

$$(d) \quad x = (A + B)y^2$$

Solution:

The equation of projectile is given as :

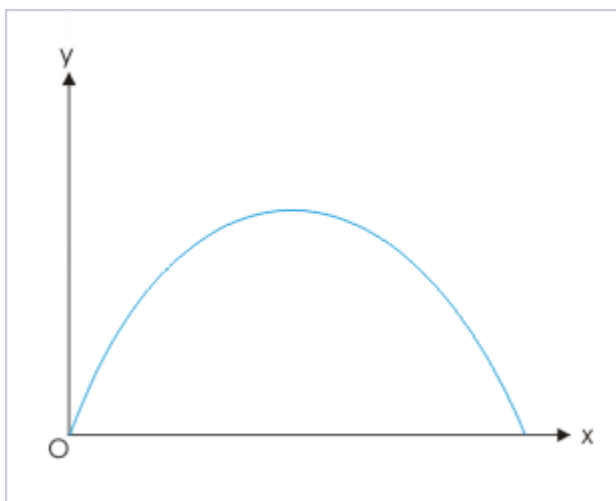
$$y = x \tan \theta - \frac{gx^2}{u^2 \cos^2 \theta}$$

This is a quadratic equation in x . The correct choice is (b).

Exercise:**Problem:**

Select correct observation(s) about the " xy " - plot of the projectile motion from the following :

Projectile motion



Projectile motion

- (a) it covers greater horizontal distance during the middle part of the motion.
 - (b) it covers greater vertical distance during the middle part of the motion.
 - (c) it covers lesser horizontal distance near the ground.
 - (d) it covers greater vertical distance near the ground.
-

Solution:

There is no component of acceleration in horizontal direction. The motion in this direction is a uniform motion and, therefore, covers equal horizontal distance in all parts of motion. It means that options (a) and (c) are incorrect.

In the vertical direction, projectile covers maximum distance when vertical component of velocity is greater. Now, projectile has greater vertical component of velocity near ground at the time of projection and at the time of return. As such, it covers maximum distance near the ground.

The correct choice is (d).

Exercise:

Problem:

A projectile is projected with an initial speed "u", making an angle " θ " to the horizontal direction along x-axis. Determine the average velocity of the projectile for the complete motion till it returns to the same horizontal plane.

Solution:

Average velocity is defined as the ratio of displacement and time. Since we treat projectile motion as two dimensional motion, we can find average velocity in two mutually perpendicular directions and then find the resultant average velocity. Projectile motion, however, is a unique case of uniform acceleration and we can find components of average velocity by averaging initial and final values.

If v_1 and v_2 be the initial and final velocities, then the average velocity for linear motion under constant acceleration is defined as :

$$v_{avg} = \frac{v_1 + v_2}{2}$$

Employing this relation in x-direction and making use of the fact that motion in horizontal direction is uniform motion, we have :

$$\begin{aligned}\Rightarrow v_{avgx} &= \frac{u_x + v_x}{2} \\ \Rightarrow v_{avgx} &= \frac{u \cos \theta + u \cos \theta}{2} = u \cos \theta\end{aligned}$$

Similarly, applying the relation of average velocity in y-direction and making use of the fact that component of velocity in vertical direction reverses its direction on return, we have :

$$\Rightarrow v_{avgy} = \frac{u_y + v_y}{2}$$

$$\Rightarrow v_{avgy} = \frac{u \sin \theta - u \sin \theta}{2} = 0$$

Hence, the resultant average velocity is :

$$\Rightarrow v_{avg} = v_{avgx} = u \cos \theta$$

Exercise:

Problem:

A projectile is projected with an initial speed "u", making an angle " θ " to the horizontal direction along x-axis. Determine change in speed of the projectile for the complete motion till it returns to the same horizontal plane.

Solution:

Both initial and final speeds are equal. Hence, there is no change in speed during the motion of projectile.

Exercise:

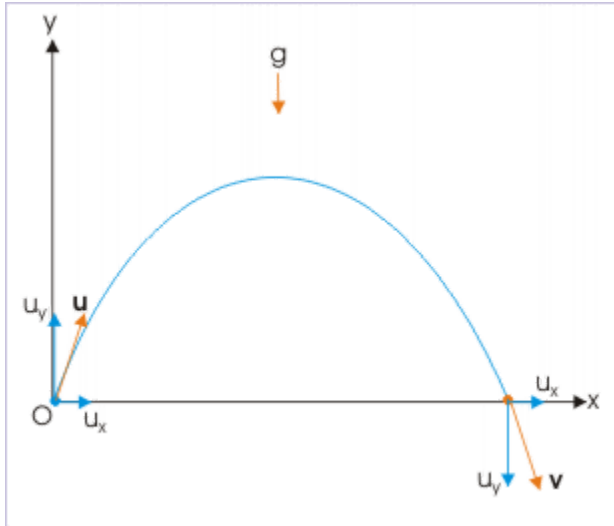
Problem:

A projectile is projected with an initial speed "u", making an angle " θ " to the horizontal direction along x-axis. Determine the change in velocity of the projectile for the complete motion till it returns to the same horizontal plane.

Solution:

Initial velocity is :

Projectile motion



Components of velocities

$$\mathbf{u} = u \cos \theta \mathbf{i} + u \sin \theta \mathbf{j}$$

Final velocity is :

$$\mathbf{v} = u \cos \theta \mathbf{i} - u \sin \theta \mathbf{j}$$

Change in velocity is given by :

$$\Rightarrow \Delta \mathbf{v} = u \cos \theta \mathbf{i} - u \sin \theta \mathbf{j} - u \cos \theta \mathbf{i} - u \sin \theta \mathbf{j}$$

$$\Rightarrow \Delta \mathbf{v} = -2u \sin \theta \mathbf{j}$$

where **j** is unit vector in y-direction.

Exercise:

Problem:

A projectile is projected at 60° to the horizontal with a speed of 10 m/s. After some time, it forms an angle 30° with the horizontal. Determine the speed (m/s) at this instant.

Solution:

We shall make use of the fact that horizontal component of projectile velocity does not change with time. Therefore, we can equate horizontal components of velocities at the time of projection and at the given instants and find out the speed at the later instant as required. Then,

$$\begin{aligned}u_x &= v_x \\ \Rightarrow u \cos 60^\circ &= v \cos 30^\circ \\ \Rightarrow \frac{10}{2} &= \frac{\sqrt{3}v}{2} \\ \Rightarrow v &= \frac{10}{\sqrt{3}} \text{ m/s}\end{aligned}$$

Exercise:**Problem:**

The horizontal and vertical components of a projectile at a given instant after projection are v_x and v_y respectively at a position specified as x (horizontal), y (vertical). Then,

- (a) The "x - t" plot is a straight line passing through origin.
- (b) The "y - t" plot is a straight line passing through origin.
- (c) The " $v_x - t$ " plot is a straight line passing through origin.
- (d) The " $v_y - t$ " plot is a straight line.

Solution:

The displacement in x-direction is given as :

$$x = v_x t$$

Since v_x is a constant, the "x-t" plot is a straight line passing through origin. Hence, option (a) is correct.

The displacement in y-direction is :

$$y = u_y t - \frac{1}{2} g t^2$$

This is an equation of parabola. The option (b), therefore, is incorrect.

The motion in horizontal direction is uniform motion. This means that component of velocity in x-direction is a constant. Therefore, " $v_x - t$ " plot should be a straight line parallel to time axis. It does not pass through origin. Hence, option (c) is incorrect.

The motion in vertical direction has acceleration due to gravity in downward direction. The component of velocity in y-direction is :

$$v_y = u_y - g t$$

This is an equation of straight line having slope of "-g" and intercept " u_y ". The " $v_y - t$ " plot, therefore, is a straight line. Hence, option (d) is correct.

Projectile motion (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the projectile motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Direction of motion on return
- Maximum height
- Equation of projectile motion
- Change in angles during motion
- Kinetic energy of a projectile
- Change in the direction of velocity vector

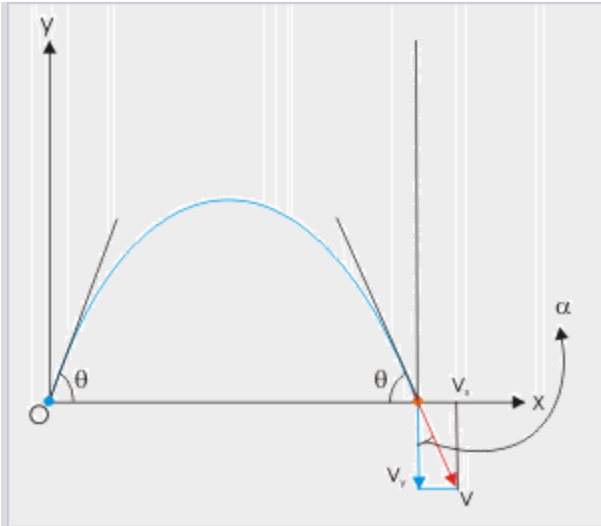
Direction of motion on return

Example:

Problem : A projectile is thrown with a speed of 15 m/s making an angle 60° with horizontal. Find the acute angle, " α ", that it makes with the vertical at the time of its return on the ground (consider $g = 10 \text{ m/s}^2$).

Solution : The vertical component of velocity of the projectile at the return on the ground is equal in magnitude, but opposite in direction. On the other hand, horizontal component of velocity remains unaltered. The figure, here, shows the acute angle that the velocity vector makes with vertical.

Projectile motion



The trajectory is symmetric about the vertical line passing through point of maximum height. From the figure, the acute angle with vertical is :

$$\alpha = 90^0 - \theta = 90^0 - 60^0 = 30^0$$

Maximum height

Example:

Problem : Motion of a projectile is described in a coordinate system, where horizontal and vertical directions of the projectile correspond to x and y axes. The velocity of the projectile is $12\mathbf{i} + 20\mathbf{j}$ m/s at an elevation of 15 m from the point of projection. Find the maximum height attained by the projectile (consider $g = 10 \text{ m/s}^2$).

Solution : Here, the vertical component of the velocity (20 m/s) is positive. It means that it is directed in positive y-direction and that the projectile is still ascending to reach the maximum height. The time to reach the maximum height is obtained using equation of motion in vertical direction :

$$v_y = u_y - gt$$

$$\Rightarrow 0 = 20 - 10t$$

$$\Rightarrow t = 2s$$

Now, the particle shall rise to a vertical displacement given by :

$$y' = u_y t - \frac{1}{2}gt^2 = 20 \times 2 - 5 \times 2^2 = 20 \text{ m}$$

The maximum height, as measured from the ground, is :

$$H = 15 + 20 = 35 \text{ m}$$

Equation of projectile motion

Example:

Problem : The equation of a projectile is given as :

$$y = \sqrt{3}x - \frac{1}{2}gx^2$$

Then, find the speed of the projection.

Solution : The general equation of projectile is :

$$y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$$

On the other hand, the given equation is :

$$y = \sqrt{3}x - \frac{1}{2}gx^2$$

Comparing two equations, we have :

$$\tan \theta = \sqrt{3}$$

$$\Rightarrow \theta = 60^\circ$$

Also,

$$\begin{aligned}
 u^2 \cos^2 \theta &= 1 \\
 \Rightarrow u^2 &= \frac{1}{\cos^2 \theta} \\
 \Rightarrow u^2 &= \frac{1}{\cos^2 60} = 4 \\
 \Rightarrow u &= 2 \text{ m/s}
 \end{aligned}$$

Change in angles during motion

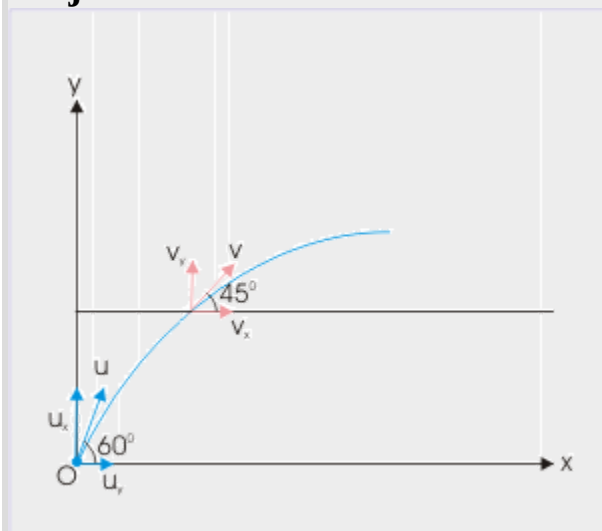
Example:

Problem : A projectile is projected at an angle 60° from the horizontal with a speed of $(\sqrt{3} + 1)$ m/s. The time (in seconds) after which the inclination of the projectile with horizontal becomes 45° is :

Solution : Let "u" and "v" be the speed at the two specified angles. The initial components of velocities in horizontal and vertical directions are :

$$\begin{aligned}
 u_x &= u \cos 60^\circ \\
 u_y &= u \sin 60^\circ
 \end{aligned}$$

Projectile motion



Similarly, the components of velocities, when projectile makes an angle 45 with horizontal, in horizontal and vertical directions are :

$$v_x = v \cos 45^0$$

$$v_y = v \sin 45^0$$

But, we know that horizontal component of velocity remains unaltered during motion. Hence,

$$v_x = u_x$$

$$\Rightarrow v \cos 45^0 = u \cos 60^0$$

$$\Rightarrow v = \frac{u \cos 60^0}{\cos 45^0}$$

Here, we know initial and final velocities in vertical direction. We can apply $v = u + at$ in vertical direction to know the time as required :

$$v \sin 45^0 = u + at = u \sin 60^0 - gt$$

$$\Rightarrow v \cos 45^0 = u \cos 60^0$$

$$\Rightarrow t = \frac{u \sin 60^0 - v \sin 45^0}{g}$$

Substituting value of "v" in the equation, we have :

$$\Rightarrow t = \frac{u \sin 60^0 - u \frac{\cos 60^0}{\cos 45^0} \times \sin 45^0}{g}$$

$$\Rightarrow t = \frac{u}{g} \sin 60^0 - \cos 60^0$$

$$\Rightarrow t = \frac{\sqrt{3}+1}{10} - \frac{\sqrt{3}-1}{2}$$

$$\Rightarrow t = \frac{2}{20} = 0.1 \text{ s}$$

Kinetic energy of a projectile

Example:

Problem : A projectile is thrown with an angle θ from the horizontal with a kinetic energy of K Joule. Find the kinetic energy of the projectile (in Joule), when it reaches maximum height.

Solution : At the time of projection, the kinetic energy is given by :

$$K = \frac{1}{2}mu^2$$

At the maximum height, vertical component of the velocity is zero. On the other hand, horizontal component of the velocity of the particle does not change. Thus, the speed of the particle, at the maximum height, is equal to the magnitude of the horizontal component of velocity. Hence, speed of the projectile at maximum height is :

$$v = u \cos \theta$$

The kinetic energy at the maximum height, therefore, is :

$$K' = \frac{1}{2}m(u \cos \theta)^2$$

Substituting value of "u" from the expression of initial kinetic energy is :

$$K' = \frac{mx2xK}{2m} \cos^2 \theta$$

$$K' = K \cos^2 \theta$$

Change in the direction of velocity vector

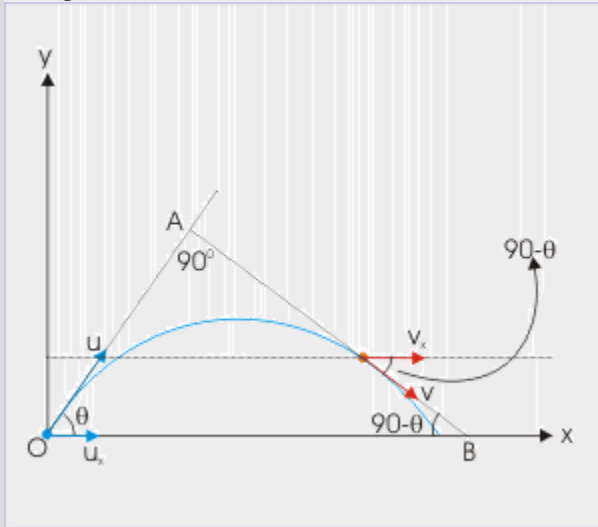
Example:

Problem : A projectile with a speed of "u" is thrown at an angle of " θ " with the horizontal. Find the speed (in m/s) of the projectile, when it is perpendicular to the direction of projection.

Solution : We need to visualize the direction of the projectile, when its direction is perpendicular to the direction of projection. Further, we may look to determine the direction of velocity in that situation.

The figure, here, shows the direction of velocity for the condition, when the direction of projectile is perpendicular to the direction of projection. From ΔOAB ,

Projectile motion



$$\angle OBA = 180^\circ - 90^\circ + \theta = 90^\circ - \theta$$

Thus, the acute angle between projectile and horizontal direction is $90^\circ - \theta$ for the given condition. Now, in order to determine the speed, we use the fact that horizontal component of velocity does not change.

$$v \cos 90^\circ - \theta = u \cos \theta$$

$$\Rightarrow v \sin \theta = u \cos \theta$$

$$\Rightarrow v = u \cot \theta$$

Features of projectile motion

The span of projectile motion in the vertical plane is determined by two factors, namely the speed of projection and angle of projection with respect to horizontal. These two factors together determine (i) how long does the projectile remain in air (time of flight, T) (ii) how far does the projectile go in the horizontal direction (range of projectile, R) and (iii) how high does the projectile reach (maximum height, H).

Further, the trajectory of the projectile is symmetric about a vertical line passing through the point of maximum height if point of projection and point of return fall on the same horizontal surface.

Time of flight, T

We have already determined the time of flight, which is given by :

Equation:

$$T = \frac{2u_y}{g} = \frac{2u \sin \theta}{g}$$

This equation was derived in the earlier module [Projectile motion](#) with the assumption that both point of projection and point of return of the projectile lie on same horizontal level. It may be also be recalled that the equation of motion in vertical direction was evaluated for the condition that net displacement during the entire motion is zero. Hence, if the points are not on the same level, then above equation will not be valid and must be determined by equation of motion for the individual case with appropriate values.

From the above equation, we see that time of flight depends on initial speed and the angle of projection (θ). We must realize here that the range of θ is $0^\circ \leq \theta \leq 90^\circ$. For this range, $\sin\theta$ is an increasing function. As such, we can say that a projection closer to vertical direction stays longer in the air for a given initial velocity. As a matter of fact, a vertical projectile for which $\theta = 90^\circ$ and $\sin\theta = 1$, stays in the air for the maximum period.

Exercise:**Problem:**

If the points of projection and return are on same level and air resistance is neglected, which of the following quantities will enable determination of the total time of flight (T) :

- (a) horizontal component of projection velocity
 - (b) projection speed and angle of projection
 - (c) vertical component of projection velocity
 - (d) speed at the highest point
-

Solution:

The total time of flight is given by :

$$T = \frac{2u_y}{g} = \frac{2u \sin \theta}{g}$$

We can see that the total time of flight can be determined if vertical component of the velocity (u_y) is given. Hence option (c) is correct. The vertical component of the velocity (u_y), in turn, is determined by the projection speed (u) and angle of projection (θ). Hence option (b) is correct.

Now speed at the highest point is equal to the horizontal component of projection velocity ($u \cos \theta$). We can not, however, determine vertical component ($u \sin \theta$) from this value, unless either " u " or " θ " is also given.

Hence, options (b) and (c) are correct.

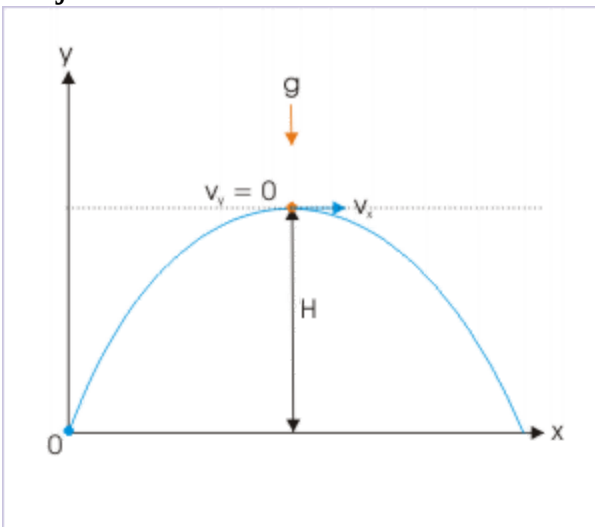
Note: We have noticed that time of flight is derived considering vertical motion. Horizontal part of the motion is not considered. Thus, the time of flight (t) at any point during the projectile motion is

dependent on vertical component of velocity or vertical part of the motion and is independent of horizontal part of the motion.

Maximum height reached by the projectile, H

The vertical component velocity of the projectile reduces to zero as motion decelerates while going up against the force due to gravity. The point corresponding to this situation, when vertical component of velocity is zero, $v_y = 0$, is the maximum height that a projectile can reach. The projectile is accelerated downward under gravity immediately thereafter. Now, considering upward motion in y-direction,

Projectile motion



Maximum height attained by
projectile

$$\begin{aligned}v_y &= u_y - gt \\ \Rightarrow 0 &= u_y - gt \\ \Rightarrow t &= \frac{u_y}{g}\end{aligned}$$

Now, using equation for displacement in the vertical direction, we can find out the vertical displacement i.e. the height as :

$$\begin{aligned}
 y &= u_y t - \frac{1}{2} g t^2 \\
 \Rightarrow H &= u_y \frac{u_y}{g} - \frac{1}{2} \frac{u_y^2}{g} \\
 \Rightarrow H &= \frac{u_y^2}{g} - \frac{u_y^2}{2g} \\
 \Rightarrow H &= \frac{u_y^2}{2g}
 \end{aligned}$$

Putting expression of component velocity ($u_y = u \sin \theta$), we have :

Equation:

$$\Rightarrow H = \frac{u^2 \sin^2 \theta}{2g}$$

We can also obtain this expression, using relation $v_y^2 = u_y^2 + 2gy$. Just like the case of the time of flight, we see that maximum height reached by the projectile depends on both initial speed and the angle of projection (θ). Greater the initial velocity and greater the angle of projection from horizontal direction, greater is the height attained by the projectile.

It is important to realize here that there is no role of the horizontal component of initial velocity as far as maximum height is concerned. It is logical also. The height attained by the projectile is purely a vertical displacement; and as motions in the two mutually perpendicular directions are independent of each other, it follows that the maximum height attained by the projectile is completely determined by the vertical component of the projection velocity.

Exercise:

Problem:

The maximum height that a projectile, thrown with an initial speed " v_0 ", can reach :

$$(a) \frac{v_0}{2g} \quad (b) \frac{v_0^2}{2g} \quad (c) \frac{v_0^2}{g} \quad (d) \frac{v_0}{g}$$

Solution:

The question only specifies speed of projection - not the angle of projection. Now, projectile rises to greatest maximum height for a given speed, when it is thrown vertically. In this case, vertical component of velocity is equal to the speed of projection itself. Further, the speed of the projectile is zero at the maximum height. Using equation of motion, we have :

$$0 = v_0^2 - 2gH$$
$$\Rightarrow H = \frac{v_0^2}{2g}$$

The assumption for the maximum height as outlined above can also be verified from the general formula of maximum height of projectile as given here :

$$H = \frac{u_y^2}{2g} = \frac{u^2 \sin^2 \theta}{2g} = \frac{v_0^2 \sin^2 \theta}{2g}$$

The numerator of the above is maximum when the angle is 90° .

$$\Rightarrow H = \frac{v_0^2}{2g}$$

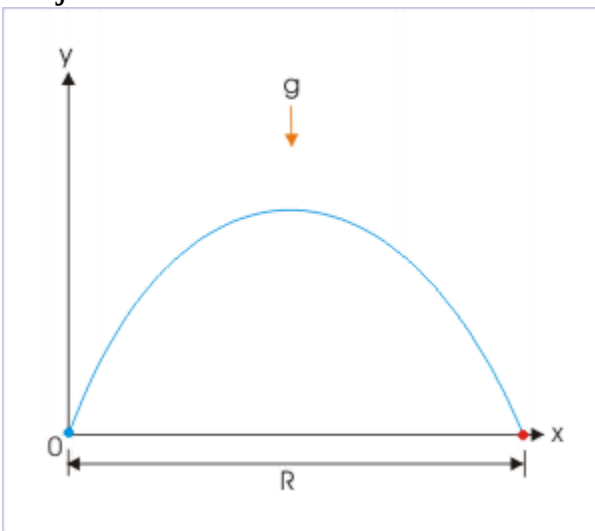
We should note that the formula of maximum height for a projectile projected at certain angle represents the maximum height of the projectile for the given angle of projection and speed. The question here, however, refers to maximum height for any angle of projection at given speed of projection. As such, we should consider an angle of projection for which projectile reaches the greatest height. This point should be kept in mind.

Hence, option (b) is correct.

Range of projectile, R

The horizontal range is the displacement in horizontal direction. There is no acceleration involved in this direction. Motion is an uniform motion. It follows that horizontal range is transversed with the horizontal component of the projection velocity for the time of flight (T). Now,

Projectile motion



The range of a projectile

$$x = u_x t$$

$$\Rightarrow R = u_x T = u \cos \theta \times T$$

where “T” is the time of flight. Putting expression for the time of flight,

$$\Rightarrow R = \frac{u \cos \theta \times 2u \sin \theta}{g}$$

Equation:

$$\Rightarrow R = \frac{u^2 \sin 2\theta}{g}$$

Horizontal range, like time of flight and maximum height, is greater for greater projection speed.

For projection above ground surface, the range of the angle of projection with respect to horizontal direction, θ , is $0^\circ \leq \theta \leq 90^\circ$ and the corresponding range of 2θ is $0^\circ \leq 2\theta \leq 180^\circ$. The “ $\sin 2\theta$ ”, as appearing in the numerator for the expression of the horizontal range, is an increasing function for $0^\circ \leq \theta \leq 45^\circ$ and a decreasing function for $45^\circ \leq \theta \leq 90^\circ$. For $\theta = 45^\circ$, $\sin 2\theta = \sin 90^\circ = 1$ (maximum).

In the nutshell, the range, R , increases with increasing angle of projection for $0^\circ \leq \theta < 45^\circ$; the range, R , is maximum when $\theta = 45^\circ$; the range, R , decreases with increasing angle of projection for $45^\circ < \theta \leq 90^\circ$.

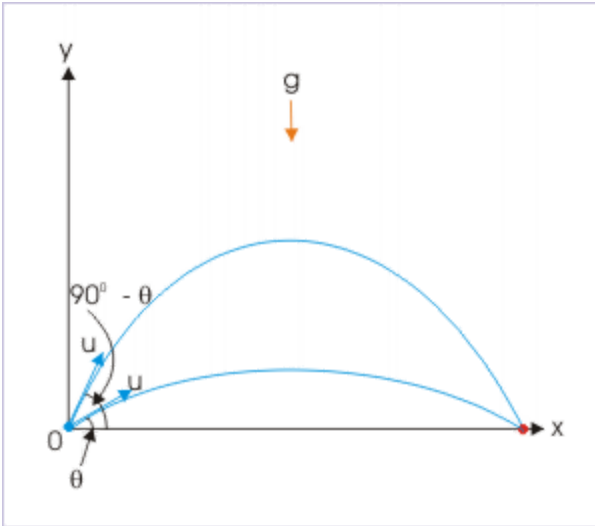
The maximum horizontal range for a given projection velocity is obtained for $\theta = 45^\circ$ as :

$$\Rightarrow R = \frac{u^2}{g}$$

There is an interesting aspect of “ $\sin 2\theta$ ” function that its value repeats for component angle i.e $(90^\circ - \theta)$. For, any value of θ ,

$$\sin 2(90^\circ - \theta) = \sin(180^\circ - 2\theta) = \sin 2\theta$$

Horizontal range



Horizontal range is same for a pair of projection angle.

It means that the range of the projectile with a given initial velocity is same for a pair of projection angles θ and $90^\circ - \theta$. For example, if the range of the projectile with a given initial velocity is 30 m for an angle of projection, $\theta = 15^\circ$, then the range for angle of projection, $\theta = 90^\circ - 15^\circ = 75^\circ$ is also 30m.

Equation of projectile motion and range of projectile

Equation of projectile motion renders to few additional forms in terms of characteristic features of projectile motion. One such relation incorporates range of projectile (R) in the expression. The equation of projectile motion is :

$$y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$$

The range of projectile is :

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 2 \sin \theta \cos \theta}{g}$$

Solving for u^2 ,

$$\Rightarrow u^2 = \frac{Rg}{2 \sin \theta \cos \theta}$$

Substituting in the equation of motion, we have :

$$\Rightarrow y = x \tan \theta - \frac{2 \sin \theta \cos \theta g x^2}{2 R g \cos^2 \theta}$$

$$\Rightarrow y = x \tan \theta - \frac{\tan \theta x^2}{R}$$

$$\Rightarrow y = x \tan \theta \left(1 - \frac{x}{R} \right)$$

It is a relatively simplified form of equation of projectile motion. Further, we note that a new variable “R” is introduced in place of “u”.

Exercise:

Problem:

If points of projection and return are on same level and air resistance is neglected, which of the following quantities will enable determination of the range of the projectile (R) :

- (a) horizontal component of projection velocity
 - (b) projection speed and angle of projection
 - (c) vertical component of projection velocity
 - (d) speed at the highest point
-

Solution:

The horizontal range is determined using formulae,

$$R = \frac{u^2 \sin 2\theta}{g}$$

Hence, horizontal range of the projectile can be determined when projection speed and angle of projection are given. The inputs required in this equation can not be made available with other given quantities.

Hence, option (b) is correct.

Note : Horizontal range (R) unlike time of flight (T) and maximum height (H), depends on both vertical and horizontal motion. This aspect is actually concealed in the term " $\sin 2\theta$ ". The formula of horizontal range (R) consists of both " $u \sin \theta$ " (for vertical motion) and " $u \cos \theta$ " (for horizontal motion) as shown here :

$$\begin{aligned}\Rightarrow R &= \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin \theta \cos \theta}{g} = \frac{2(u \cos \theta)(u \sin \theta)}{g} \\ \Rightarrow R &= \frac{2u_x u_y}{g}\end{aligned}$$

Exercise:**Problem:**

A projectile is thrown with a velocity $6\mathbf{i} + 20\mathbf{j} \text{ m/s}$. Then, the range of the projectile (R) is :

- (a) 12 m (b) 20 m (c) 24 m (d) 26 m

Solution:

We shall not use the standard formulae as it would be difficult to evaluate angle of projection from the given data. Now, the range of the projectile (R) is given by :

$$R = u_x T$$

Here,

$$u_x = 6 \text{ m/s}$$

We need to know the total time of flight, T. For motion in vertical direction, the vertical displacement is zero. This consideration gives the time of flight as :

$$T = \frac{2u_y}{g} = \frac{2 \times 20}{10} = 4 \text{ s}$$

Hence, range of the flight is :

$$\Rightarrow R = u_x T = 6 \times 4 = 24$$

Hence, option (c) is correct.

Impact of air resistance

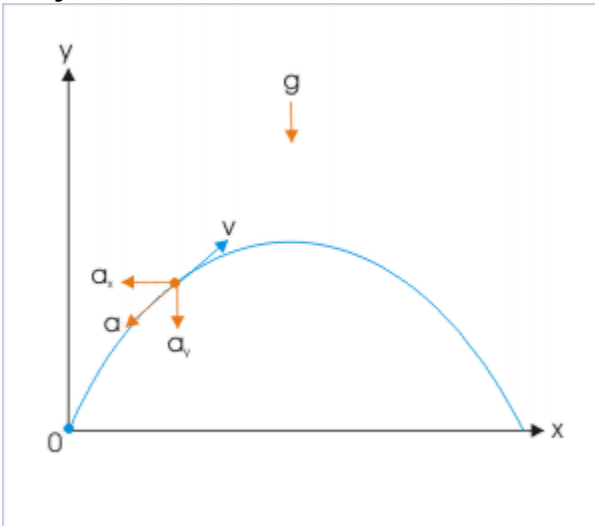
We have so far neglected the effect of air resistance. It is imperative that if air resistance is significant then the features of a projectile motion like time of flight, maximum height and range are modified. As a matter of fact, this is the case in reality. The resulting motion is generally adversely affected as far as time of flight, maximum height and the range of the projectile are concerned.

Air resistance is equivalent to friction force for solid (projectile) and fluid (air) interface. Like friction, air resistance is self adjusting in certain ways. It adjusts to the relative speed of the projectile. Generally, greater the speed greater is air resistance. Air resistance also adjusts to the direction of motion such that its direction is opposite to the direction of relative velocity of two entities. In the nutshell, air resistance opposes motion and is

equivalent to introducing a variable acceleration (resistance varies with the velocity in question) in the direction opposite to that of velocity.

For simplicity, if we consider that resistance is constant, then the vertical component of acceleration (a_y) due to resistance acts in downward direction during upward motion and adds to the acceleration due to gravity. On the other hand, vertical component of air resistance acts in upward direction during downward motion and negates to the acceleration due to gravity. Whereas the horizontal component of acceleration due to air resistance (a_x) changes the otherwise uniform motion in horizontal direction to a decelerated motion.

Projectile motion with air resistance



Acceleration due to air
resistance

With air resistance, the net or resultant acceleration in y direction depends on the direction of motion. During upward motion, the net or resultant vertical acceleration is " $-g - a_y$ ". Evidently, greater vertical acceleration acting downward reduces speed of the particle at a greater rate. This, in turn, reduces maximum height. During downward motion, the net or resultant vertical acceleration is " $-g + a_y$ ". Evidently, lesser vertical acceleration acting downward increases speed of the particle at a slower

rate. Clearly, accelerations of the projectile are not equal in upward and downward motions. As a result, projection velocity and the velocity of return are not equal.

On the other hand, the acceleration in x direction is " $-a_x$ ". Clearly, the introduction of horizontal acceleration opposite to velocity reduces the range of the projectile (R).

Situations involving projectile motion

There are classic situations relating to the projectile motion, which needs to be handled with appropriate analysis. We have quite a few ways to deal with a particular situation. It is actually the nature of problem that would determine a specific approach from the following :

1. We may approach a situation analyzing as two mutually perpendicular linear motions. This is the basic approach.
2. In certain cases, the equations obtained for the specific attributes of the projectile motion such as range or maximum height can be applied directly.
3. In some cases, we would be required to apply the equation of path, which involves displacements in two directions (x and y) simultaneously in one equation.
4. We always have the option to use composite vector form of equations for two dimensional motion, where unit vectors are involved.

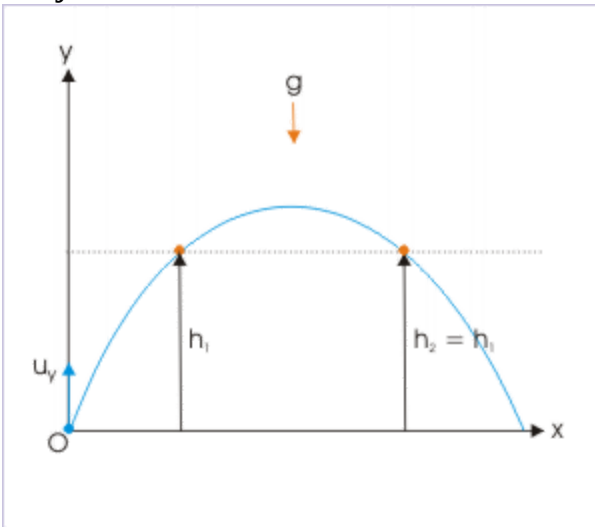
Besides, we may require combination of approached as listed above. In this section, we shall study these classic situations involving projectile motion.

Clearing posts of equal height

A projectile can clear posts of equal height, as projectile retraces vertical displacement attained during upward flight while going down. We can approach such situation in two alternative ways. The equation of motion for displacement yields two values for time for a given vertical displacement (height) : one corresponds to the time for upward flight and other for the

downward flight as shown in the figure below. Corresponding to these two time values, we determine two values of horizontal displacement (x).

Projectile motion



The projectile retraces vertical displacement.

Alternatively, we may use equation of trajectory of the projectile. The y coordinate has a quadratic equation in " x ". it again gives two values of " x " for every value of " y ".

Example:

Problem : A projectile is thrown with a velocity of $25\sqrt{2}$ m/s and at an angle 45° with the horizontal. The projectile just clears two posts of height 30 m each. Find (i) the position of throw on the ground from the posts and (ii) separation between the posts.

Solution : Here, we first use the equation of displacement for the given height in the vertical direction to find the values of time when projectile reaches the specified height. The equation of displacement in vertical direction (y) under constant acceleration is a quadratic equation in time (t). Its solution yields two values for time. Once two time instants are known,

we apply the equation of motion for uniform motion in horizontal direction to determine the horizontal distances as required. Here,

$$u_y = 25\sqrt{2} \times \sin 45^\circ = 25 \text{ m / s}$$

$$\begin{aligned} y &= u_y t - \frac{1}{2} g t^2 \\ \Rightarrow 30 &= 25t - \frac{1}{2} \times 10 \times t^2 \\ \Rightarrow t^2 - 5t + 6 &= 0 \\ \Rightarrow t &= 2 \text{ s or } 3 \text{ s} \end{aligned}$$

Horizontal motion (refer the figure) :

$$OA = u_x t = 25\sqrt{2} \times \cos 45^\circ \times 2 = 50 \text{ m}$$

Thus, projectile needs to be thrown from a position 50 m from the pole. Now,

$$OB = u_x t = 25\sqrt{2} \times \cos 45^\circ \times 3 = 75 \text{ m}$$

Hence, separation, d, is :

$$d = OB - OA = 75 - 50 = 25 \text{ m}$$

Alternatively

Equation of vertical displacement (y) is a quadratic equation in horizontal displacement (x). Solution of equation yields two values of "x" corresponding to two positions having same elevation. Now, equation of projectile is given by :

$$y = x \tan \theta - \frac{g x^2}{2 u^2 \cos^2 \theta}$$

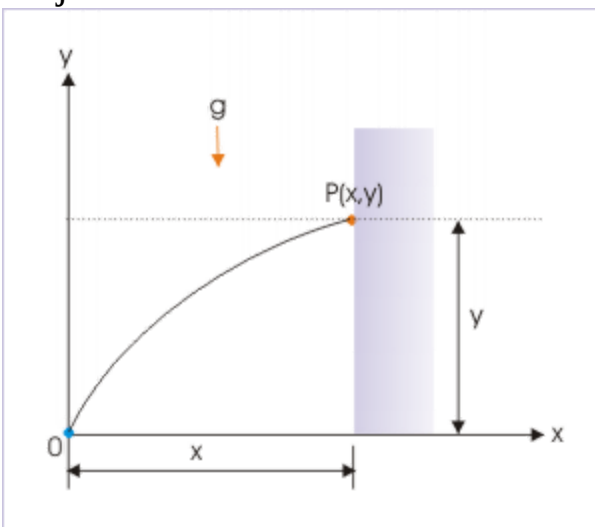
Putting values,

$$\begin{aligned}
30 &= x \tan 45^\circ - \frac{10x^2}{2\sqrt{25}^2 \cos^2 45^\circ} \\
\Rightarrow 30 &= x - \frac{10x^2}{225\sqrt{2}^2 \cos^2 45^\circ} \\
\Rightarrow 30 &= x - \frac{x^2}{125} \\
\Rightarrow x^2 - 125x + 3750 &= 0 \\
\Rightarrow (x - 50)(x - 75) &= 0 \\
\Rightarrow x &= 50 \text{ or } 75 \text{ m} \\
\Rightarrow d &= 75 - 50 = 25 \text{ m}
\end{aligned}$$

Hitting a specified target

An archer aims a bull's eye; a person throws a pebble to strike an object placed at height and so on. The motion involved in these situations is a projectile motion – not a straight line motion. The motion of the projectile (arrow or pebble) has an arched trajectory due to gravity. We need to aim higher than line of sight to the object in order to negotiate the loss of height during flight.

Projectile motion



The projectile hits the wall at a given point.

Hitting a specified target refers to a target whose coordinates (x,y) are known. There are two different settings of the situation. In one case, the angle of projection is fixed. We employ equation of the projectile to determine the speed of projectile. In the second case, speed of the projectile is given and we need to find the angle(s) of projection. The example here illustrates the first case.

Example:

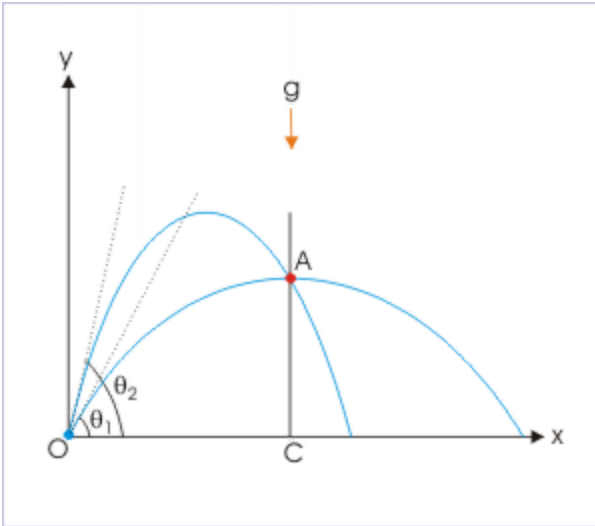
Problem : A projectile, thrown at an angle 45° from the horizontal, strikes a building 30 m away at a point 15 above the ground. Find the velocity of projection.

Solution : As explained, the equation of projectile path suits the description of motion best. Here,
 $x = 30 \text{ m}$, $y = 15 \text{ m}$ and $\theta = 45^\circ$. Now,

$$\begin{aligned}
 y &= x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta} \\
 \Rightarrow 15 &= 30 \tan 45^\circ - \frac{10 \times 30^2}{2u^2 \cos^2 45^\circ} \\
 \Rightarrow 15 &= 30 \times 1 - \frac{10 \times 2 \times 30^2}{2u^2} \\
 \Rightarrow 15 &= 30 - \frac{9000}{u^2} \\
 \Rightarrow u^2 &= 600 \\
 \Rightarrow u &= 24.49 \text{ m/s}
 \end{aligned}$$

As pointed out earlier, we may need to determine the angle(s) for a given speed such that projectile hits a specified target having known coordinates. This presents two possible angles with which projectile can be thrown to hit the target. This aspect is clear from the figure shown here :

Projectile motion



Projectile motion

Clearly, we need to use appropriate form of equation of motion which yields two values of angle of projection. This form is :

$$y = x \tan \theta - \frac{gx^2(1+\tan^2 \theta)}{2u^2}$$

This equation, when simplified, form a quadratic equation in " $\tan \theta$ ". This in turn yields two values of angle of projection. Smaller of the angles gives the projection for least time of flight.

Example:

Problem : A person standing 50 m from a vertical pole wants to hit the target kept on top of the pole with a ball. If the height of the pole is 13 m and his projection speed is $10\sqrt{g}$ m/s, then what should be the angle of projection of the ball so that it strikes the target in minimum time?

Solution : Equation of projectile having square of " $\tan \theta$ " is :

$$\Rightarrow y = x \tan \theta - \frac{gx^2}{2u^2} (1 + \tan^2 \theta)$$

Putting values,

$$\Rightarrow 13 = 50 \tan \theta - \frac{10 \times 50^2}{2 \times (10\sqrt{g})^2} (1 + \tan^2 \theta)$$

$$\Rightarrow 25 \tan^2 \theta - 100 \tan \theta + 51 = 0$$

$$\Rightarrow \tan \theta = 17/5 \quad \text{or} \quad 3/5$$

Taking the smaller angle of projection to hit the target,

$$\Rightarrow \theta = \tan^{-1} \left(\frac{3}{5} \right)$$

Determining attributes of projectile trajectory

A projectile trajectory under gravity is completely determined by the initial speed and the angle of projection or simply by the initial velocity (direction is implied). For the given velocity, maximum height and the range are unique – notably independent of the mass of the projectile.

Thus, a projectile motion involving attributes such as maximum height and range is better addressed in terms of the equations obtained for the specific attributes of the projectile motion.

Example:

Problem : Determine the angle of projection for which maximum height is equal to the range of the projectile.

Solution : We equate the expressions of maximum height and range ($H = R$) as :

$$\frac{u^2 \sin^2 \theta}{2g} = \frac{u^2 \sin 2\theta}{g}$$

$$\begin{aligned}u^2 \sin^2 \theta &= 2u^2 \sin 2\theta \\ \Rightarrow \sin^2 \theta &= 2 \sin 2\theta = 4 \sin \theta \times \cos \theta \\ \Rightarrow \sin \theta &= 4 \cos \theta \\ \Rightarrow \tan \theta &= 4 \\ \Rightarrow \theta &= \tan^{-1}(4)\end{aligned}$$

Exercises

Exercise:

Problem:

Which of the following is/ are independent of the angle of projection of a projectile :

- (a) time of flight
- (b) maximum height reached
- (c) acceleration of projectile
- (d) horizontal component of velocity

Solution:

The time of flight is determined by considering vertical motion. It means that time of flight is dependent on speed and the angle of projection.

$$T = \frac{2u_y}{g} = \frac{2u \sin \theta}{g}$$

Maximum height is also determined, considering vertical motion. As such, maximum height also depends on the angle of projection.

$$H = \frac{u_y^2}{2g} = \frac{u^2 \sin^2 \theta}{2g}$$

Horizontal component, being component of velocity, depends on the angle of projection.

$$u_y = u \cos \theta$$

It is only the acceleration of projectile, which is equal to acceleration due to gravity and is, therefore, independent of the angle of projection. Hence, option (c) is correct.

Exercise:

Problem:

Two particles are projected with same initial speeds at 30° and 60° with the horizontal. Then

- (a) their maximum heights will be equal
- (b) their ranges will be equal
- (c) their time of flights will be equal
- (d) their ranges will be different

Solution:

The maximum heights, ranges and time of lights are compared, using respective formula as :

(i)Maximum Height

$$\Rightarrow \frac{H_1}{H_2} = \frac{u^2 \sin^2 \theta_1}{u^2 \sin^2 \theta_2} = \frac{\sin^2 30^\circ}{\sin^2 60^\circ} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{3}$$

Thus, the maximum heights attained by two projectiles are unequal.

(ii) Range :

$$\Rightarrow \frac{R_1}{R_2} = \frac{u^2 \sin 2\theta_1}{u^2 \sin 2\theta_2} = \frac{\sin^2 60^\circ}{\sin^2 120^\circ} = \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{1}$$

Thus, the ranges of two projectiles are equal.

(iii) Time of flight

$$\Rightarrow \frac{T_1}{T_2} = \frac{2u \sin \theta_1}{2u \sin \theta_2} = \frac{\sin 30^\circ}{\sin 60^\circ} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}}$$

Thus, the times of flight of two projectiles are unequal.

Hence, option (b) is correct.

Exercise:

Problem:

The velocity of a projectile during its flight at an elevation of 8 m from the ground is $3\mathbf{i} - 5\mathbf{j}$ in the coordinate system, where x and y directions represent horizontal and vertical directions respectively. The maximum height attained (H) by the particle is :

$$(a) 10.4 \text{ m} \quad (b) 8.8 \text{ m} \quad (c) 9.25 \text{ m} \quad (d) 9 \text{ m}$$

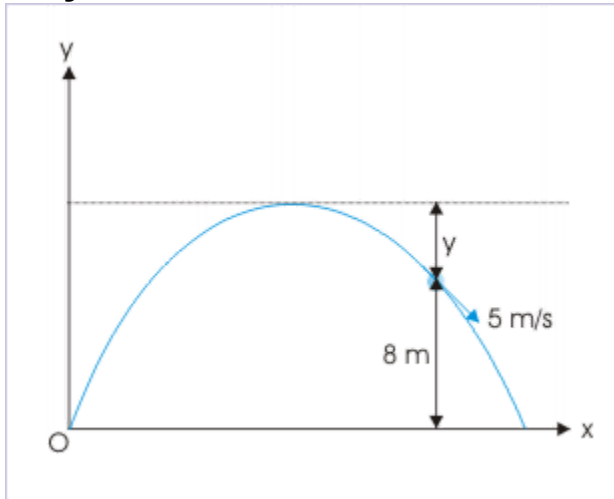
Solution:

We note that vertical component is negative, meaning that projectile is moving towards the ground. The vertical component of velocity 8 m above the ground is

$$v_y = -5 \text{ m/s}$$

The vertical displacement (y) from the maximum height to the point 8 m above the ground as shown in the figure can be obtained, using equation of motion.

Projectile motion



Projectile motion

$$v_y^2 = u_y^2 + 2ay$$

Considering the point under consideration as origin and upward direction as positive direction.

$$\Rightarrow (-5)^2 = 0 + 2X - 10Xh$$

$$\Rightarrow h = -25/20 = -1.25 \quad m$$

Thus, the maximum height, H , attained by the projectile is :

$$\Rightarrow H = 8 + 1.25 = 9.25 \quad m$$

Hence, option (c) is correct.

Exercise:

Problem:

A projectile is thrown with a given speed so as to cover maximum range (R). If "H" be the maximum height attained during the throw, then the range "R" is equal to :

(a) $4H$ (b) $3H$ (c) $\sqrt{2}H$ (d) H

Solution:

The projectile covers maximum range when angle of projection is equal to 45° . The maximum range "R" is given by :

$$\Rightarrow R = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin 90^\circ}{g} = \frac{u^2}{g}$$

On the other hand, the maximum height attained by the projectile for angle of projection, 45° , is :

$$\Rightarrow H = \frac{u_y^2}{2g} = \frac{u^2 \sin^2 45^\circ}{2g} = \frac{u^2}{4g}$$

Comparing expressions of range and maximum height, we have :

$$\Rightarrow R = 4H$$

Hence, option (a) is correct.

Exercise:**Problem:**

The speed of a projectile at maximum height is half its speed of projection, "u". The horizontal range of the projectile is :

(a) $\frac{\sqrt{3}u^2}{g}$ (b) $\frac{\sqrt{3}u^2}{2g}$ (c) $\frac{u^2}{4g}$ (d) $\frac{u^2}{2g}$

Solution:

The horizontal range of the projectile is given as :

$$R = \frac{u^2 \sin 2\theta}{g}$$

In order to evaluate this expression, we need to know the angle of projection. Now, the initial part of the question says that the speed of a projectile at maximum height is half its speed of projection, "u". However, we know that speed of the projectile at the maximum height is equal to the horizontal component of projection velocity,

$$u \cos \theta = \frac{u}{2}$$

$$\Rightarrow \cos \theta = \frac{1}{2} = \cos 30^\circ$$

$$\Rightarrow \theta = 30^\circ$$

The required range is :

$$\Rightarrow R = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin 60^\circ}{g} = \frac{\sqrt{3}u^2}{2g}$$

Hence, option (b) is correct.

Exercise:**Problem:**

Let " T_1 " and " T_2 " be the times of flights of a projectile for projections at two complimentary angles for which horizontal range is "R". The product of times of flight, " $T_1 T_2$ ", is equal to :

$$(a) \frac{R}{g} \quad (b) \frac{R^2}{g} \quad (c) \frac{2R}{g} \quad (d) \frac{R}{2g}$$

Solution:

Here, we are required to find the product of times of flight. Let " θ " and " $90^\circ - \theta$ " be two angles of projections. The times of flight are given as :

$$T_1 = \frac{2u \sin \theta}{g}$$

and

$$T_2 = \frac{2u \sin(90^\circ - \theta)}{g} = \frac{2u \cos \theta}{g}$$

Hence,

$$\Rightarrow T_1 T_2 = \frac{4u^2 \sin \theta \cos \theta}{g^2}$$

But,

$$\Rightarrow R = \frac{2u^2 \sin \theta \cos \theta}{g}$$

Combining two equations,

$$\Rightarrow T_1 T_2 = \frac{2R}{g}$$

Hence, option (c) is correct.

Exercise:**Problem:**

A projectile is projected with a speed " u " at an angle " θ " from the horizontal. The magnitude of average velocity between projection and the time, when projectile reaches the maximum height.

$$(a) u \cos^2 \theta \quad (b) \sqrt{(3u^2 \cos \theta + 1)}$$

$$(c) \frac{\sqrt{(3u^2 \cos \theta + 1)}}{2u} \quad (d) \frac{u \sqrt{(3u^2 \cos \theta + 1)}}{2}$$

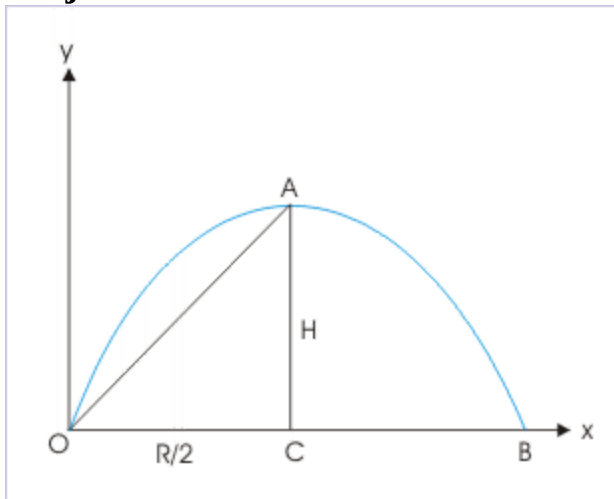
Solution:

The magnitude of average velocity is given as :

$$v_{avg} = \frac{\text{Displacement}}{\text{Time}}$$

Now the displacement here is OA as shown in the figure. From right angle triangle OAC,

Projectile motion



Projectile motion

$$OA = \sqrt{(OC^2 + AC^2)} = \sqrt{\left\{ \left(\frac{R}{2} \right)^2 + H^2 \right\}}$$

$$\Rightarrow OA = \sqrt{\left(\frac{R^2}{4} + H^2\right)}$$

Hence, magnitude of average velocity is :

$$\Rightarrow v_{avg} = \frac{\sqrt{\left(\frac{R^2}{4} + H^2\right)}}{T}$$

Substituting expression for each of the terms, we have :

$$\Rightarrow v_{avg} = \frac{\sqrt{\left(\frac{u^4 \sin^2 2\theta}{4g^2} + \frac{u^4 \sin^4 \theta}{4g^2}\right)}}{\frac{2u \sin \theta}{g}}$$

$$v_{avg} = u (3 \cos^2 \theta + 1)/2$$

$$\Rightarrow v_{avg} = \frac{u \sqrt{(3 \cos^2 \theta + 1)}}{2}$$

Hence, option (d) is correct.

Exercise:

Problem:

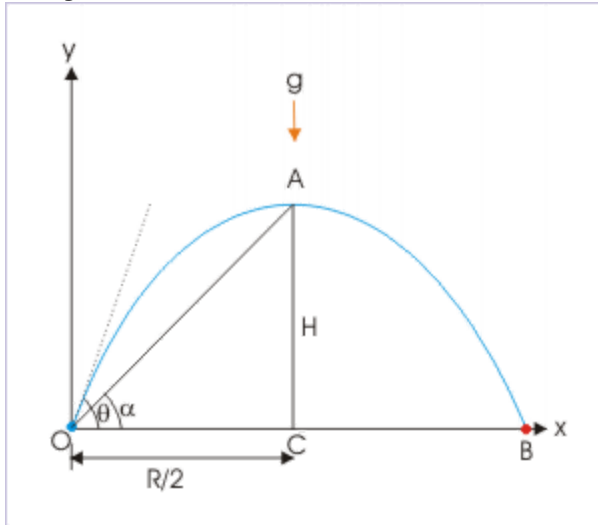
A projectile is projected at an angle " θ " from the horizon. The tangent of angle of elevation of the highest point as seen from the position of projection is :

$$(a) \frac{\tan \theta}{4} \quad (b) \frac{\tan \theta}{2} \quad (c) \tan \theta \quad (d) \frac{3 \tan \theta}{2}$$

Solution:

The angle of elevation of the highest point " θ " is shown in the figure. Clearly,

Projectile motion



Projectile motion

$$\tan \alpha = \frac{AC}{OC} = \frac{H}{\frac{R}{2}} = \frac{2H}{R}$$

Putting expressions of the maximum height and range of the flight, we have :

$$\Rightarrow \tan \alpha = \frac{2u^2 \sin^2 \theta \times g}{2g \times u^2 \sin 2\theta} = \frac{\sin^2 \theta}{2 \sin \theta \cos \theta} = \frac{\tan \theta}{2}$$

Hence, option (b) is correct.

Exercise:

Problem:

In a firing range, shots are taken at different angles and in different directions. If the speed of the bullets is " u ", then find the area in which bullets can spread.

$$(a) \frac{2\pi u^4}{g^2} \quad (b) \frac{u^4}{g^2} \quad (c) \frac{\pi u^4}{g^2} \quad (d) \frac{\pi u^2}{g^2}$$

Solution:

The bullets can spread around in a circular area of radius equal to maximum horizontal range. The maximum horizontal range is given for angle of projection of 45° .

$$R_{\max} = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin 2 \times 45^\circ}{g} = \frac{u^2 \sin 90^\circ}{g} = \frac{u^2}{g}$$

The circular area corresponding to the radius equal to maximum horizontal range is given as :

$$\Rightarrow A = \pi R_{\max}^2 = \pi \left(\frac{u^2}{g} \right)^2 = \frac{\pi u^4}{g^2}$$

Hence, option (c) is correct.

More exercises

Check the module titled "[Features of projectile motion \(application\)](#) to work out more problems.

Features of projectile motion (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

- 1: In general, we should rely on analysis in two individual directions as linear motion.
- 2: Wherever possible, we should use the formula directly as available for time of flight, maximum height and horizontal range.
- 3: We should be aware that time of flight and maximum height are two attributes of projectile motion, which are obtained by analyzing motion in vertical direction. For determining time of flight, the vertical displacement is zero; whereas for determining maximum height, vertical component of velocity is zero.
- 4: However, if problem has information about motion in horizontal direction, then it is always advantageous to analyze motion in horizontal direction. It is so because motion in horizontal direction is uniform motion and analysis in this direction is simpler.
- 5: The situation, involving quadratic equations, may have three possibilities : (i) quadratic in time " t " (ii) quadratic in displacement or position " x " and (iii) quadratic in " $\tan\theta$ " i.e. " θ ". We should use appropriate equations in each case as discussed in the module titled "[Features of projectile motion](#)".

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the features of projectile motion. The questions are categorized in terms of

the characterizing features of the subject matter :

- Time of flight
- Horizontal range
- Maximum height
- Height attained by a projectile
- Composition of motion
- Projectile motion with wind/drag force

Time of flight

Example:

Problem : The speed of a particle, projected at 60° , is 20 m/s at the time of projection. Find the time interval for projectile to loose half its initial speed.

Solution : Here, we see that final and initial speeds (not velocity) are subject to given condition. We need to use the given condition with appropriate expressions of speeds for two instants.

$$u = \sqrt{(u_x^2 + u_y^2)}$$

$$v = \sqrt{(v_x^2 + v_y^2)}$$

According to question,

$$u = 2v$$

$$\Rightarrow u^2 = 4v^2$$

$$\Rightarrow u_x^2 + u_y^2 = 4(v_x^2 + v_y^2)$$

But, horizontal component of velocity remains same. Hence,

$$\Rightarrow u_x^2 + u_y^2 = 4(u_x^2 + v_y^2)$$

Rearranging for vertical velocity :

$$v_y^2 = u_y^2 - 3u_x^2 = (20 \sin 60^\circ)^2 - 3(20 \cos 60^\circ)^2$$

$$v_y^2 = 20^2 \frac{3}{4} - 3 \times 20^2 \frac{1}{4} = 0$$

The component of velocity in vertical direction becomes zero for the given condition. This means that the projectile has actually reached the maximum height for the given condition. The time to reach maximum height is half of the time of flight :

$$t = \frac{u \sin \theta}{g} = \frac{20 \sin 60^\circ}{10} = \sqrt{3} \text{ s}$$

Horizontal range

Example:

Problem : A man can throw a ball to a greatest height denoted by "h". Find the greatest horizontal distance that he can throw the ball (consider $g = 10 \text{ m/s}^2$).

Solution : The first part of the question provides the information about the initial speed. We know that projectile achieves greatest height in vertical throw. Let "u" be the initial speed. We can, now, apply equation of motion " $v^2 = u^2 + 2gy$ " for vertical throw. We use this form of equation as we want to relate initial speed with the greatest height.

Here, $v = 0$; $a = -g$

$$\Rightarrow 0 = u^2 - 2gh$$

$$u^2 = 2gh$$

The projectile, on the other hand, attains greatest horizontal distance for the angle of projection, $\theta = 45^\circ$. Accordingly, the greatest horizontal distance is :

$$R_{\max} = \frac{u^2 \sin 2 \times 45^\circ}{g} = \frac{u^2 \sin 90^\circ}{g} = \frac{u^2}{g} = \frac{2gh}{g} = 2h$$

Example:

Problem : A bullet from a gun is fired at a muzzle speed of 50 m/s to hit a target 125 m away at the same horizontal level. At what angle from horizontal should the gun be aimed to hit the target (consider $g = 10 \text{ m/s}^2$) ?

Solution : Here, horizontal range is given. We can find out the angle of projection from horizontal direction, using expression of horizontal range :

$$\begin{aligned} R &= \frac{u^2 \sin 2\theta}{g} \\ \Rightarrow \sin 2\theta &= \frac{gR}{u^2} = \frac{10 \times 125}{50^2} = \frac{1}{2} \\ \Rightarrow \sin 2\theta &= \sin 30^\circ \\ \Rightarrow \theta &= 15^\circ \end{aligned}$$

Example:

Problem : A projectile has same horizontal range for a given projection speed for the angles of projections θ_1 and θ_2 ($\theta_2 > \theta_1$) with the horizontal. Find the ratio of the times of flight for the two projections.

Solution : The ratio of time of flight is :

$$\frac{T_1}{T_2} = \frac{u \sin \theta_1}{u \sin \theta_2} = \frac{\sin \theta_1}{\sin \theta_2}$$

For same horizontal range, we know that :

$$\theta_2 = (90^\circ - \theta_1)$$

Putting this, we have :

$$\frac{T_1}{T_2} = \frac{\sin \theta_1}{\sin(90^\circ - \theta_1)} = \tan \theta_1$$

Example:

Problem : A projectile, thrown at an angle 15° with the horizontal, covers a horizontal distance of 1000 m. Find the maximum distance the projectile can cover with the same speed (consider $g = 10 \text{ m/s}^2$).

Solution : The range of the projectile is given by :

$$R = \frac{u^2 \sin 2\theta}{g}$$

Here, $R = 1000 \text{ m}$, $g = 10 \text{ m/s}^2$, $\theta = 15^\circ$

$$\Rightarrow 1000 = \frac{u^2 \sin(2 \times 15^\circ)}{g} = \frac{u^2}{2g}$$

$$\Rightarrow \frac{u^2}{2g} = 1000 \text{ m}$$

Now, the maximum range (for $\theta = 45^\circ$) is :

$$\Rightarrow R_{\max} = \frac{u^2}{g} = 2000 \text{ m}$$

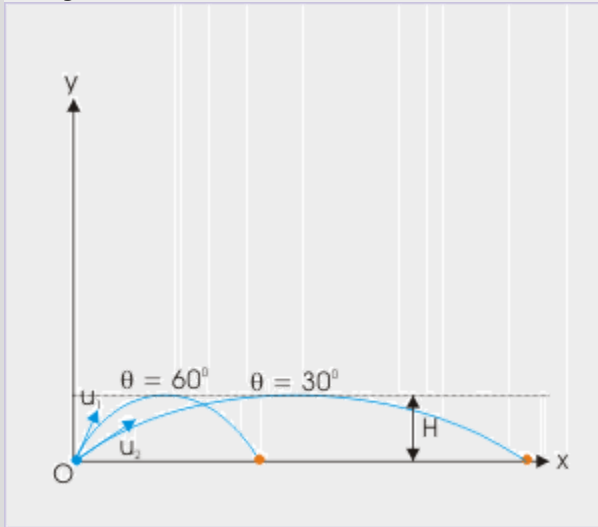
Maximum height

Example:

Problem : Two balls are projected from the same point in the direction inclined at 60° and 30° respectively with the horizontal. If they attain the same height, then the ratio of speeds of projection is :

Solution : Since the projectiles attain same height,

Projectile motion



$$H_1 = H_2 = H$$

$$\frac{u_1^2 \sin^2 60^\circ}{2g} = \frac{u_2^2 \sin^2 30^\circ}{2g}$$

$$\frac{u_1^2 \times 3}{4} = \frac{u_2^2 \times 1}{4}$$

$$u_1^2 : u_2^2 = 1 : 3$$

$$\Rightarrow u_1 : u_2 = 1 : \sqrt{3}$$

Example:

Problem : A projectile is thrown vertically up, whereas another projectile is thrown at an angle θ with the vertical. Both of the projectiles stay in the air for the same time (neglect air resistance). Find the ratio of maximum heights attained by two projectiles.

Solution : Let u_1 and u_2 be the speeds of projectiles for vertical and non-vertical projections. The times of the flight for vertical projectile is given by :

$$T_1 = \frac{2u_1}{g}$$

We note here that the angle is given with respect to vertical - not with respect to horizontal as the usual case. As such, the expression of time of

flight consists of cosine term :

$$T_2 = \frac{2u_2 \cos \theta}{g}$$

As, time of flight is same,

$$\frac{2u_1}{g} = \frac{2u_2 \cos \theta}{g}$$

$$u_1 = u_2 \cos \theta$$

On the other hand, the maximum heights attained in the two cases are :

$$H_1 = \frac{u_1^2}{2g}$$

$$H_2 = \frac{u_2^2 \cos^2 \theta}{2g}$$

Using the relation $u_1 = u_2 \cos \theta$ as obtained earlier, we have :

$$H_2 = \frac{u_1^2}{2g}$$

$$\Rightarrow H_1 = H_2$$

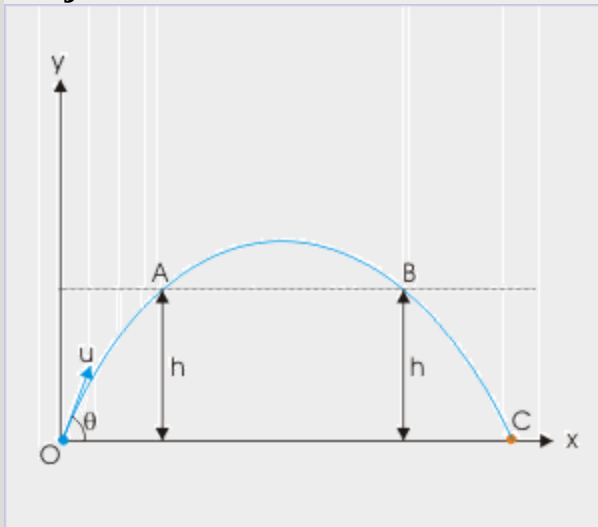
Note: The result is intuitive about the nature of projectile. The time of flight and vertical height both are consideration of motion in vertical direction. Since times of flight in both cases are same, the vertical components of two projectiles should be same. Otherwise, times of flight will be different. Now, if vertical component are same, then maximum heights have to be same.

Height attained by a projectile

Example:

Problem : The times for attaining a particular vertical elevation during projectile motion are t_1 and t_2 . Find time of flight, T , in terms of t_1 and t_2 .

Solution : We can answer this question analytically without using formula. Let the positions of the projectile at two time instants be "A" and "B", as shown in the figure. The time periods t_1 and t_2 denotes time taken by the projectile to reach points "A" and "B" respectively. Clearly, time of flight, T , is equal to time taken to travel the curve OAB (t_2) plus the time taken to travel the curve BC.

Projectile motion

Now, projectile takes as much time to travel the curve OA, as it takes to travel curve BC. This is so, because the time of travel of equal vertical displacement in either direction (up or down) in vertical motion under gravity is same. Since the time of travel for curve OA is t_1 , the time of travel for curve BC is also t_1 . Thus, the total time of flight is :

$$T = t_1 + t_2$$

Alternatively,

As the heights attained are equal,

$$h_1 = h_2$$

$$\Rightarrow u_y t_1 - \frac{1}{2} g t_1^2 = u_y t_2 - \frac{1}{2} g t_2^2$$

$$\Rightarrow u_y (t_2 - t_1) = \frac{1}{2} g (t_2^2 - t_1^2) = \frac{1}{2} g (t_2 + t_1) (t_2 - t_1)$$

$$\Rightarrow t_2 + t_1 = \frac{2u_y}{g} = T$$

Composition of motion

Example:

Problem : The position of a projectile projected from the ground is :

$$x = 3t$$

$$y = (4t - 2t^2)$$

where “x” and “y” are in meters and “t” in seconds. The position of the projectile is (0,0) at the time of projection. Find the speed with which the projectile hits the ground.

Solution : When the projectile hits the ground, $y = 0$,

$$0 = (4t - 2t^2)$$

$$\Rightarrow 2t^2 - 4t = t(2t - 2) = 0$$

$$\Rightarrow t = 0, t = 2 \text{ s}$$

Here $t = 0$ corresponds to initial condition. Thus, projectile hits the ground in 2 s. Now velocities in two directions are obtained by differentiating given functions of the coordinates,

$$v_x = \frac{dx}{dt} = 3$$

$$v_y = \frac{dy}{dt} = 4 - 4t$$

Now, the velocities for $t = 2$ s,

$$v_x = \frac{dx}{dt} = 3 \text{ m/s}$$

$$v_y = \frac{dy}{dt} = 4 - 4 \times 2 = -4 \text{ m/s}$$

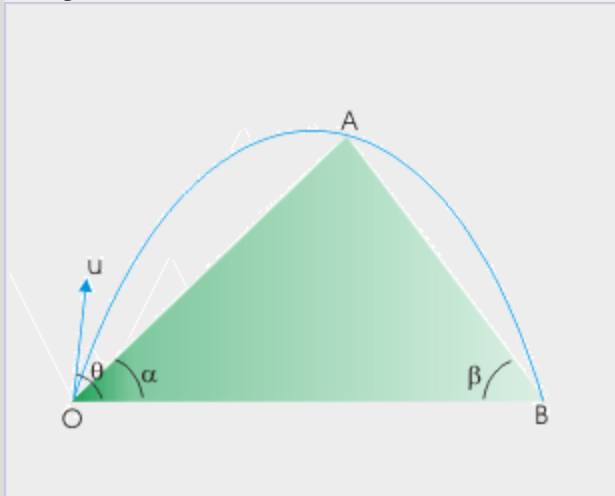
The resultant velocity of the projectile,

$$v = \sqrt{(v_x^2 + v_y^2)} = \sqrt{\{3^2 + (-4)^2\}} = 5 \text{ m/s}$$

Example:

Problem : A projectile, thrown from the foot of a triangle, lands at the edge of its base on the other side of the triangle. The projectile just grazes the vertex as shown in the figure. Prove that :

Projectile motion



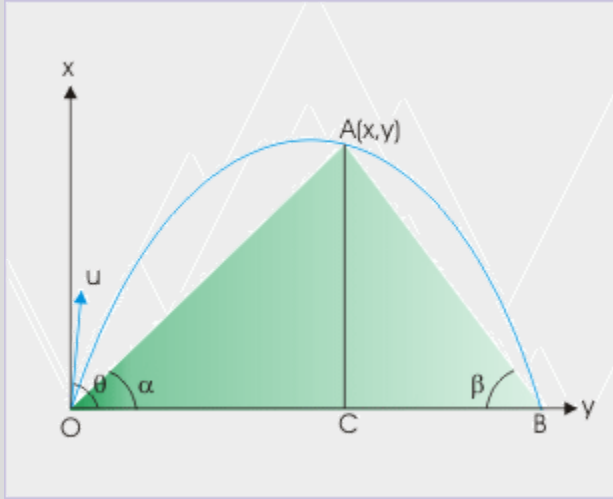
The projectile grazes the vertex of the triangle.

$$\tan \alpha + \tan \beta = \tan \theta$$

where “ θ ” is the angle of projection as measured from the horizontal.

Solution : In order to expand trigonometric ratio on the left side, we drop a perpendicular from the vertex of the triangle “A” to the base line OB to meet at a point C. Let x, y be the coordinate of vertex “A”, then,

Projectile motion



The projectile grazes the vertex of the triangle.

$$\tan \alpha = \frac{AC}{OC} = \frac{y}{x}$$

and

$$\tan \beta = \frac{AC}{BC} = \frac{y}{(R - x)}$$

Thus,

$$\tan \alpha + \tan \beta = \frac{y}{x} + \frac{y}{(R - x)} = \frac{yR}{x(R - x)}$$

Intuitively, we know the expression is similar to the expression involved in the equation of projectile motion that contains range of projectile,

$$\Rightarrow y = x \tan \theta \left(1 - \frac{x}{R} \right)$$

$$\Rightarrow \tan \theta = \frac{yR}{x(R - x)}$$

Comparing equations,

$$\Rightarrow \tan \alpha + \tan \beta = \tan \theta$$

Projectile motion with wind/drag force

Example:

Problem : A projectile is projected at angle “ θ ” from the horizontal at the speed “ u ”. If an acceleration of “ $g/2$ ” is applied to the projectile due to wind in horizontal direction, then find the new time of flight, maximum height and horizontal range.

Solution : The acceleration due to wind affects only the motion in horizontal direction. It would, therefore, not affect attributes like time of flight or maximum height that results exclusively from the consideration of motion in vertical direction. The generic expressions of time of flight, maximum height and horizontal range of flight with acceleration are given as under :

$$T = \frac{2u_y}{g}$$

$$H = \frac{u_y^2}{2g} = \frac{gT^2}{4}$$

$$R = \frac{u_x u_y}{g}$$

The expressions above revalidate the assumption made in the beginning. We can see that it is only the horizontal range that depends on the component of motion in horizontal direction. Now, considering accelerated motion in horizontal direction, we have :

$$x = R' = u_x T + \frac{1}{2} a_x T^2$$

$$\Rightarrow R' = u_x T + \frac{1}{2} \left(\frac{g}{2} \right) T^2$$

$$R' = R + H$$

Example:

Problem : A projectile is projected at angle “ θ ” from the horizontal at the speed “ u ”. If an acceleration of $g/2$ is applied to the projectile in horizontal direction and a deceleration of $g/2$ in vertical direction, then find the new time of flight, maximum height and horizontal range.

Solution : The acceleration due to wind affects only the motion in horizontal direction. It would, therefore, not affect attributes resulting exclusively from the consideration in vertical direction. It is only the horizontal range that will be affected due to acceleration in horizontal direction. On the other hand, deceleration in vertical direction will affect all three attributes.

1: Time of flight

Let us work out the effect on each of the attribute. Considering motion in vertical direction, we have :

$$y = u_y T + \frac{1}{2} a_y T^2$$

For the complete flight, $y = 0$ and $t = T$. Also,

$$a_y = - \left(g + \frac{g}{2} \right) = - \frac{3g}{2}$$

Putting in the equation,

$$\Rightarrow 0 = u_y T - \frac{1}{2} X \frac{3g}{2} X T^2$$

Neglecting $T = 0$,

$$\Rightarrow T = \frac{4u_y}{3g} = \frac{4u \sin \theta}{3g}$$

2: Maximum height

For maximum height, $v_y = 0$,

$$0 = u_y^2 - 2X \frac{3g}{2} X H$$

$$H = \frac{u_y^2}{3g} = \frac{u^2 \sin^2 \theta}{3g}$$

2: Horizontal range

Now, considering accelerated motion in horizontal direction, we have :

$$x = R = u_x T + \frac{1}{2} a_x T^2$$

$$R = u_x \left(\frac{4u_y}{g} \right) + \frac{1}{2} \left(\frac{g}{2} \right) \left(\frac{4u_y}{g} \right)^2$$

$$\Rightarrow R = \left(\frac{4u_y}{g} \right) \left[u_x + \frac{1}{2} \left(\frac{g}{2} \right) \left(\frac{4u_y}{g} \right) \right]$$

$$\Rightarrow R = \left(\frac{4u_y}{g} \right) \left\{ u_x + u_y \right\}$$

$$\Rightarrow R = \left(\frac{4u^2 \sin \theta}{g} \right) \left[\cos \theta + \sin \theta \right]$$

Projectile motion types

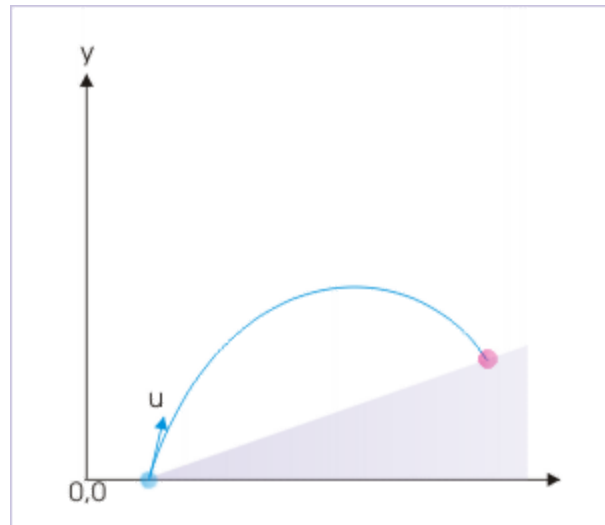
So far, we have limited our discussion to the classic mode of a projectile motion, where points of projection and return are on the same horizontal plane. This situation, however, may be altered. The projection level may be at an elevation with respect to the plane where projectile returns or the projection level may be at a lower level with respect to the plane where projectile returns. The two situations are illustrated in the figures below.

Projection and return at different levels

Projection from higher elevation

The return of projectile at higher elevation

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The two variants are basically the same parabolic motion. These motion types are inherently similar to the one where points of projection and return are on same level. If we look closely, then we find that the motion of the projectile from an elevation and from a lower level are either an extension or a curtailment of normal parabolic motion.

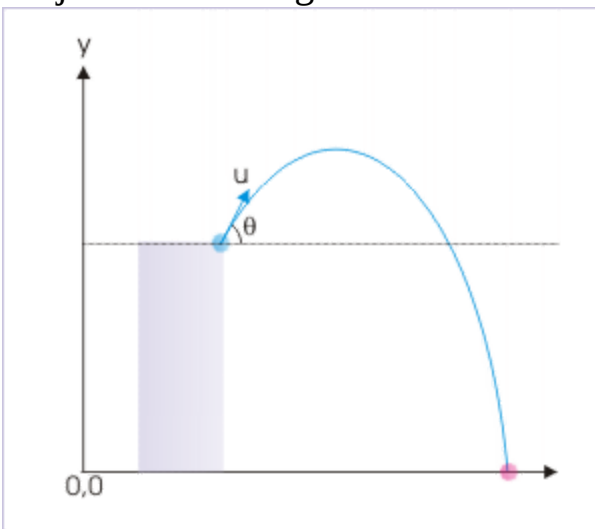
The projection on an incline (where point of return is on higher level) is a shortened projectile motion as if projectile has been stopped before returning to the normal point of return. This motion is also visualized as if the projectile is thrown over an incline or a wedge as shown in the figure. Again, there are two possibilities : (i) the projectile can be thrown up the

incline or (ii) the projectile can be thrown down the incline. For the sake of convenience and better organization, we shall study projectile motion on an incline in a separate module. In this module, we shall restrict ourselves to the first case in which projectile is projected from an elevation.

Projection from a higher level

The projection of a projectile from a higher point results in a slightly different parabolic trajectory. We can visually recognize certain perceptible differences from the normal case as listed here :

Projection from higher level



The trajectory extends beyond the normal point of return.

- The upward trajectory is smaller than downward trajectory.
- Time of ascent is smaller than the time of descent.
- The speed of projection is not equal to speed of return on the ground.
- The velocity of return is more aligned to vertical as the motion progresses.

It is evident that the expressions derived earlier for time of flight (T), maximum height (H) and range (R) are not valid in the changed scenario.

But the basic consideration of the analysis is necessarily same. The important aspect of projectile motion that motions in two mutually perpendicular directions are independent of each other, still, holds. Further, the nature of motion in two directions is same as before : the motion in vertical direction is accelerated due to gravity, whereas motion in horizontal direction has no acceleration.

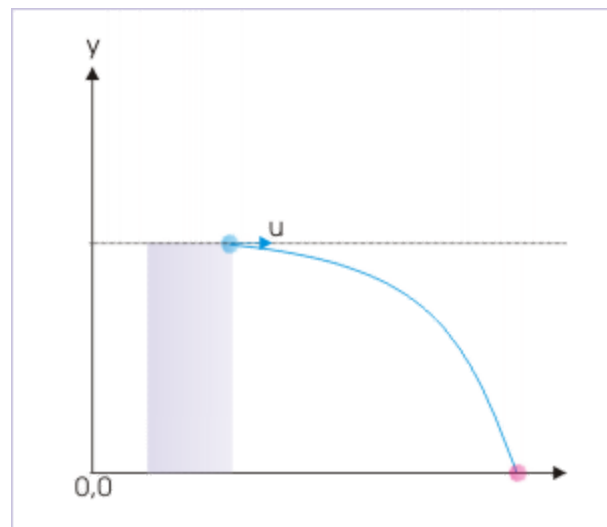
Now, there are two important variations of this projectile motion, when projected from an elevated level. The projectile may either be projected at certain angle (up or down) with the horizontal or it may be projected in the horizontal direction.

The projection from higher elevation

Projection from higher elevation at an angle

Projection from higher elevation in horizontal direction

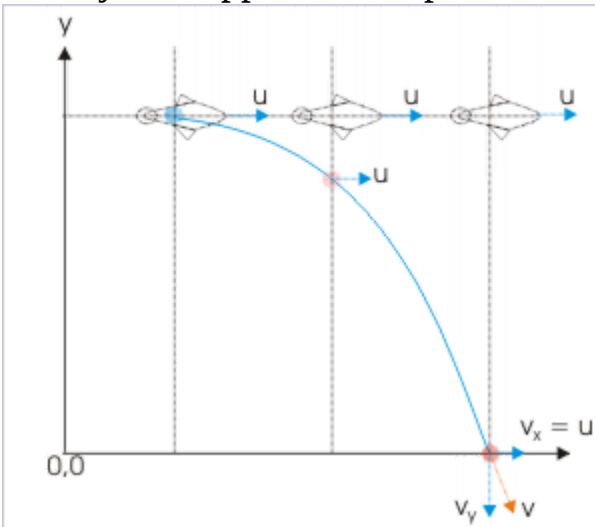
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There are many real time situations that resemble horizontal projection. When an object is dropped from a plane flying parallel to the ground at certain height, then the object acquires horizontal velocity of the plane when the object is released. As the object is simply dropped, the velocity in vertical direction is zero. This horizontal velocity of the object, as acquired from the plane, is then modified by the force of gravity, whereby the object follows a parabolic trajectory before hitting the ground.

This situation is analogous to projection from ground except that we track motion from the highest point. Note that vertical velocity is zero and horizontal velocity is tangential to the path at the time of projection. This is exactly the same situation as when projectile is projected from the ground and reaches highest point. In the nutshell, the description of motion here is same as the description during descent when projected from the ground.

An object dropped from a plane moving in horizontal direction

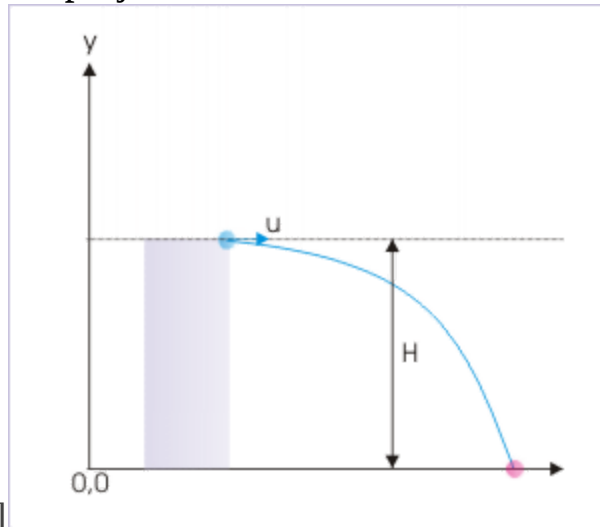


The position of plane is above object as both moves with same velocity in horizontal direction.

The interesting aspect of the object dropped from plane is that both plane and object are moving with same horizontal velocity. Hence, plane is always above the dropped object, provided plane maintains its velocity.

The case of projection from a higher level at certain angle (up or down) to the horizontal is different to the one in which projectile is projected horizontally. The projectile has a vertical component of initial velocity when thrown at an angle with horizontal. This introduces the difference between two cases. The projectile thrown up attains a maximum height above the projection level. On the return journey downward, it travels past its level of projection. The difference is visually shown in the two adjoining figures below.

Maximum height attained by the projectile



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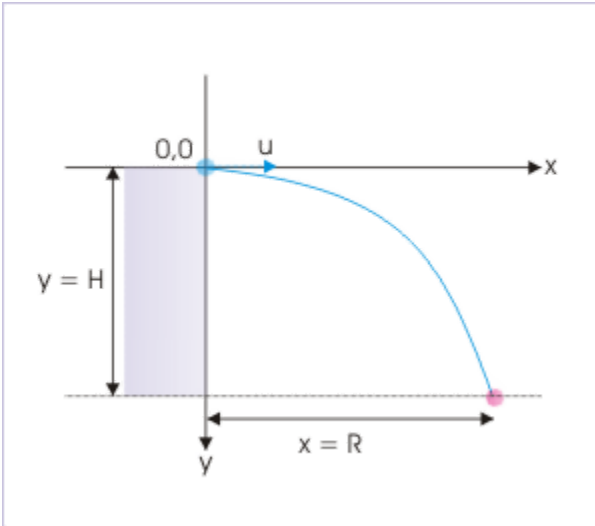
The resulting trajectory in the first case has both upward and downward motions. On the other hand, the motion in upward direction is completely missing in the horizontal projection as the projectile keeps losing altitude all the time.

Projectile thrown in horizontal direction

We can easily analyze projectile motion following the technique of component motions in two mutually perpendicular directions (horizontal and vertical). Typically, we consider vertical component of motion to determine time of flight (T). The initial velocity in vertical direction is zero.

We consider point of projection as origin of coordinate system. Further, we choose x-axis in horizontal direction and y-axis in the vertically downward direction for the convenience of analysis. Then,

An object projected in horizontal direction



$$y = H = u_y T + \frac{1}{2} g T^2$$

But $u_y = 0$,

$$\Rightarrow H = \frac{1}{2} g T^2$$

$$\Rightarrow T = \sqrt{\left(\frac{2H}{g} \right)}$$

Note the striking similarity here with the free fall of a body under gravity from a height “H”. The time taken in free fall is same as the time of flight of projectile in this case. Now, the horizontal range of the projectile is given as :

$$x = R = u_x T = u \sqrt{\left(\frac{2H}{g} \right)}$$

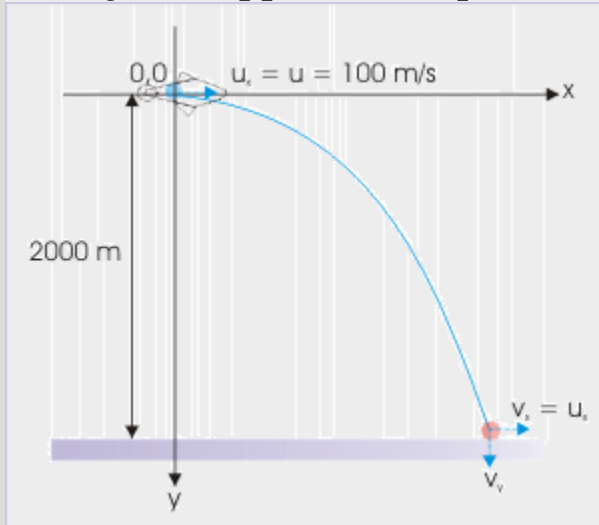
Example:

Problem : A plane flying at the speed of 100 m/s parallel to the ground drops an object from a height of 2 km. Find (i) the time of flight (ii)

velocity of the object at the time it strikes the ground and (iii) the horizontal distance traveled by the object.

Solution : The basic approach to solve the problem involves consideration of motion in two mutually perpendicular direction. Here, we consider a coordinate system with the point of release as the origin and down ward direction as the positive y-direction.

An object dropped from a plane moving in horizontal direction



(i) Time of flight, T

In vertical direction :

$$u_y = 0, a = 10 \text{ m/s}^2, y = 2000 \text{ m}, T = ?$$

Using equation, $y = u_y T + \frac{1}{2} g T^2$, we have :

$$\Rightarrow y = \frac{1}{2} g t^2$$

$$\Rightarrow T = \sqrt{\left(\frac{2y}{g}\right)}$$

$$\Rightarrow T = \sqrt{\left(\frac{2 \times 2000}{10}\right)} = 20 \text{ s}$$

(ii) Velocity at the ground

We can find the velocity at the time of strike with ground by calculating component velocities at that instant in the two mutually perpendicular directions and finding the resultant (composite) velocity as :

$$v = \sqrt{(v_x^2 + v_y^2)}$$

Initial vertical component of initial velocity (u_y) is zero and the object is accelerated down with the acceleration due to gravity. Hence,

$$v_y = u_y + gt = 0 + gt = gt$$

The component of velocity in the horizontal direction remains unchanged as there is no acceleration in this direction.

$$v_x = u_x = u$$

$$\Rightarrow v = \sqrt{(v_x^2 + v_y^2)} = \sqrt{(u^2 + g^2 t^2)}$$

Putting values,

$$\Rightarrow v = \sqrt{(100^2 + 10^2 \times 20^2)} = \sqrt{50000} = 100\sqrt{5} \text{ m/s}$$

(iii) Horizontal distance traveled

From consideration of uniform motion in horizontal direction, we have :

$$x = u_x t = ut$$

Putting values,

$$\Rightarrow x = R = 100 \times 20 = 2000 \text{ m}$$

Exercise:

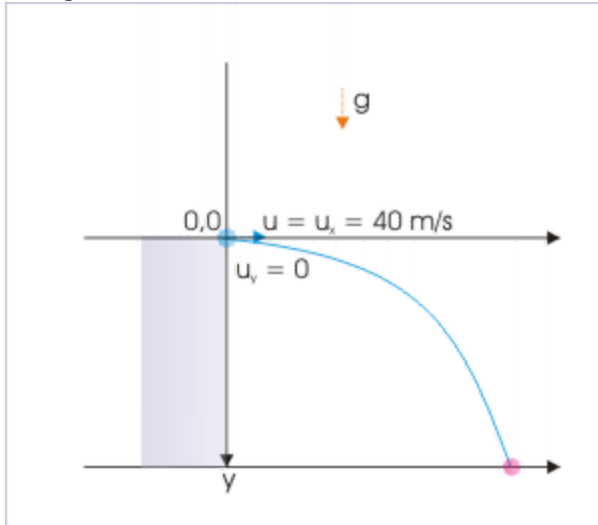
Problem:

A ball is thrown horizontally from a tower at a speed of 40 m/s. The speed of the projectile (in m/s) after 3 seconds, before it touches the ground, is (consider $g = 10 \text{ m/s}^2$) :

- (a)30 (b)40 (c)50 (d)60

Solution:

Here, we consider a reference system whose origin coincides with the point of projection. The downward direction is along y - direction as shown in the figure.

Projectile motion

Projectile motion

The initial speed of the projectile is equal to the horizontal component of the velocity, which remains unaltered during projectile motion. On the other hand, vertical component of velocity at the start of motion is zero. Thus,

$$u_x = 40 \text{ m/s}, \quad u_y = 0$$

Using equation of motion, we have :

$$v_y = u_y + a_y t$$

$$\Rightarrow v_y = 0 + gt = 10 \times 3 = 30 \text{ m/s}$$

Since the horizontal component of velocity remains unaltered, the speed, after 3 second, is :

$$\Rightarrow v = \sqrt{v_x^2 + v_y^2} = \sqrt{40^2 + 30^2} = 50 \text{ m/s}$$

Hence, option (c) is correct.

Exercise:

Problem:

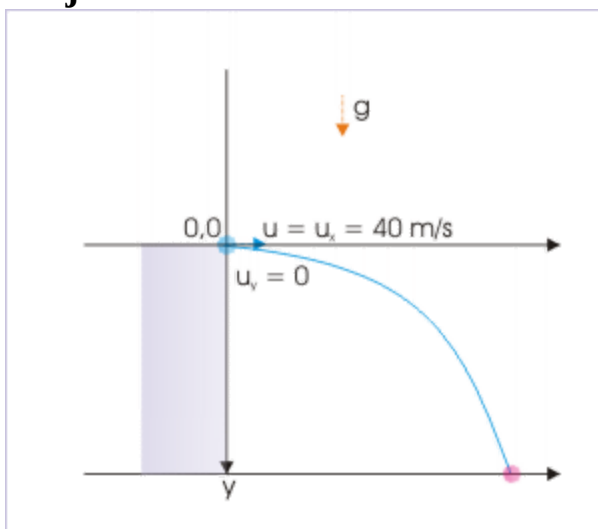
A ball is projected horizontally from a height at a speed of 30 m/s. The time after which the vertical component of velocity becomes equal to horizontal component of velocity is : (consider $g = 10 \text{ m/s}^2$) :

- (a) 1s (b) 2s (c) 3s (d) 4s

Solution:

The ball does not have vertical component of velocity when projected. The ball, however, is accelerated downward and gains speed in vertical direction. At certain point of time, the vertical component of velocity equals horizontal component of velocity. At this instant, the angle that the velocity makes with the horizontal is :

Projectile motion



Projectile motion

$$\tan \theta = \frac{v_y}{v_x} = 1$$

$$\Rightarrow \theta = 45^\circ$$

We should note that this particular angle of 45° at any point during the motion, as a matter of fact, signifies that two mutually perpendicular components are equal.

But we know that horizontal component of velocity does not change during the motion. It means that vertical component of velocity at this instant is equal to horizontal component of velocity i.e.

$$v_y = v_x = u_x = 30 \text{ m/s}$$

Further, we know that we need to analyze motion in vertical direction to find time as required,

$$v_y = u_y + a_y t$$

$$\Rightarrow 30 = 0 + 10t$$

$$\Rightarrow t = 3s$$

Hence, option (c) is correct.

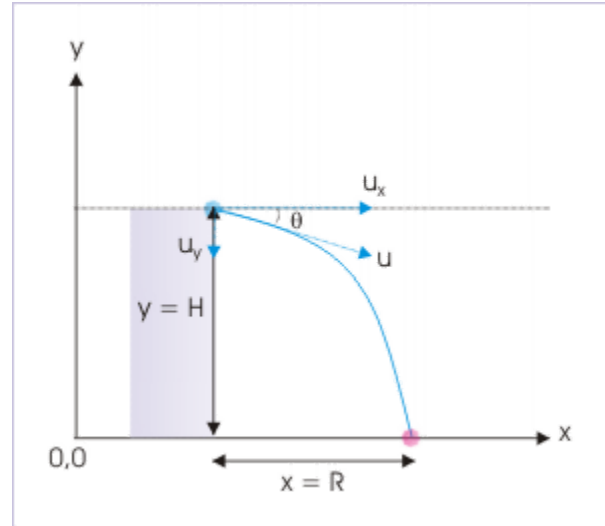
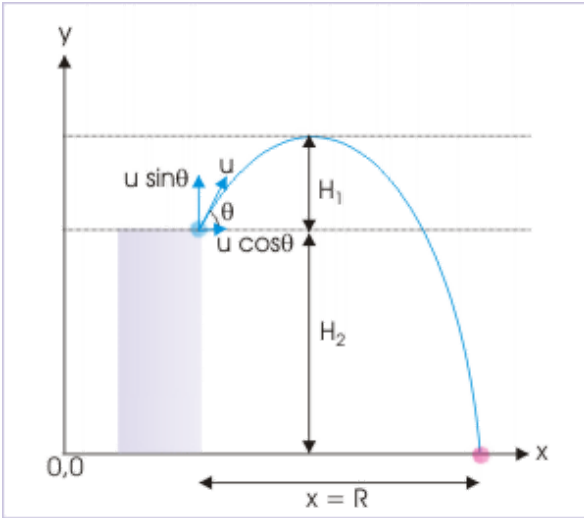
Projectile thrown at an angle with horizontal direction

There are two possibilities. The projectile can be projected up or down as shown in the figure here :

Projection from an elevated level

Projection upwards from an
elevated level

Projection downwards from an
elevated level



Projectile thrown up at an angle with horizontal direction

The time of flight is determined by analyzing motion in vertical direction. The net displacement during the motion is equal to the elevation of point of projection above ground i.e. H_2 . To analyze the motion, we consider point of projection as origin, horizontal direction as x-axis and upward vertical direction as y-axis.

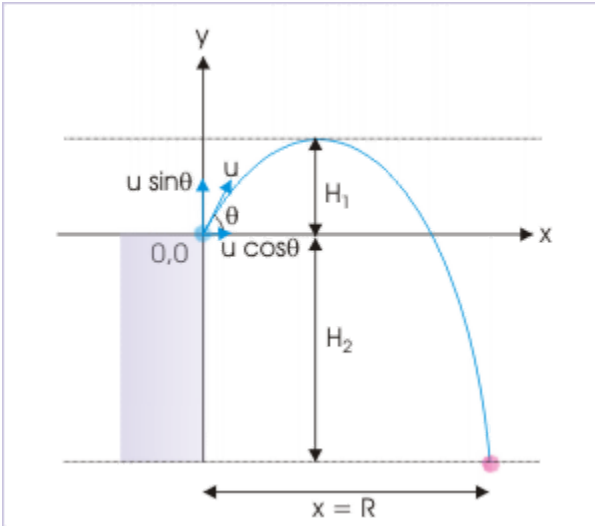
$$y = -H_2 = u_y T - \frac{1}{2} g T^2$$

Rearranging,

$$T^2 - \left(\frac{2u_y}{g} \right) T + \left(\frac{2H_2}{g} \right) = 0$$

This is a quadratic equation in “T”. Solving we get two values of T, one of which gives the time of flight.

An object projected up at an angle from an elevation



Horizontal range is given by analyzing motion in horizontal direction as :

$$x = R = u_x T$$

While calculating maximum height, we can consider motion in two parts. The first part is the motion above the projection level. On the other hand second part is the projectile motion below projection level. The total height is equal to the sum of the magnitudes of vertical displacements :

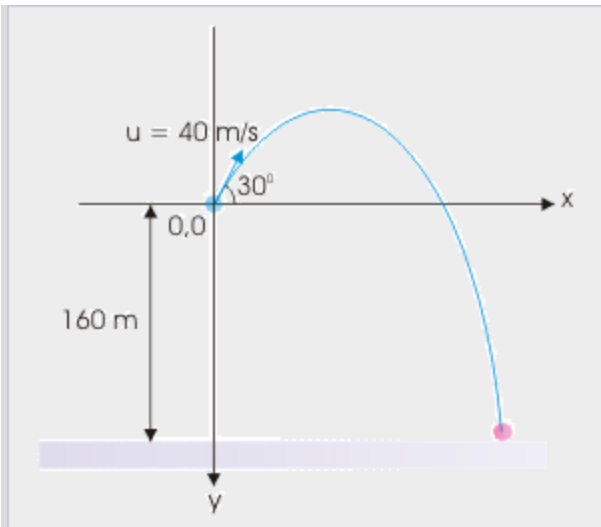
$$H = |H_1| + |H_2|$$

Example:

Problem : A projectile is thrown from the top of a building 160 m high, at an angle of 30° with the horizontal at a speed of 40 m/s. Find (i) time of flight (ii) Horizontal distance covered at the end of journey and (iii) the maximum height of the projectile above the ground.

Solution : Unlike horizontal projection, the projectile has a vertical component of initial velocity. This vertical component is acting upwards, which causes the projectile to rise above the point of projection. Here, we choose the point of projection as the origin and downward direction as the positive y – direction.

An object projected at an angle from an elevation



(i) Time of flight, T

Here,

$$u_y = u \sin \theta = -40 \sin 30^\circ = -20 \text{ m/s};$$

$$y = 160 \text{ m};$$

$$g = 10 \text{ m/s}^2$$

Using equation, $y = u_y t + \frac{1}{2} g t^2$, we have :

$$\Rightarrow 160 = -20t + \frac{1}{2} 10t^2$$

$$\Rightarrow 5t^2 - 20t - 160 = 0$$

$$\Rightarrow t^2 - 4t - 32 = 0$$

$$\Rightarrow t^2 - 8t + 4t - 32 = 0$$

$$\Rightarrow t(t - 8) + 4(t - 8) = 0$$

$$\Rightarrow t = -4 \text{ s or } t = 8 \text{ s}$$

Neglecting negative value of time, $T = 8 \text{ s}$.

(ii) Horizontal distance, R

There is no acceleration in horizontal direction. Using equation for uniform motion,

$$x = u_x T$$

Here,

$$u_x = u \cos \theta = 40 \cos 30^\circ = 20\sqrt{3} \text{ m/s};$$

$$T = 8 \text{ s}$$

$$\Rightarrow x = u_x T = 20\sqrt{3} \times 8 = 160\sqrt{3} \text{ m}$$

(iii) Maximum height, H

The maximum height is the sum of the height of the building (H_2) and the height attained by the projectile above the building (H_1).

$$H = H_1 + H_2$$

We consider vertical motion to find the height attained by the projectile above the building (H_1).

$$H_1 = \frac{u^2 \sin^2 \theta}{2g}$$

$$\Rightarrow H_1 = \frac{(40)^2 \sin^2 30^\circ}{2 \times 10}$$

$$\Rightarrow H_1 = \frac{(40)^2 \times \left(\frac{1}{4}\right)}{2 \times 10} = 20 \text{ m}$$

Thus maximum height, H, is :

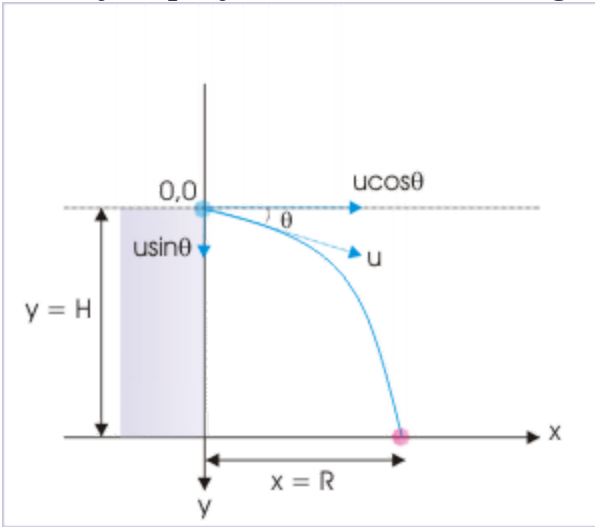
$$H = H_1 + H_2 = 20 + 160 = 180 \text{ m}$$

Projectile thrown down at an angle with horizontal direction

Projectile motion here is similar to that of projectile thrown horizontally. The only difference is that the projectile has a finite component of velocity in downward direction against zero vertical velocity.

For convenience, the point of projection is considered as origin of reference and the positive x and y directions of the coordinate system are considered

in horizontal and vertically downward directions.
An object projected down at an angle from an elevation



The time of flight is obtained considering motion in vertical direction as :

$$y = H = u_y T + \frac{1}{2} g T^2$$

Rearranging, we get a quadratic equation in “T”,

$$T^2 + \left(\frac{2u_y}{g} \right) T - \left(\frac{2H_1}{g} \right) = 0$$

One of the two values of "T" gives the time of flight in this case. The horizontal range, on the other hand, is given as :

$$x = R = u_x T$$

Exercises

Exercise:

Problem:

Two balls of masses " m_1 " and " m_2 "are thrown from a tower in the horizontal direction at speeds " u_1 " and " u_2 " respectively at the same time. Which of the two balls strikes the ground first?

- (a) the ball thrown with greater speed
 - (b) the ball thrown with lesser speed
 - (c) the ball with greater mass
 - (d) the balls strike the ground simultaneously
-

Solution:

The time to strike the ground is obtained by considering motion in vertical direction.

Here, $u_y = 0$ and $y=T$ (total time of flight)

$$y = \frac{1}{2} g T^2$$

$$\Rightarrow T = \sqrt{\left(\frac{2y}{g}\right)}$$

Thus, we see that time of flight is independent of both mass and speed of the projectile in the horizontal direction. The two balls, therefore, strike the ground simultaneously.

Hence, option (d) is correct.

Exercise:

Problem:

A body dropped from a height "h" strikes the ground with a velocity 3 m/s. Another body of same mass is projected horizontally from the same height with an initial speed of 4 m/s. The final velocity of the second body (in m/s), when it strikes the earth will be :

- (a)3 (b)4 (c)5 (d)7

Solution:

We can use the fact that velocities in vertical direction, attained by two bodies through same displacement, are equal. As such, velocity of the second body, on reaching the ground, would also be 3 m/s. Now, there is no acceleration in horizontal direction. The horizontal component of velocity of the second body, therefore, remains constant.

$$v_x = 4 \text{ m/s}$$

and

$$v_y = 3 \text{ m/s}$$

The resultant velocity of two component velocities in mutually perpendicular directions is :

$$\Rightarrow v = \sqrt{v_x^2 + v_y^2} = \sqrt{4^2 + 3^2} = 5 \text{ m/s}$$

Exercise:**Problem:**

A projectile (A) is dropped from a height and another projectile (B) is projected in horizontal direction with a speed of 5 m/s from the same height. Then the correct statement(s) is(are) :

- (a) The projectile (A) reach the ground earlier than projectile (B).
- (b) Both projectiles reach the ground with the same speed.
- (c) Both projectiles reach the ground simultaneously.
- (d) The projectiles reach the ground with different speeds.

Solution:

First three options pertain to the time of flight. The time of flight depends only the vertical displacement and vertical component of projection velocity. The vertical components of projection in both cases are zero. The time of flight (T) in either case is obtained by considering motion in vertical direction as :

$$y = u_y T + \frac{1}{2} a_y T^2$$

$$\Rightarrow H = 0 + \frac{1}{2} g T^2$$

$$\Rightarrow T = \sqrt{\left(\frac{2H}{g}\right)}$$

Hence, both projectiles reach the ground simultaneously. On the other hand, component of velocity in vertical direction, on reaching the ground is :

$$v_y = u_y + a_y$$

$$\Rightarrow T = 0 + gT$$

Putting value of T, we have :

$$\Rightarrow v_y = \sqrt{2gH}$$

Projectile (A) has no component of velocity in horizontal direction, whereas projectile (B) has finite component of velocity in horizontal direction. As such, velocities of projectiles and hence the speeds of the particles on reaching the ground are different.

Hence, options (c) and (d) are correct.

Exercise:

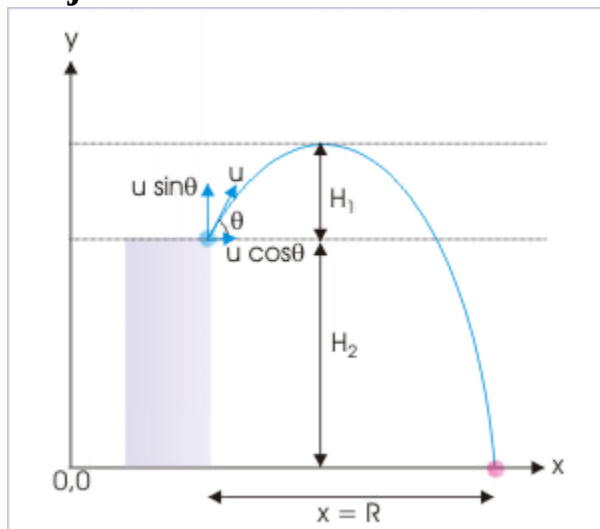
Problem:

Four projectiles, "A", "B", "C" and "D" are projected from top of a tower with velocities (in m/s) $10 \mathbf{i} + 10 \mathbf{j}$, $10 \mathbf{i} - 20 \mathbf{j}$, $-10 \mathbf{i} - 10 \mathbf{j}$ and $-20 \mathbf{i} + 10 \mathbf{j}$ in the coordinate system having point of projection as origin. If "x" and "y" coordinates are in horizontal and vertical directions respectively, then :

- (a) Time of flight of C is least.
 - (b) Time of flight of B is least.
 - (c) Times of flights of A and D are greatest.
 - (d) Times of flights of A and D are least.
-

Solution:

The time of flight of a projectile solely depends on vertical component of velocity.

Projectile motion

Projectile motion

The projectile "A" is thrown up from an elevated point with vertical component of velocity 10 m/s. It travels to the maximum height (H_1) and the elevation from the ground(H_2).

The projectile "B" is thrown down from an elevated point with vertical component of velocity 20 m/s. It travels only the elevation from the ground (H_2).

The projectile "C" is thrown down from an elevated point with vertical component of velocity 10 m/s. It travels only the elevation from the ground (H_2).

The projectile "D" is thrown up from an elevated point with vertical component of velocity 15 m/s. It travels to the maximum height (H_1) and the elevation from the ground(H_2).

Clearly, projectile "B" and "C" travel the minimum vertical displacement. As vertical downward component of velocity of "B" is greater than that of "C", the projectile "B" takes the least time.

The projectiles "A" and "D" are projected up with same vertical components of velocities i.e. 10 m/s and take same time to travel to reach the ground. As the projectiles are projected up, they take more time than projectiles projected down.

Hence, options (b) and (c) are correct.

Acknowledgment

Author wishes to thank Scott Kravitz, Programming Assistant, Connexions for making suggestion to remove syntax error in the module.

Projectile motion types (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1: Identify projectile motion types. The possible variants are :

- Projectile is thrown in horizontal direction. In this case, initial vertical component of velocity is zero. Consider horizontal direction as positive x-direction and vertically downward direction as positive y-direction.
- Projectile is thrown above horizontal level. The projectile first goes up and then comes down below the level of projection
- Projectile is thrown below horizontal level. Consider horizontal direction as positive x-direction and vertically downward direction as positive y-direction.

2: We can not use standard equations of time of flight, maximum height and horizontal range. We need to analyze the problem in vertical direction for time of flight and maximum height. Remember that determination of horizontal range will involve analysis in both vertical (for time of flight) and horizontal (for the horizontal range) directions.

3: However, if problem has information about motion in horizontal direction, then it is always advantageous to analyze motion in horizontal direction.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the projectile motion types. The questions are categorized in terms of the

characterizing features of the subject matter :

- Time of flight
- Range of flight
- Initial velocity
- Final velocity

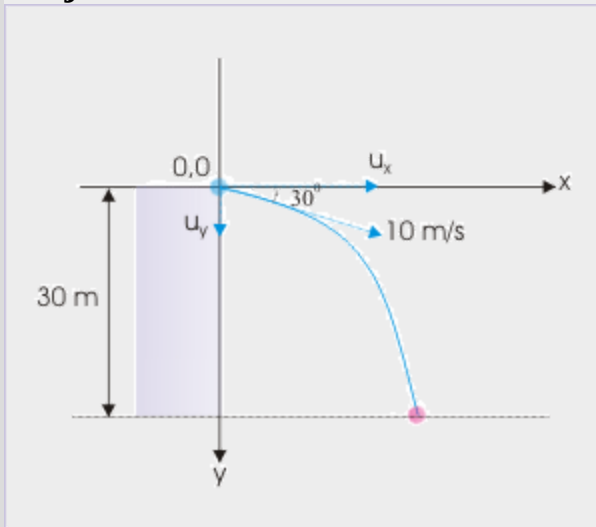
Time of flight

Example:

Problem : A ball from a tower of height 30 m is projected down at an angle of 30° from the horizontal with a speed of 10 m/s. How long does ball take to reach the ground? (consider $g = 10 \text{ m/s}^2$)

Solution : Here, we consider a reference system whose origin coincides with the point of projection. Further, we consider that the downward direction is positive y - direction.

Projectile motion



Motion in vertical direction :

Here, $u_y = u \sin \theta = 10 \sin 30^\circ = 5 \text{ m/s}$; $y = 30 \text{ m}$; . Using $y = u_y t + \frac{1}{2} a_y t^2$, we have :

$$\begin{aligned}\Rightarrow 30 &= 5t + \frac{1}{2}10t^2 \\ \Rightarrow t^2 + t - 6 &= 0 \\ \Rightarrow t(t+3) - 2(t+3) &= 0 \\ \Rightarrow t &= -3 \text{ s or } t = 2 \text{ s}\end{aligned}$$

Neglecting negative value of time, $t = 2 \text{ s}$

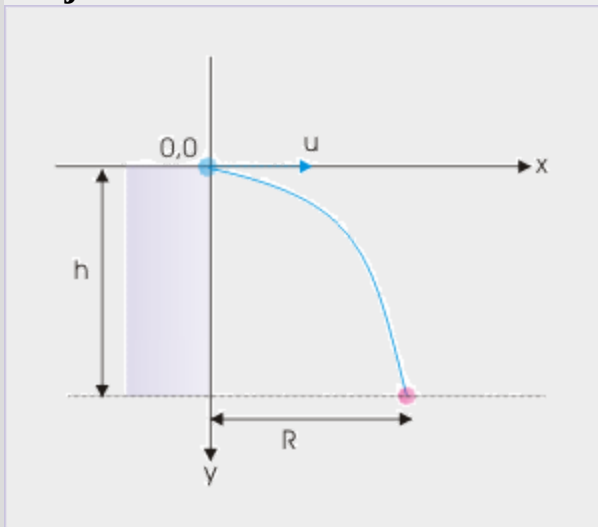
Range of flight

Example:

Problem : A ball is thrown from a tower of height “h” in the horizontal direction at a speed “u”. Find the horizontal range of the projectile.

Solution : Here, we consider a reference system whose origin coincides with the point of projection. we consider that the downward direction is positive y - direction.

Projectile motion



$$\Rightarrow x = R = u_x T = uT$$

Motion in the vertical direction :

Here, $u_y = 0$ and $t = T$ (total time of flight)

$$\Rightarrow h = \frac{1}{2} g T^2$$

$$\Rightarrow T = \sqrt{\frac{2h}{g}}$$

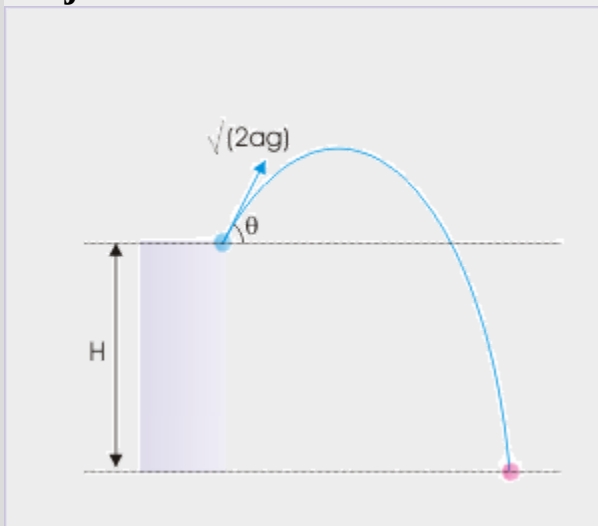
Putting expression of total time of flight in the expression for horizontal range, we have :

$$\Rightarrow R = u \sqrt{\frac{2h}{g}}$$

Example:

Problem : A projectile is projected up with a velocity $\sqrt{2ag}$ at an angle “ θ ” from an elevated position as shown in the figure. Find the maximum horizontal range that can be achieved.

Projectile motion

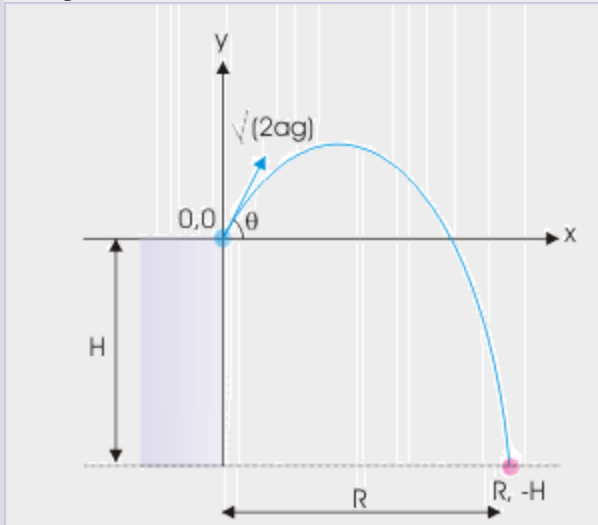


Projectile projected from an elevated point .

Solution : In order to determine the maximum horizontal range, we need to find an expression involving horizontal range. We shall use the equation

of projectile as we have the final coordinates of the motion as shown in the figure below :

Projectile motion



Projectile projected from an elevated point .

$$y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$$

Substituting and changing trigonometric ratio with the objective to create a quadratic equation in “tan θ ” :

$$\Rightarrow -H = R \tan \theta - \frac{gR^2}{2 \cos^2 \theta} \quad 1 + \tan^2 \theta$$

Rearranging, we have :

$$R^2 \tan^2 \theta - 4aR \tan \theta + R^2 - 4aH = 0$$

$$\Rightarrow \tan \theta = \frac{4aR \pm \sqrt{(4aR)^2 - 4R^2(R^2 - 4aH)}}{2R^2}$$

For $\tan \theta$ to be real, it is required that

$$16a^2 R^2 \geq 4R^2 R^2 - 4aH$$

$$\Rightarrow 4a^2 \geq R^2 - 4aH$$

$$\Rightarrow R^2 \leq 4a(a + H)$$

$$\Rightarrow R \leq \pm 2 \sqrt{a(a + H)}$$

Hence, maximum possible range is :

$$\Rightarrow R = 2 \sqrt{a(a + H)}$$

Initial velocity

Example:

Problem : A ball is thrown horizontally from the top of the tower to hit the ground at an angle of 45° in 2 s. Find the speed of the ball with which it was projected.

Solution : The question provides the angle at which the ball hits the ground. A hit at 45° means that horizontal and vertical speeds are equal.

$$\tan 45^\circ = \frac{v_y}{v_x} = 1$$

$$\Rightarrow v_x = v_y$$

However, we know that horizontal component of velocity does not change with time. Hence, final velocity in horizontal direction is same as initial velocity in that direction.

$$\Rightarrow v_x = v_y = u_x$$

We can now find the vertical component of velocity at the time projectile hits the ground by considering motion in vertical direction. Here, $u_y = 0$, $t = 2s$.

Using equation of motion in vertical direction, assuming downward direction as positive :

$$v_y = u_y + at$$

$$\Rightarrow v_y = 0 + 10 \times 2 = 20 \text{ m/s}$$

Hence, the speed with which the ball was projected in horizontal direction is :

$$\Rightarrow u_x = v_y = 20 \text{ m/s}$$

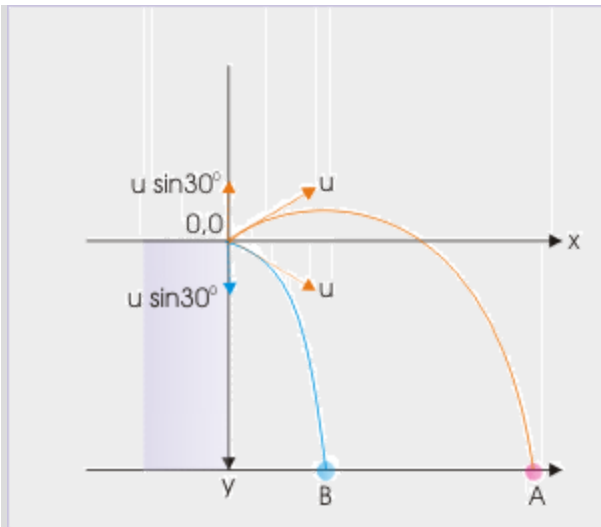
Final velocity

Example:

Problem : A ball “A” is thrown from the edge of building “h”, at an angle of 30° from the horizontal, in upward direction. Another ball ”B” is thrown at the same speed from the same position, making same angle with horizontal, in vertically downward direction. If "u" be the speed of projection, then find their speed at the time of striking the ground.

Solution : The horizontal components of velocity for two projectiles are equal. Further, horizontal component of velocity remains unaltered during projectile motion. The speed of the projectile at the time of striking depends solely on vertical component of velocity. For the sake of convenience of analysis, we consider point of projection as the origin of coordinate system and vertically downward direction as positive y - direction.

Projectile motion



Motion in vertical direction :

For A, $u_{yA} = -u \sin 30^\circ = -\frac{u}{2}$

For B, $u_{yB} = u \sin 30^\circ = \frac{u}{2}$

Thus, velocities in vertical direction are equal in magnitude, but opposite in direction. The ball, "A", which is thrown upward, returns after reaching the maximum vertical height. For consideration in vertical direction, the ball returns to the point of projection with same speed it was projected.

What it means that the vertical component of velocity of ball "A" on return at the point projection is " $u/2$ ". This further means that two balls "A" and "B", as a matter of fact, travel down with same downward vertical component of velocity.

In other words, the ball "A" returns to its initial position acquiring same speed " $u/2$ " as that of ball "B" before starting its downward journey. Thus, speeds of two balls are same i.e " $u/2$ " for downward motion. Hence, two balls strike the ground with same speed. Let the final speed is " v ".

Now, we apply equation of motion to determine the final speed in the vertical direction.

$$v_y^2 = u_y^2 + 2gh$$

Putting values, we have :

$$v_y^2 = \left(\frac{u}{2}\right)^2 + 2gh = \frac{u^2 + 8gh}{4}$$

$$v_y = \frac{(u^2 + 8gh)}{2}$$

The horizontal component of velocity remains same during the journey. It is given as :

$$v_x = u \cos 30^\circ = \frac{\sqrt{3}u}{2}$$

The resultant of two mutually perpendicular components is obtained, using Pythagoras theorem :

$$\Rightarrow v^2 = v_x^2 + v_y^2 = \frac{3u^2}{4} + \frac{u^2 + 8gh}{4}$$

$$\Rightarrow v^2 = \frac{3u^2 + u^2 + 8gh}{4} = u^2 + 2gh$$

$$\Rightarrow v = \sqrt{(u^2 + 2gh)}$$

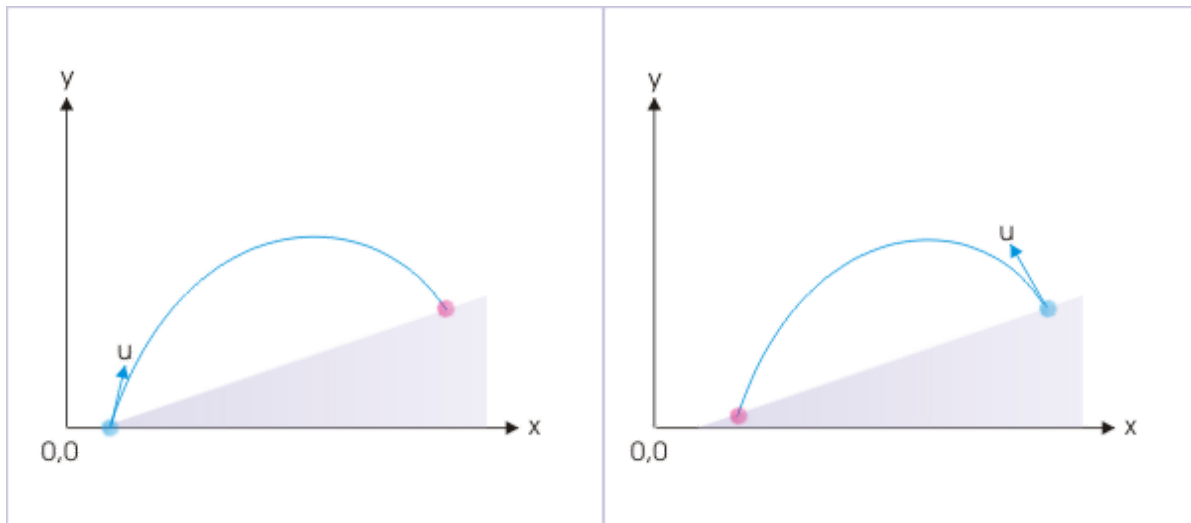
Projectile motion on an incline

Projectile motion on an incline plane is one of the various projectile motion types. The main distinguishing aspect is that points of projection and return are not on the same horizontal plane. There are two possibilities : (i) the point of return is at a higher level than the point of projection i.e projectile is thrown up the incline and (ii) Point of return is at a lower level than point of projection i.e. projectile is thrown down the incline.

Projection on the incline

Projection up the incline

Projection down the incline



We have so far studied the projectile motion, using technique of component motions in two mutually perpendicular directions – one which is horizontal and the other which is vertical. We can simply extend the methodology to these types of projectile motion types as well. Alternatively, we can choose coordinate axes along the incline and in the direction of perpendicular to the incline. The analysis of projectile motion in two coordinate systems differs in the detail of treatment.

For convenience of comparison, we shall refer projectile motion on a horizontal surface as the “normal case”. The reference to “normal case” enables us to note differences and similarities between “normal case” and the case of projectile motion on an incline plane.

Analyzing alternatives

As pointed out, there are two different approaches of analyzing projectile motion on an incline plane. The first approach could be to continue analyzing motion in two mutually perpendicular horizontal and vertical directions. The second approach could be to analyze motion by changing the reference orientation i.e. we set up our coordinate system along the incline and a direction along the perpendicular to incline.

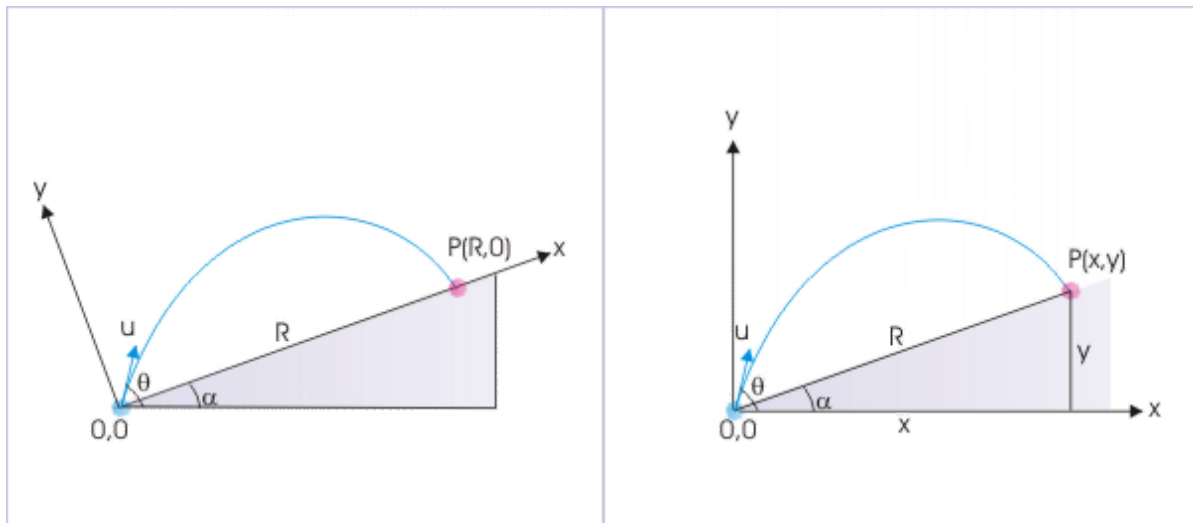
The analysis alternatives are, therefore, distinguished on the basis of coordinate system that we choose to employ :

- planar coordinates along incline (x) and perpendicular to incline (y)
- planar coordinates in horizontal (x) and vertical (y) directions

Coordinate systems

With reference to incline

With reference to horizontal



The two alternatives, as a matter of fact, are entirely equivalent. However, we shall study both alternatives separately for the simple reason that they provide advantage in analyzing projectile motion in specific situation.

Projection up the incline

As pointed out, the projection up the incline can be studied in two alternative ways. We discuss each of the approach, highlighting intricacies of each approach in the following sub-section.

Coordinates along incline (x) and perpendicular to incline (y)

This approach is typically superior approach in so far as it renders measurement of time of flight in a relatively simpler manner. However, before we proceed to analyze projectile motion in this new coordinate set up, we need to identify and understand attributes of motion in mutually perpendicular directions.

Measurement of angle of projection is one attribute that needs to be handled in a consistent manner. It is always convenient to follow certain convention in referring angles involved. We had earlier denoted the angle of projection as measured from the horizontal and denoted the same by the symbol “ θ ”. It is evident that it would be reasonable to extend the same convention and also retain the same symbol for the angle of projection.

It also follows that we measure other angles from the horizontal – even if we select x-coordinate in any other direction like along the incline. This convention avoids confusion. For example, the angle of incline “ α ” is measured from the horizontal. The horizontal reference, therefore, is actually a general reference for measurement of angles in the study of projectile motion.

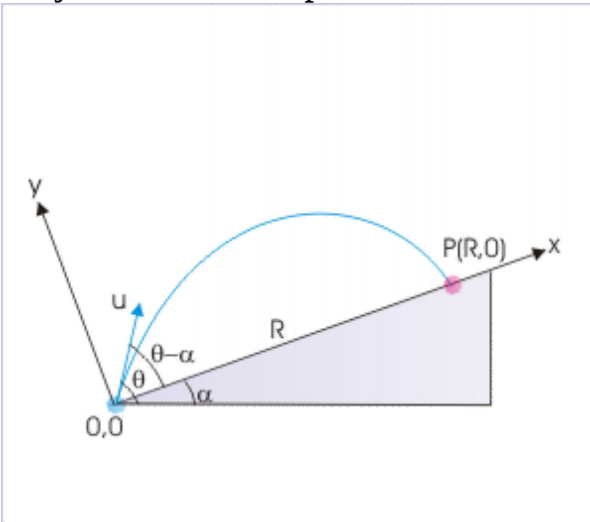
Now, let us have a look at other characterizing aspects of new analysis set up :

1: The coordinate “x” is along the incline – not in the horizontal direction; and the coordinate “y” is perpendicular to incline – not in the vertical direction.

2: Angle with the incline

From the figure, it is clear that the angle that the velocity of projection makes with x-axis (i.e. incline) is “ $\theta - \alpha$ ”.

Projectile motion up an incline



The projection from lower level.

3: The point of return

The point of return is specified by the coordinate $R,0$ in the coordinate system, where “R” is the range along the incline.

4: Components of initial velocity

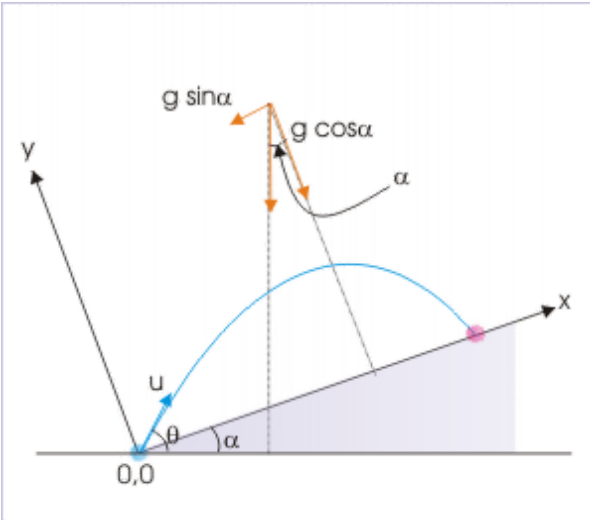
$$u_x = u \cos(\theta - \alpha)$$

$$u_y = u \sin(\theta - \alpha)$$

5: The components of acceleration

In order to determine the components of acceleration in new coordinate directions, we need to know the angle between acceleration due to gravity and y-axis. We see that the direction of acceleration is perpendicular to the base of incline (i.e. horizontal) and y-axis is perpendicular to the incline.

Components of acceleration due to gravity



The acceleration due to gravity forms an angle with y-axis, which is equal to angle of incline.

Thus, the angle between acceleration due to gravity and y – axis is equal to the angle of incline i.e. “ α ”. Therefore, components of acceleration due to gravity are :

$$a_x = -g \sin \alpha$$

$$a_y = -g \cos \alpha$$

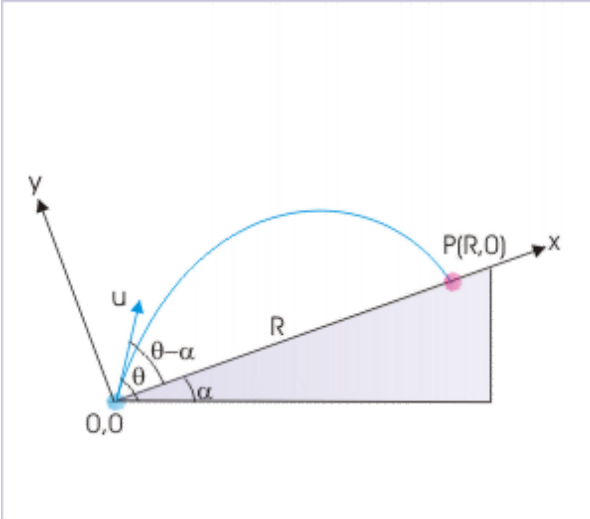
The negative signs precede the expression as two components are in the opposite directions to the positive directions of the coordinates.

6: Unlike in the normal case, the motion in x-direction i.e. along the incline is not uniform motion, but a decelerated motion. The velocity is in positive x-direction, whereas acceleration is in negative x-direction. As such, component of motion in x-direction is decelerated at a constant rate “ $g \sin \alpha$ ”.

Time of flight

The time of flight (T) is obtained by analyzing motion in y -direction (which is no more vertical as in the normal case). The displacement in y -direction after the projectile has returned to the incline, however, is zero as in the normal case. Thus,

Projectile motion up an incline



The projection from lower level.

$$y = u_y T + \frac{1}{2} a_y T^2 = 0$$

$$\Rightarrow u \sin(\theta - \alpha) T + \frac{1}{2} (-g \cos \alpha) T^2 = 0$$

$$\Rightarrow T \left\{ u \sin(\theta - \alpha) + \frac{1}{2} (-g \cos \alpha) T \right\} = 0$$

Either,

$$T = 0$$

or,

$$\Rightarrow T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

The first value represents the initial time of projection. Hence, second expression gives us the time of flight as required. We should note here that the expression of time of flight is alike normal case in a significant manner.

In the generic form, we can express the formula of the time of flight as :

$$T = \left| \frac{2u_y}{a_y} \right|$$

In the normal case, $u_y = u \sin \theta$ and $a_y = -g$. Hence,

$$T = \frac{2u \sin \theta}{g}$$

In the case of projection on incline plane, $u_y = u \sin(\theta - \alpha)$ and $a_y = -g \cos \alpha$. Hence,

$$T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

This comparison and understanding of generic form of the expression for time of flight helps us write the formula accurately in both cases.

Range of flight

First thing that we should note that we do not call “horizontal range” as the range on the incline is no more horizontal. Rather we simply refer the displacement along x-axis as “range”. We can find range of flight by considering motion in both “x” and “y” directions. Note also that we needed the same approach even in the normal case. Let “R” be the range of projectile motion.

The motion along x-axis is no more uniform, but decelerated. This is the major difference with respect to normal case.

$$x = u_x T - \frac{1}{2} a_x T^2$$

Substituting value of “T” as obtained before, we have :

$$R = \frac{u \cos(\theta - \alpha) \times 2u \sin(\theta - \alpha)}{g \cos \alpha} - \frac{g \sin \alpha \times 4u^2 \sin^2(\theta - \alpha)}{2g^2 \cos^2 \alpha}$$

$$\Rightarrow R = \frac{u^2}{g \cos^2 \alpha} \left\{ 2 \cos(\theta - \alpha) \sin(\theta - \alpha) \cos \alpha - \sin \alpha \times 2 \sin^2(\theta - \alpha) \right\}$$

Using trigonometric relation, $2 \sin^2(\theta - \alpha) = 1 - \cos 2(\theta - \alpha)$,

$$\Rightarrow R = \frac{u^2}{g \cos^2 \alpha} \left[\sin 2(\theta - \alpha) \cos \alpha - \sin \alpha \left\{ 1 - \cos 2(\theta - \alpha) \right\} \right]$$

$$\Rightarrow R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin 2(\theta - \alpha) \cos \alpha - \sin \alpha + \sin \alpha \cos 2(\theta - \alpha) \right\}$$

We use the trigonometric relation,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B ,$$

$$\Rightarrow R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin(2\theta - 2\alpha + \alpha) - \sin \alpha \right\}$$

$$\Rightarrow R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin(2\theta - \alpha) - \sin \alpha \right\}$$

This is the expression for the range of projectile on an incline. We can see that this expression reduces to the one for the normal case, when $\alpha = 0$,

$$\Rightarrow R = \frac{u^2 \sin 2\theta}{g}$$

Maximum range

The range of a projectile thrown up the incline is given as :

$$R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin(2\theta - \alpha) - \sin \alpha \right\}$$

We see here that the angle of incline is constant. The range, therefore, is maximum for maximum value of “ $\sin(2\theta - \alpha)$ ”. Thus, range is maximum for the angle of projection as measured from horizontal direction, when :

$$\sin(2\theta - \alpha) = 1$$

$$\Rightarrow \sin(2\theta - \alpha) = \sin \pi/2$$

$$\Rightarrow 2\theta - \alpha = \pi/2$$

$$\Rightarrow \theta = \pi/2 + \alpha/2$$

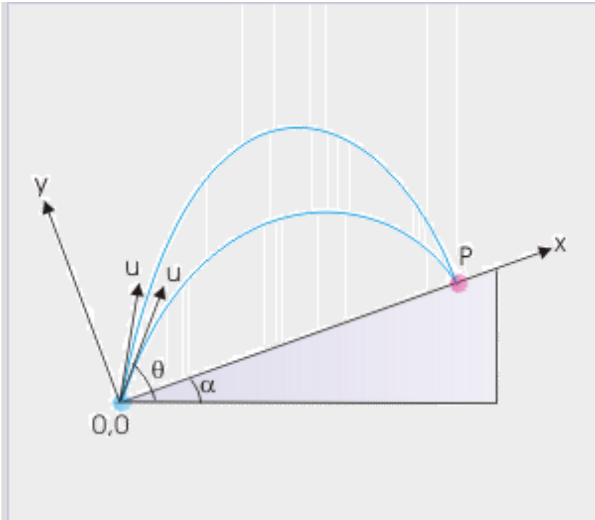
The maximum range, therefore, is :

$$\Rightarrow R_{\max} = \frac{u^2}{g \cos^2 \alpha} (1 - \sin \alpha)$$

Example:

Problem : Two projectiles are thrown with same speed, “ u ”, but at different angles from the base of an incline surface of angle “ α ”. The angle of projection with the horizontal is “ θ ” for one of the projectiles. If two projectiles reach the same point on incline, then determine the ratio of times of flights for the two projectiles.

Projectile motion up an incline



Two projectiles reach the same point on the incline.

Solution : We need to find the ratio of times of flights. Let T_1 and T_2 be the times of flights. Now, the time of flight is given by :

$$T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

Here, the angle of projection of one of the projectiles, “ θ ”, is given. However, angle of projection of other projectile is not given. Let “ θ' ” be the angle of projection of second projectile.

$$\Rightarrow \frac{T_1}{T_2} = \frac{2u \sin(\theta - \alpha)}{2u \sin(\theta' - \alpha)}$$

We need to know “ θ' ” to evaluate the above expression. For this, we shall make use of the fact that projectiles have same range for two angles of projections. We can verify this by having a look at the expression of range, which is given as :

$$\Rightarrow R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin(2\theta - \alpha) - \sin \alpha \right\}$$

Since other factors remain same, we need to analyze motions of two projectiles for same range in terms of angle of projection only. We have noted in the case of normal projectile motion that there are complimentary angle for which horizontal range is same. Following the same line of argument and making use of the trigonometric relation $\sin\theta = \sin(\pi - \theta)$, we analyze the projectile motions of equal range. Here,

$$\sin(2\theta' - \alpha) = \sin\{\pi - (2\theta - \alpha)\} = \sin(\pi - 2\theta + \alpha)$$

$$2\theta' - \alpha = \pi - 2\theta + \alpha$$

$$\Rightarrow 2\theta' = \pi - 2\theta + 2\alpha$$

$$\Rightarrow \theta' = \frac{\pi}{2} - \theta + \alpha$$

Putting this value in the expression for the ratio of times of flights, we have :

$$\Rightarrow \frac{T_1}{T_2} = \frac{2u \sin(\theta - \alpha)}{2u \sin(\pi/2 - \theta + \alpha - \alpha)}$$

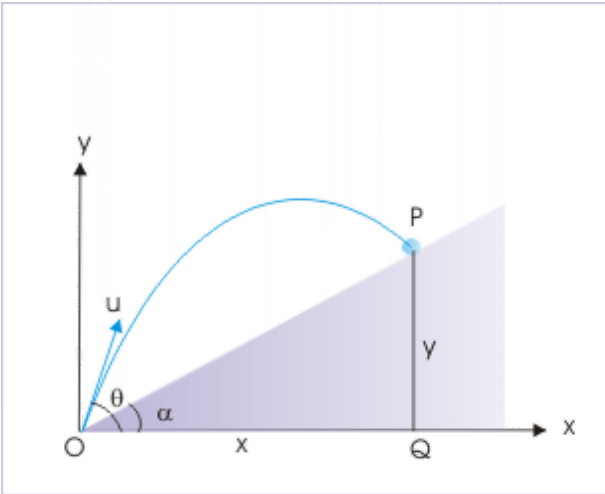
$$\Rightarrow \frac{T_1}{T_2} = \frac{\sin(\theta - \alpha)}{\sin(\pi/2 - \theta)}$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{\sin(\theta - \alpha)}{\cos \theta}$$

Coordinates in horizontal (x) and vertical (y) directions

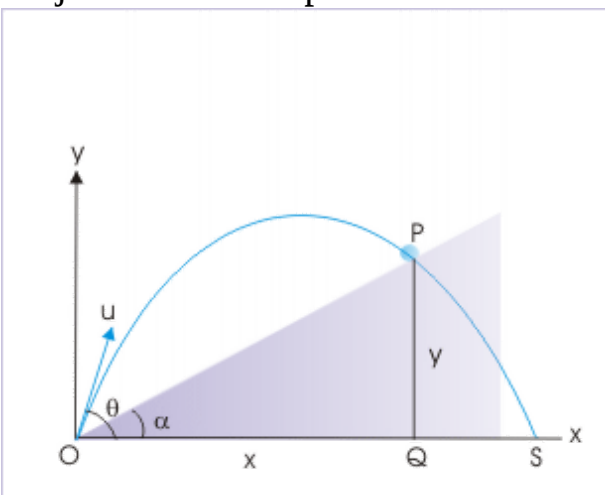
This approach retains the coordinates used in the normal case (in which projectile returns to the same horizontal level). In this consideration, the description of projectile motion is same as normal case except that motion is aborted in the mid-air by the incline. Had incline been not there, the projectile would have continued with its motion as shown in the figure.

Projectile motion up an incline



Projectile motion up is curtailed
by incline.

When the projectile is allowed to return to the projection level, then the point of return is $(OQ,0)$, where OQ is the horizontal range. This position of point of return changes to a new point (x,y) , specified by the angle of elevation “ α ” of the wedge with respect to horizontal as shown in the figure.
Projectile motion up an incline



Projectile motion described with x -
axis in horizontal direction and
 y -axis in vertical direction.

From the triangle OPQ,

$$\cos \alpha = \frac{x}{OP} = \frac{x}{R}$$

The range of the projectile is given by :

$$\Rightarrow R = \frac{x}{\cos \alpha}$$

The strategy here is to determine “x” i.e. “OQ” considering the motion as normal projectile motion. Thus, we shall first determine “x” and then using above relation, we obtain the relation for the range of flight along the incline. Now, considering motion in horizontal direction , we have :

$$x = u \cos \theta T$$

where “T” is the time of flight of projectile motion on the incline. It is given as determined earlier :

$$T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

Substituting in the expression of “x”, we have :

$$\Rightarrow x = \frac{u \cos \theta \times 2u \sin(\theta - \alpha)}{g \cos \alpha}$$

$$\Rightarrow x = \frac{u^2 2 \sin(\theta - \alpha) \cos \theta}{g \cos \alpha}$$

We simplify this relation, using trigonometric relation as given here :

$$\sin C - \sin D = 2 \sin \left(\frac{C - D}{2} \right) \cos \left(\frac{C + D}{2} \right)$$

Comparing right hand side of the equation with the expression in the numerator of the equation of “x”, we have :

$$C - D = 2\theta - 2\alpha$$

$$C + D = 2\theta$$

Adding, we have :

$$\Rightarrow C = 2\theta - \alpha$$

$$\Rightarrow D = \alpha$$

Thus, we can write :

$$\Rightarrow 2 \sin(\theta - \alpha) \cos \theta = \sin(2\theta - \alpha) - \sin \alpha$$

Substituting the expression in the equation of “x”,

$$\Rightarrow x = \frac{u^2}{g \cos \alpha} \left\{ \sin(2\theta - \alpha) - \sin \alpha \right\}$$

Using the relation connecting horizontal range “x” with the range on incline, “R”, we have :

$$R = \frac{x}{\cos \alpha}$$

$$\Rightarrow R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin(2\theta - \alpha) - \sin \alpha \right\}$$

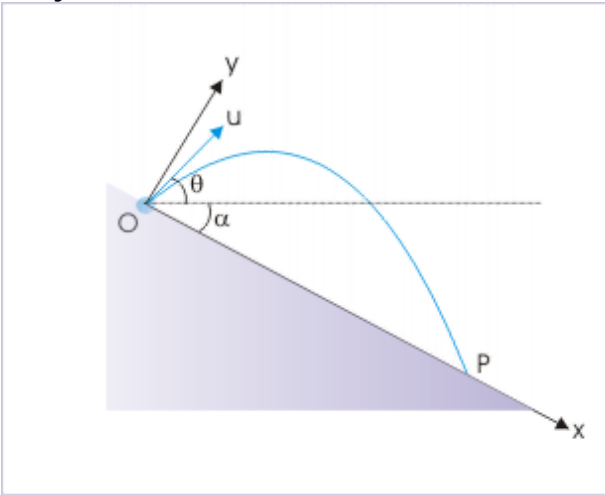
Thus, we get the same expression for range as expected. Though the final expressions are same, but the understanding of two approaches is important as they have best fit application in specific situations.

Projection down the incline

A typical projection down the incline is shown with a reference system in which “x” and “y” axes are directions along incline and perpendicular to incline. The most important aspect of the analysis of this category of

projectile motion is the emphasis that we put on the convention for measuring angles.

Projectile motion down an incline



Projectile motion thrown from a higher point.

The angle of projection and angle of incline both are measured from a horizontal line. The expression for the time of flight is obtained by analyzing motion in vertical directions. Here we present the final results without working them out as the final forms of expressions are suggestive.

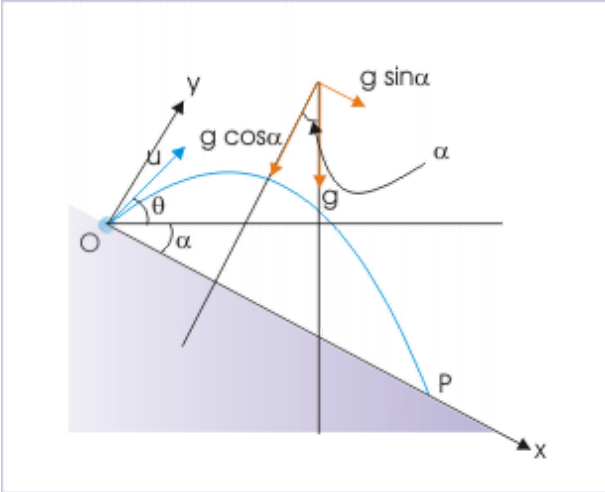
1: Components of initial velocity

$$u_x = u \cos(\theta + \alpha)$$

$$u_y = u \sin(\theta + \alpha)$$

2: Components of acceleration

Components of acceleration due to gravity



Acceleration due to gravity forms an angle with y-direction, which is equal to angle of incline.

$$a_x = g \sin \alpha$$

$$a_y = -g \cos \alpha$$

3: Time of flight

The expression of time of flight differs only with respect to angle of sine function in the numerator of the expression :

$$T = \frac{2u \sin(\theta + \alpha)}{g \cos \alpha}$$

4: Range of flight

The expression of range of flight differs only with respect to angle of sine function :

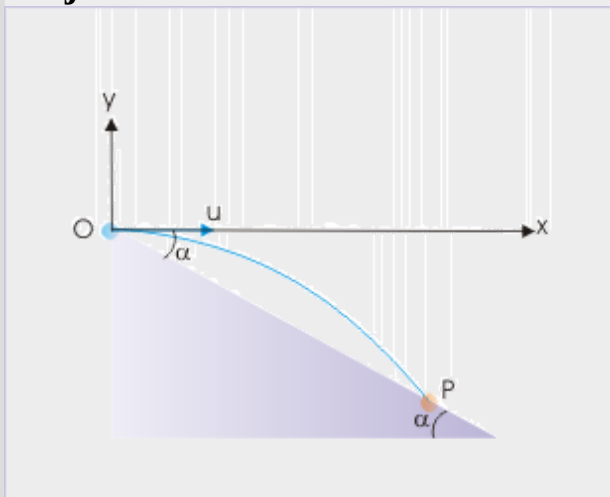
$$R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin(2\theta + \alpha) + \sin \alpha \right\}$$

It is very handy to note that expressions have changed only with respect of the sign of “ α ” for the time of flight and the range. We only need to exchange “ α ” by “ $-\alpha$ ” .

Example:

Problem : A ball is projected in horizontal direction from an elevated point “O” of an incline of angle “ α ” with a speed “ u ”. If the ball hits the incline surface at a point “P” down the incline, find the coordinates of point “P”.

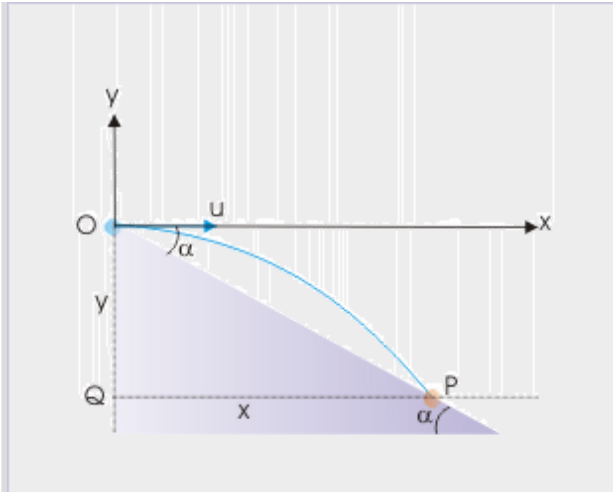
Projectile motion down an incline



A ball is projected in horizontal direction.

Solution : We can answer this question, using the relation of coordinates with range “R” as :

Projectile motion down an incline



A ball is projected in horizontal direction.

$$x = R \cos \alpha$$

$$y = -R \sin \alpha$$

Now, range of the flight for the downward flight is given as :

$$R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin(2\theta + \alpha) + \sin \alpha \right\}$$

The important thing to realize here is that the ball is projected in horizontal direction. As we measure angle from the horizontal line, it is evident that the angle of projection is zero. Hence,

$$\theta = 0^\circ$$

Putting in the equation for the range of flight, we have :

$$R = \frac{2u^2 \sin \alpha}{g \cos^2 \alpha}$$

Therefore, coordinates of the point of return, “P”, is :

$$\Rightarrow x = R \cos \alpha = \frac{2u^2 \sin \alpha}{g \cos \alpha}$$

$$\Rightarrow x = \frac{2u^2 \tan \alpha}{g}$$

Similarly,

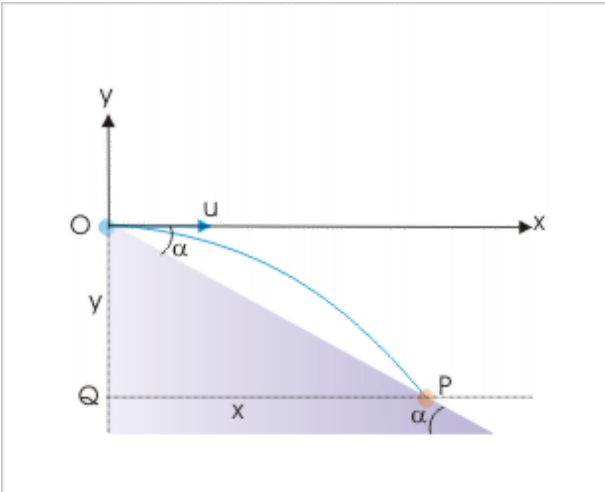
$$\Rightarrow y = -R \sin \alpha = -\frac{2u^2 \sin \alpha}{g \cos^2 \alpha} \sin \alpha$$

$$\Rightarrow y = -\frac{2u^2 \tan^2 \alpha}{g}$$

This example illustrated how to use formulae of the range of flight. We should, however, know that actually, we have the options to analyze projectile motion down an incline without using derived formula.

As a matter of fact, we can consider projectile motion down an incline as equivalent to projectile motion from an elevated point as studied in the previous module with out any reference to an incline or wedge. We need to only shift the horizontal base line to meet the point of return. The line joining the point of projection and point of return, then, represents the incline surface.

Projectile motion down an incline



A ball is projected in horizontal direction.

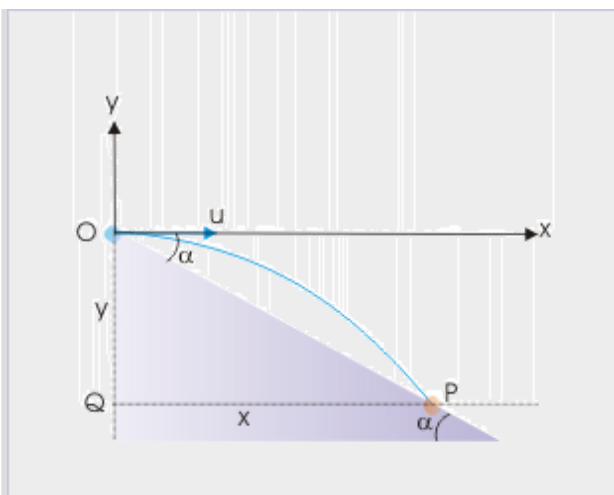
Observe the projectile motion from a height as shown in the figure. Let the projectile returns to a point “P”. The line “OP” then represents the incline surface. We can analyze this motion in rectangular coordinates “x” and “y” in horizontal and vertical direction, using general technique of analysis in component directions.

Here, we work with the same example as before to illustrate the working in this alternative manner.

Example:

Problem : A ball is projected in horizontal direction from an elevated point “O” of an incline of angle “ α ” with a speed “u”. If the ball hits the incline surface at a point “P” down the incline, find the coordinates of point “P”.

Projectile motion down from a height



A ball is projected in horizontal direction.

Solution : We draw or shift the horizontal base and represent the same by the line QP as shown in the figure below.

From the general consideration of projectile motion, the vertical displacement, “y”, is :

$$y = u_y T + \frac{1}{2} a_y T^2$$

$$\Rightarrow y = 0 - \frac{1}{2} g T^2$$

Considering the magnitude of vertical displacement only, we have :

$$\Rightarrow y = \frac{1}{2} g T^2$$

On the other hand, consideration of motion in x-direction yields,

$$x = u_x T = u T$$

Since, we aim to find the coordinates of point of return “P”, we eliminate “T” from the two equations. This gives us :

$$y = \frac{gx^2}{2u^2}$$

From the triangle OPQ, we have :

$$\tan \alpha = \frac{y}{x}$$

$$\Rightarrow y = x \tan \alpha$$

Combining two equations, we have :

$$y = x \tan \alpha = \frac{gx^2}{2u^2}$$

$$\Rightarrow \tan \alpha = \frac{gx}{2u^2}$$

$$\Rightarrow x = \frac{2u^2 \tan \alpha}{g}$$

and

$$\Rightarrow y = x \tan \alpha = \frac{2u^2 \tan^2 \alpha}{g}$$

The y-coordinate, however, is below origin and is negative. Thus, we put a negative sign before the expression :

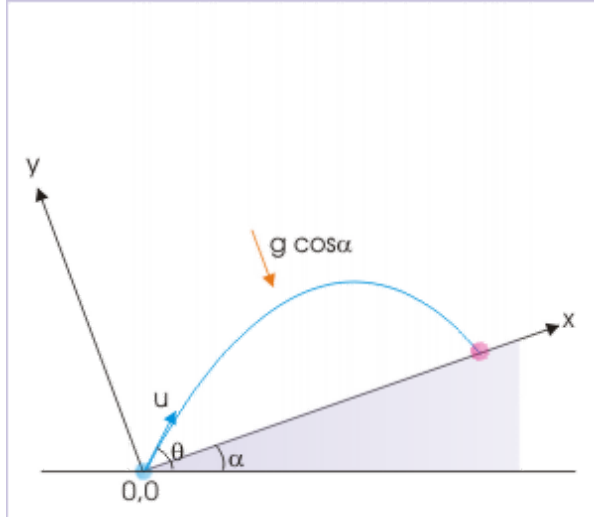
$$\Rightarrow y = -\frac{2u^2 \tan^2 \alpha}{g}$$

Exercises

Exercise:

Problem:

A projectile is thrown from the base of an incline of angle 30° as shown in the figure. It is thrown at an angle of 60° from the horizontal direction at a speed of 10 m/s . The total time of flight is (consider $g = 10 \text{ m/s}^2$) :

Projectile motion on an incline

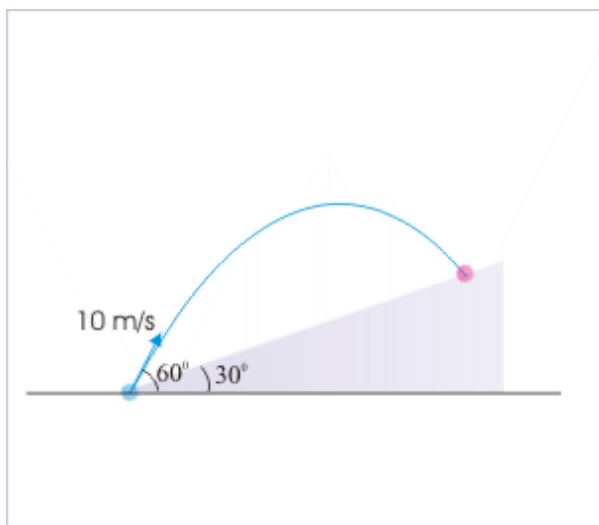
Projectile motion on an incline

$$(a) 2 \quad (b) \sqrt{3} \quad (c) \frac{\sqrt{3}}{2} \quad (d) \frac{2}{\sqrt{3}}$$

Solution:

This situation can be handled with a reoriented coordinate system as shown in the figure. Here, angle of projection with respect to x - direction is $(\theta - \alpha)$ and acceleration in y - direction is " $g \cos \alpha$ ". Now, total time of flight for projectile motion, when points of projection and return are on same level, is :

Projectile motion on an incline



Projectile motion on an incline

$$T = \frac{2u \sin \theta}{g}$$

Replacing " θ " by " $(\theta - \alpha)$ " and " g " by " $g \cos \alpha$ ", we have formula of time of flight over the incline :

$$\Rightarrow T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

Now, $\theta = 60^\circ$, $\alpha = 30^\circ$, $u = 10$ m/s. Putting these values,

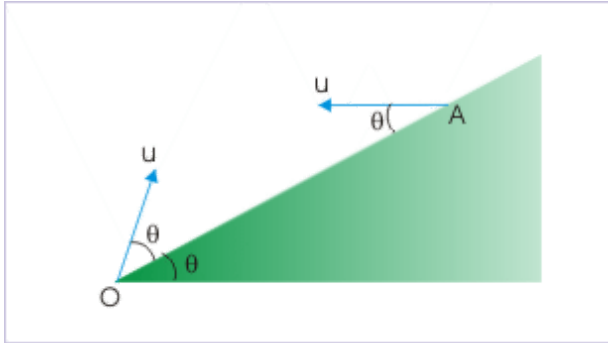
$$\Rightarrow T = \frac{2 \times 10 \sin(60^\circ - 30^\circ)}{g \cos 30^\circ} = \frac{20 \sin 30^\circ}{10 \cos 30^\circ} = \frac{2}{\sqrt{3}}$$

Hence, option (d) is correct.

Exercise:

Problem:

Two projectiles are thrown with the same speed from point "O" and "A" so that they hit the incline. If t_O and t_A be the time of flight in two cases, then :

Projectile motion on an incline

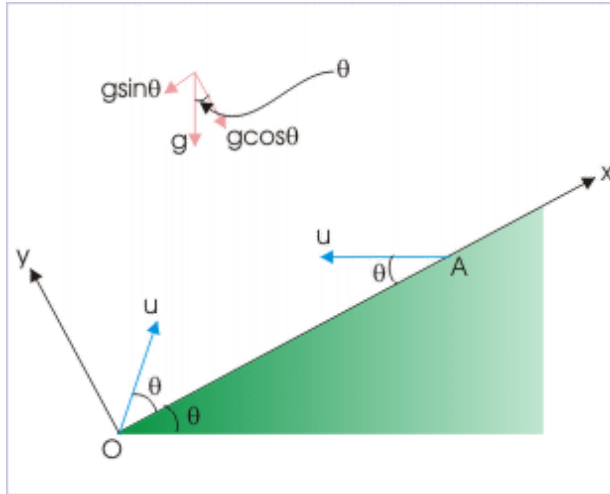
Projectile motion on an incline

$$(a)t_O = t_A \quad (b)t_O < t_A \quad (c)t_O > t_A \quad (d)t_O = t_A = \frac{u \tan \theta}{g}$$

Solution:

We have discussed that projectile motion on an incline surface can be rendered equivalent to projectile motion on plane surface by reorienting coordinate system as shown here :

Projectile motion on an incline



Projectile motion on an incline

In this reoriented coordinate system, we need to consider component of acceleration due to gravity along y-direction. Now, time of flight is given by :

$$t = \frac{2u_y}{g \cos \theta}$$

Let us first consider the projectile thrown from point "O". Considering the angle the velocity vector makes with the horizontal, the time of flight is given as :

$$\Rightarrow t_O = \frac{2u \sin(2\theta - \theta)}{g \cos \theta}$$

$$\Rightarrow t_O = \frac{2u \tan \theta}{g}$$

For the projectile thrown from point "A", the angle with horizontal is zero. Hence, the time of flight is :

$$\Rightarrow t_A = \frac{2u \sin(2 \times 0 + \theta)}{g \cos \theta} = \frac{2u \tan \theta}{g}$$

Thus, we see that times of flight in the two cases are equal.

$$\Rightarrow t_A = t_O$$

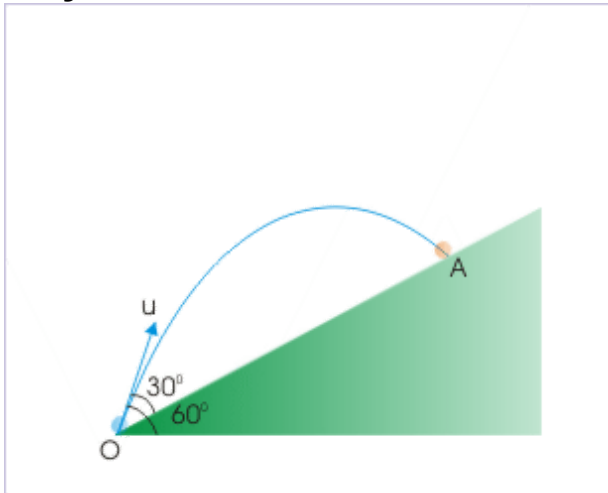
Hence, option (a) is correct.

Exercise:

Problem:

A ball is projected on an incline of 30° from its base with a speed 20 m/s, making an angle 60° from the horizontal. The magnitude of the component of velocity, perpendicular to the incline, at the time ball hits the incline is :

Projectile motion on an incline



Projectile motion on an incline

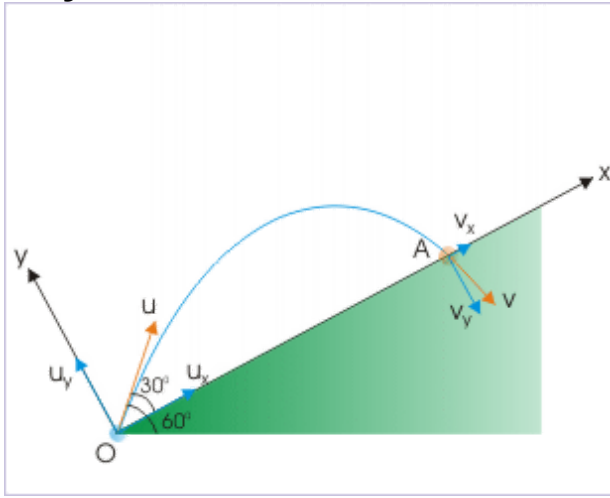
- (a) 10 m/s (b) $10\sqrt{3} \text{ m/s}$ (c) $20\sqrt{3} \text{ m/s}$ (d) $20\sqrt{3} \text{ m/s}$

Solution:

The velocity in y-direction can be determined making use of the fact that a ball under constant acceleration like gravity returns to the ground

with the same speed, but inverted direction. The component of velocity in y-direction at the end of the journey, in this case, is :

Projectile motion on an incline



Projectile motion on an incline

$$v_y = u_y = -20 \sin 30^\circ = -20 \times \frac{1}{2} = -10 \text{ m/s}$$

$$|v_y| = 10 \text{ m/s}$$

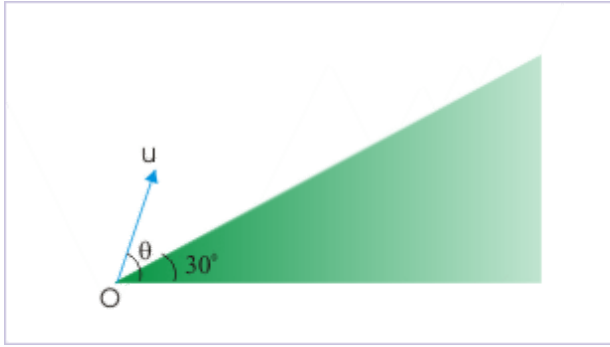
Hence, option (a) is correct.

Exercise:

Problem:

A projectile is projected from the foot of an incline of angle 30° . What should be the angle of projection, as measured from the horizontal direction so that range on the incline is maximum?

Projectile motion on an incline



Projectile motion on an incline

(a) 45° (b) 60° (c) 75° (d) 90°

Solution:

We can interpret the equation obtained for the range of projectile :

$$R = \frac{u^2}{g \cos^2 \alpha} \left[\sin(2\theta - \alpha) - \sin \alpha \right]$$

The range is maximum for the maximum value of " $\sin(2\theta - \alpha)$ " :

$$\sin(2\theta - \alpha) = 1 = \sin 90^\circ$$

$$\Rightarrow 2\theta - 30^\circ = 90^\circ$$

$$\Rightarrow \theta = 60^\circ$$

Hence, option (b) is correct.

Exercise:

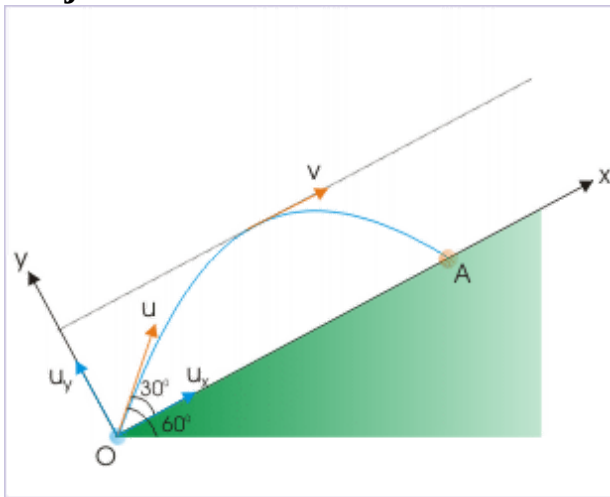
Problem:

A projectile is projected from the foot of an incline of angle 30° with a velocity 30 m/s . The angle of projection as measured from the horizontal is 60° . What would be its speed when the projectile is parallel to the incline?

- (a) 10 m/s (b) $2\sqrt{3} \text{ m/s}$ (c) $5\sqrt{3} \text{ m/s}$ (d) $10\sqrt{3} \text{ m/s}$
-

Solution:

In the coordinate system of incline and perpendicular to incline, motion parallel to incline denotes a situation when component of velocity in y -direction is zero. Note that this is an analogous situation to the point of maximum height in the normal case when projectile returns to same level.

Projectile motion on an incline

Projectile motion on an incline

We shall analyze the situation, taking advantage of this fact. Since component of velocity in y -direction is zero, it means that velocity of

projectile is same as that of component velocity in x-direction. For consideration of motion in y -direction, we have :

$$v_y = u_y + a_y t$$

$$\Rightarrow 0 = 30 \sin 30^\circ - g \cos 30^\circ X t$$

$$\Rightarrow t = \frac{15X2}{10X\sqrt{3}} = \sqrt{3}s$$

For consideration of motion in x -direction, we have :

$$v_x = u_x + a_x t = 30 \cos 30^\circ - g \sin 30^\circ X t$$

$$\Rightarrow v_x = 30X \frac{\sqrt{3}}{2} - 10X \frac{1}{2} X \sqrt{3}$$

$$\Rightarrow v_x = 15X\sqrt{3} - 5X\sqrt{3} = 10\sqrt{3}$$

$$\Rightarrow v_x = 10\sqrt{3} \quad m/s$$

But, component of velocity in y-direction is zero. Hence,

$$v = v_A = 10\sqrt{3} \quad m/s$$

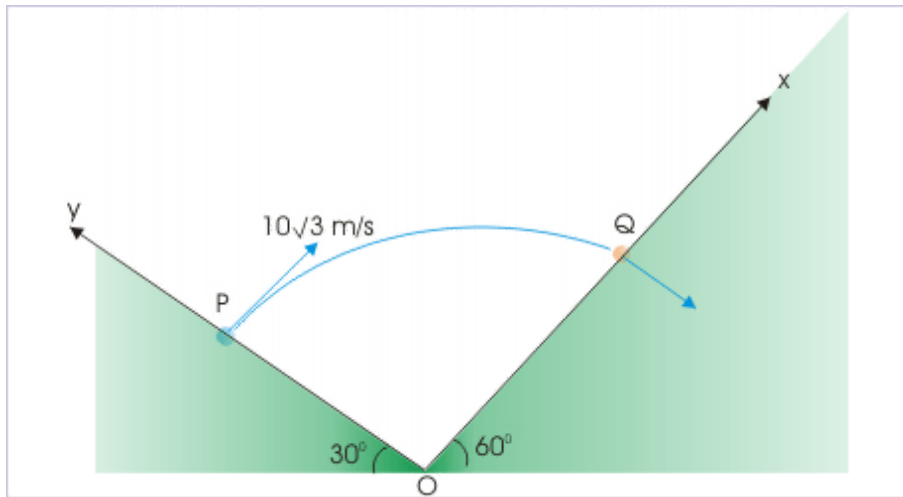
Hence, option (d) is correct.

Exercise:

Problem:

Two incline planes of angles 30° and 60° are placed touching each other at the base as shown in the figure. A projectile is projected at right angle with a speed of $10\sqrt{3}$ m/s from point "P" and hits the other incline at point "Q" normally. If the coordinates are taken along the inclines as shown in the figure, then

Projectile motion on an incline



Projectile motion on an incline

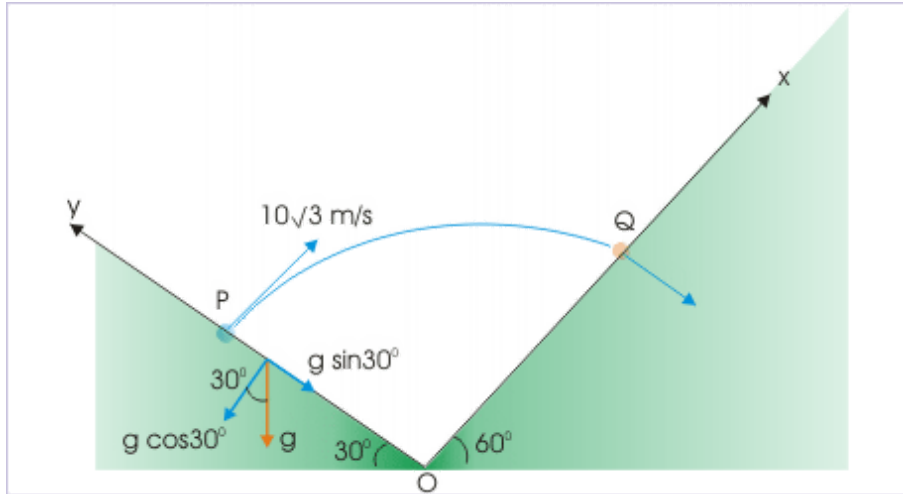
- (a) component of acceleration in x-direction is $-5\sqrt{3}m/s^2$
- (b) component of acceleration in x-direction is $-10\sqrt{3}m/s^2$
- (c) component of acceleration in y-direction is $-5\sqrt{3}m/s^2$
- (d) component of acceleration in y-direction is $-5m/s^2$

Solution:

This arrangement is a specific case in which the incline planes are at right angles to each other. We have taken advantage of this fact in assigning our coordinates along the planes, say y-axis along the first incline and x-axis along the second incline.

The acceleration due to gravity is acting in the vertically downward direction. We can get the component accelerations either using the angle of the first or second incline. Either of the considerations will yield the same result. Considering the first incline,

Projectile motion on an incline



Projectile motion on an incline

$$a_x = -g \cos 30^\circ = -10X \frac{\sqrt{3}}{2} = -5\sqrt{3}m/s^2$$

$$a_y = -g \sin 30^\circ = -10X \frac{1}{2} = -5 \quad m/s^2$$

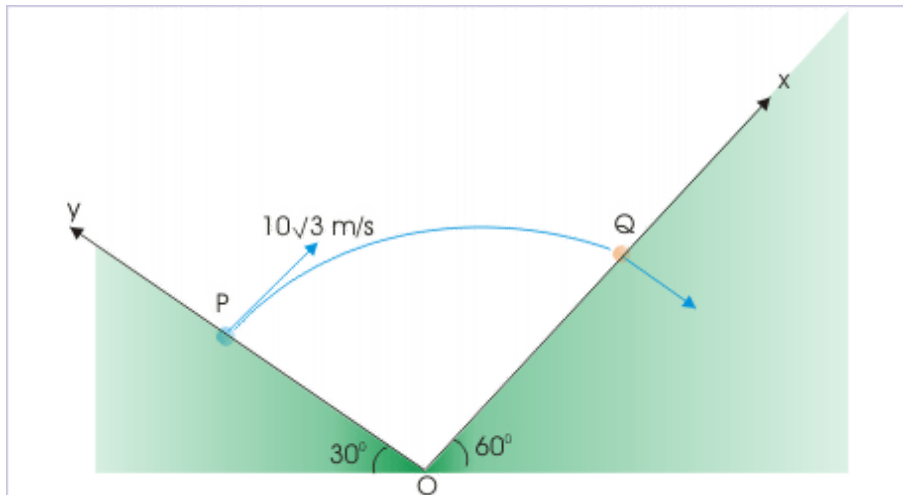
Hence, options (a) and (d) are correct.

Exercise:

Problem:

Two incline planes of angles 30° and 60° are placed touching each other at the base as shown in the figure. A projectile is projected at right angle with a speed of $10\sqrt{3}$ m/s from point "P" and hits the other incline at point "Q" normally. Then, the time of flight is :

Projectile motion on an incline



Projectile motion on an incline

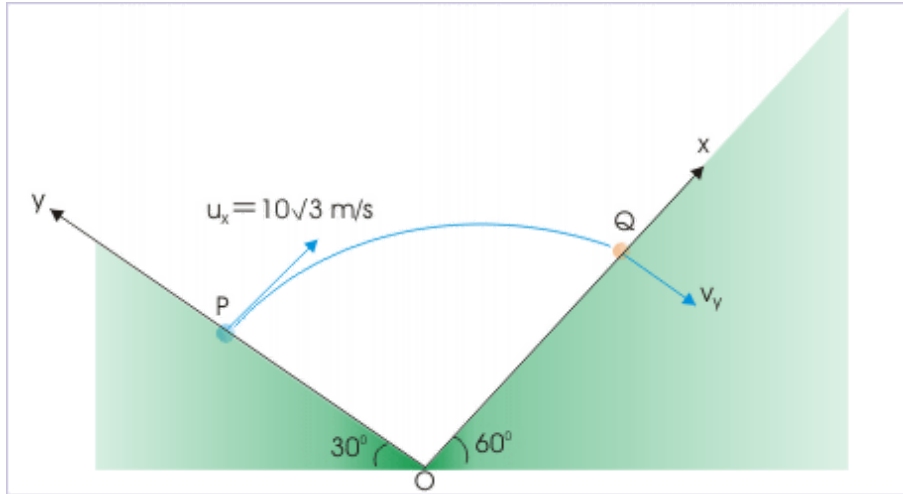
- (a) $1s$ (b) $2s$ (c) $3s$ (d) $4s$

Solution:

This arrangement is an specific case in which incline plane are right angle to each other. We have actually taken advantage of this fact in assigning our coordinates along the planes, say y-axis along first incline and x-axis against second incline.

In order to find the time of flight, we can further use the fact that projectile hits the other plane at right angle i.e. parallel to y-axis. This means that component of velocity in x-direction i.e. along the second incline is zero. This, in turn, suggests that we can analyze motion in x-direction to obtain time of flight.

Projectile motion on an incline



Projectile motion on an incline

In x-direction,

$$v_x = u_x + a_x T$$

$$\Rightarrow 0 = u_x + a_x T$$

$$\Rightarrow T = -\frac{u_x}{a_x}$$

We know need to know "ux" and "ax" in this coordinate system. The acceleration due to gravity is acting in vertically downward direction. Considering first incline,

$$a_x = -g \cos 30^\circ = -10 \times \frac{\sqrt{3}}{2} = -5\sqrt{3} \text{ m/s}^2$$

$$a_y = -g \sin 30^\circ = -10 \times \frac{1}{2} = -5 \text{ m/s}^2$$

Also, we observe that projectile is projected at right angle. Hence, component of projection velocity in x - direction is :

$$u_x = 10\sqrt{3} \text{ m/s}$$

Putting values in the equation and solving, we have :

$$\Rightarrow T = -\frac{u_x}{a_x} = \frac{-10\sqrt{3}}{-5\sqrt{3}} = 2 \text{ s}$$

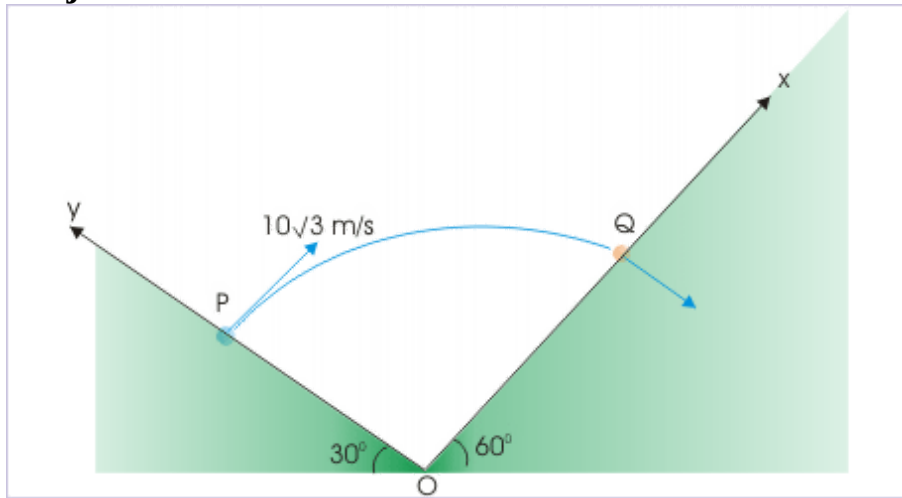
Hence, option (b) is correct.

Exercise:

Problem:

Two incline planes of angles 30° and 60° are placed touching each other at the base as shown in the figure. A projectile is projected at right angle with a speed of $10\sqrt{3}$ m/s from point "P" and hits the other incline at point "Q" normally. The speed with which the projectile hits the incline at "Q" is :

Projectile motion on an incline



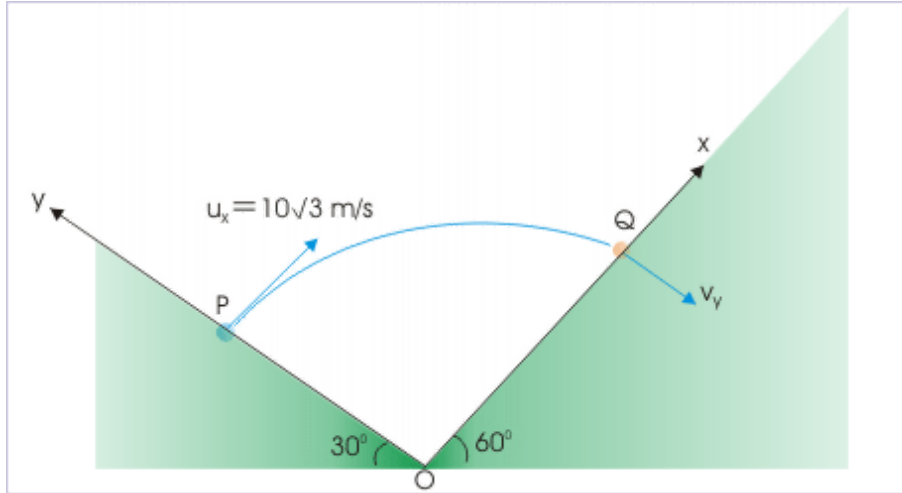
Projectile motion on an incline

- (a) 5 m/s (b) 10 m/s (c) $10\sqrt{3}$ m/s (d) 20 m/s

Solution:

We notice here that initial velocity in y-direction is zero. On the other hand, final velocity in the y-direction is equal to the velocity with which projectile hits at "Q". The x-component of velocity at "Q" is zero. The analysis of motion in y - direction gives us the relation for component of velocity in y-direction as :

Projectile motion on an incline



Projectile motion on an incline

In y-direction,

$$v = v_y = u_y + a_y$$

$$\Rightarrow T = 0 + a_y T$$

$$\Rightarrow v = v_y = a_y T$$

Thus, we need to know component of acceleration in y-direction and time of flight. As far as components of acceleration are concerned, the acceleration due to gravity is acting in vertically downward direction. Considering first incline, we have :

$$a_x = -g \cos 30^\circ = -10 \times \frac{\sqrt{3}}{2} = -5\sqrt{3} \text{ m/s}^2$$

$$a_y = -g \sin 30^\circ = -10 \times \frac{1}{2} = -5 \text{ m/s}^2$$

In order to find the time of flight, we can further use the fact that the component of velocity in x-direction i.e. along the second incline is zero. This, in turn, suggests that we can analyze motion in x-direction to obtain time of flight.

In x-direction,

$$\begin{aligned} v_x &= u_x + a_x T \\ \Rightarrow 0 &= u_x + a_x T \\ \Rightarrow T &= -\frac{u_x}{a_x} \end{aligned}$$

We now need to know " u_x " in this coordinate system. We observe that projectile is projected at right angle. Hence, component of projection velocity in x - direction is :

$$u_x = 10\sqrt{3} \text{ m/s}$$

Putting values in the equation and solving, we have :

$$\Rightarrow T = -\frac{u_x}{a_x} = \frac{-10\sqrt{3}}{-5\sqrt{3}} = 2 \text{ s}$$

Thus putting values for the expression for the speed of the projectile with which it hits the incline is :

$$\Rightarrow v = v_y = a_y T = -5 \times 2 = -10 \text{ m/s}$$

Thus, speed is 10 m/s.

Hence, option (b) is correct.

Projectile motion on an incline (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints for solving problems

Problems based on projectile motion over an incline are slightly difficult. The analysis is complicated mainly because there are multitudes of approaches available. First there is issue of coordinates, then we might face the conflict to either use derived formula or analyze motion independently in component directions and so on. We also need to handle motion up and down the incline in an appropriate manner. However, solutions get easier if we have the insight into the working with new set of coordinate system and develop ability to assign appropriate values of accelerations, angles and component velocities etc.

Here, we present a simple set of guidelines in a very general way :

- 1:** Analyze motion independently along the selected coordinates. Avoid using derived formula to the extent possible.
- 2:** Make note of information given in the question like angles etc., which might render certain component of velocity zero in certain direction.
- 3:** If range of the projectile is given, we may try the trigonometric ratio of the incline itself to get the answer.
- 4:** If we use coordinate system along incline and in the direction perpendicular to it, then always remember that component motion along both incline and in the direction perpendicular to it are accelerated motions. Ensure that we use appropriate components of acceleration in the equations of motion.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the concept of projectile motion on an incline. The questions are categorized in terms of the characterizing features of the subject matter :

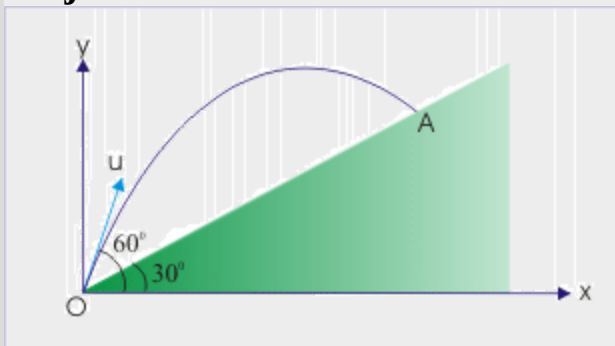
- Range of the flight
- Angle of projection
- Final Speed of the projectile
- Elastic collision with the incline
- Projectile motion on double inclines

Range of the flight

Example:

Problem : A projectile is thrown with a speed " u " at an angle 60° over an incline of 30° . If the time of flight of the projectile is " T ", then find the range of the flight.

Projectile motion on an incline

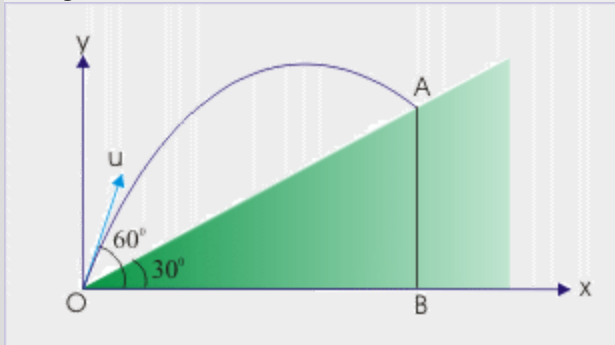


The time of flight is " T ".

Solution : We can see here that time of flight is already given. We can find range considering projectile motion in the coordinates of horizontal and vertical axes. The range of the projectile " R " is obtained by using

trigonometric ratio in triangle OAB. The range is related to horizontal base “OB” as :

Projectile motion on an incline



The time of flight is “T”.

$$\cos 30^0 = \frac{OB}{OA}$$

$$R = OA = \frac{OB}{\cos 30^0} = OB \sec 30^0$$

Now, we can find OB by considering motion in horizontal direction :

$$\Rightarrow R = OB = u_x T = u \cos 60^0 T = \frac{uT}{2}$$

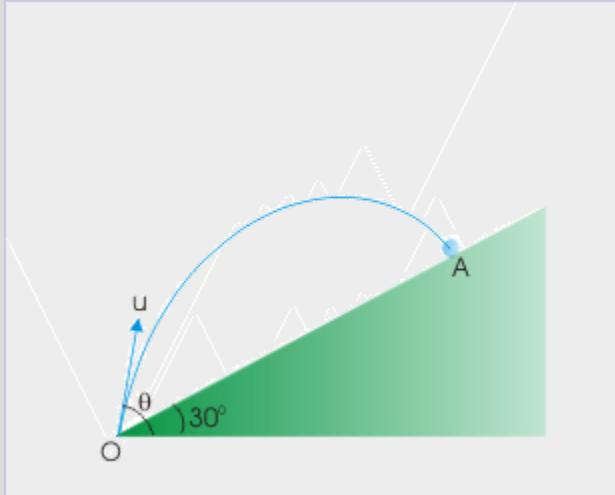
Thus, the range of the projectile, OA, is :

$$\Rightarrow R = OA = \frac{uT \sec 30^0}{2} = \frac{uT}{\sqrt{3}}$$

Angle of projection

Example:

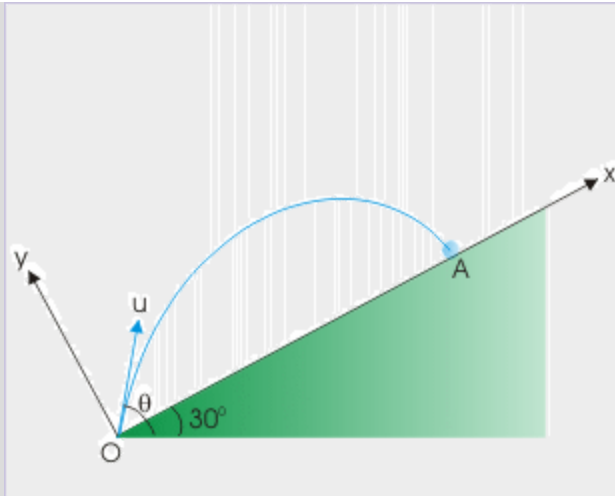
Problem : A particle is projected from the foot of an incline of angle “ 30° ” at a certain velocity so that it strikes the incline normally. Find the angle of projection (θ) as measured from the horizontal.

Projectile motion on an incline

The projectile hits the incline normally.

Solution : Here, projectile hits the incline normally. It means that component of velocity along the incline is zero. We should remember that the motion along the incline is not uniform motion, but a decelerated motion. In order to take advantage of the fact that final component velocity along incline is zero, we consider motion in a coordinate system along the incline and along a direction perpendicular to it.

Projectile motion on an incline



The projectile hits the incline normally.

$$v_x = u_x + a_x T$$

$$\Rightarrow 0 = u \cos(\theta - 30^\circ) - g \sin 30^\circ T$$

$$T = \frac{u \cos(\theta - 30^\circ)}{g \sin 30^\circ}$$

Now, time of flight is also given by the formulae :

$$\Rightarrow T = \frac{2u \sin(\theta - 30^\circ)}{g \cos 30^\circ}$$

Equating two expressions for time of flight, we have :

$$\Rightarrow \frac{2u \sin(\theta - 30^\circ)}{g \cos 30^\circ} = \frac{u \cos(\theta - 30^\circ)}{g \sin 30^\circ}$$

$$\Rightarrow \tan(\theta - 30^\circ) = \frac{1}{2 \tan 30^\circ} = \frac{\sqrt{3}}{2}$$

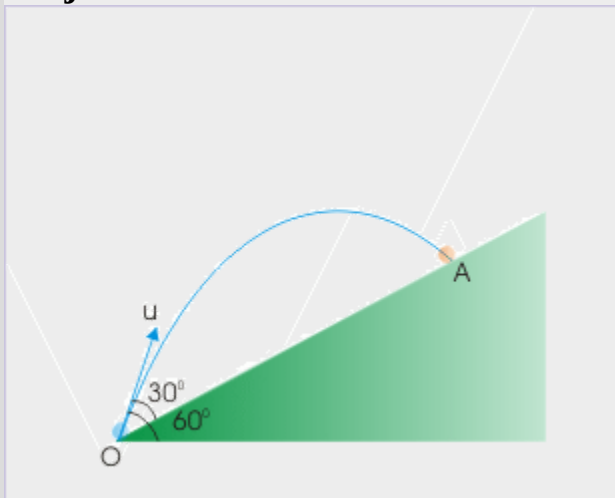
$$\Rightarrow \theta = 30^\circ + \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Final speed of the projectile

Example:

Problem : A ball is projected on an incline of 30° from its base with a speed 20 m/s, making an angle 60° from the horizontal. Find the speed with which the ball hits the incline.

Projectile motion on an incline



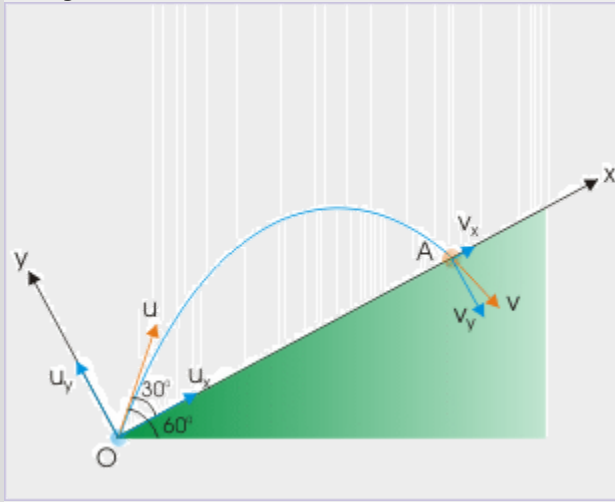
The projectile is projected from base of the incline.

Solution : We analyze this problem in the coordinates along the incline (x-axis) and in the direction perpendicular to the incline (y-axis). In order to find the speed at the end of flight, we need to find the component velocities in “x” and “y” directions.

The velocity in y-direction can be determined making use of the fact that a ball under constant acceleration like gravity returns to the ground with the

same speed, but in opposite direction. The component of velocity in y-direction at the end of the journey, therefore, is :

Projectile motion on an incline



The projectile is projected from base of the incline.

$$v_y = -u_y = -20 \sin 30^\circ = -20 \times \frac{1}{2} = -10 \text{ m/s}$$

Now, we should attempt to find the component of velocity in x-direction. We should, however, recall that motion in x-direction is not a uniform motion, but has deceleration of “ $-g \sin \alpha$ ”. Using equation of motion in x-direction, we have :

$$v_x = u_x + a_x T$$

Putting values,

$$v_x = u \cos 30^\circ - g \sin 30^\circ T = 20 \times \frac{\sqrt{3}}{2} - 10 \times \frac{1}{2} T = 10\sqrt{3} - 5T$$

Clearly, we need to know time of flight to know the component of velocity in x-direction. The time of flight is given by :

$$T = \frac{2u_y}{a_y} = \frac{2u \sin 30^0}{g \cos 30^0} = \frac{2 \times 20}{\sqrt{3} \times 10} = \frac{4}{\sqrt{3}}$$

Hence, component of velocity in x-direction is :

$$\Rightarrow vx = 10\sqrt{3} - 5T = 10\sqrt{3} - 5 \times \frac{4}{\sqrt{3}}$$

$$vx = \frac{30 - 20}{\sqrt{3}} = \frac{10}{\sqrt{3}}$$

The speed of the projectile is equal to resultant of components :

$$\Rightarrow v = \sqrt{\left\{ (-10)^2 + \left(\frac{10}{\sqrt{3}} \right)^2 \right\}}$$

$$v = \sqrt{\left(\frac{400}{3} \right)} = \frac{20}{\sqrt{3}} \text{ m/s}$$

Elastic collision with the incline

Example:

Problem : A ball falls through a height “H” and impacts an incline elastically. Find the time of flight between first and second impact of the projectile on the incline.

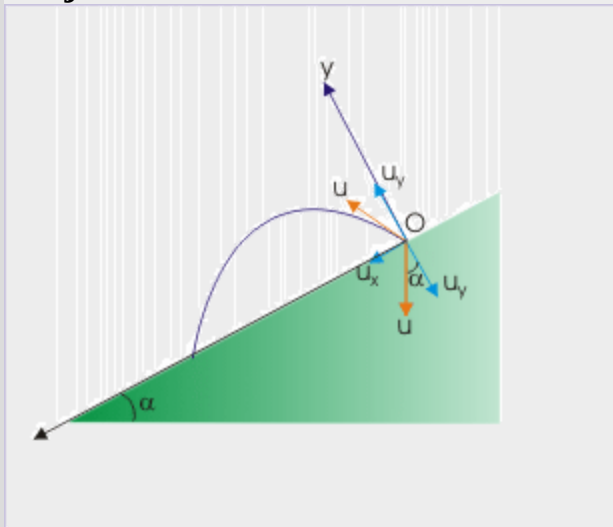
Solution : The ball falls vertically through a distance "H". We can get the value of initial speed by considering free fall of the ball before impacting incline.

$$\Rightarrow 0 = u^2 - 2gH$$

$$\Rightarrow u = \sqrt{(2gH)}$$

In order to answer this question, we need to identify the velocity with which projectile rebounds. Since impact is considered elastic, the projectile is rebounded without any loss of speed. The projectile is rebounded such that angle of incidence i.e. the angle with the normal is equal to angle of reflection.

Projectile motion on an incline



The ball impacts incline elastically.

The components of velocity along the incline and perpendicular to it are shown in the figure. The motion of ball, thereafter, is same as that of a projectile over an incline. Here, we shall analyze motion in y-direction (normal to the incline) to find the time of flight. We note that the net displacement between two strikes is zero in y – direction.

Applying equation of motion

$$y = u_y T + \frac{1}{2} a_y T^2$$

$$\Rightarrow 0 = u \cos \alpha T - \frac{1}{2} g \cos \alpha T^2$$

$$T = 0$$

Or

$$T = \frac{2u \cos \alpha}{g \cos \alpha} = \frac{2u}{g}$$

Putting value of initial speed in the equation of time of flight, we have :

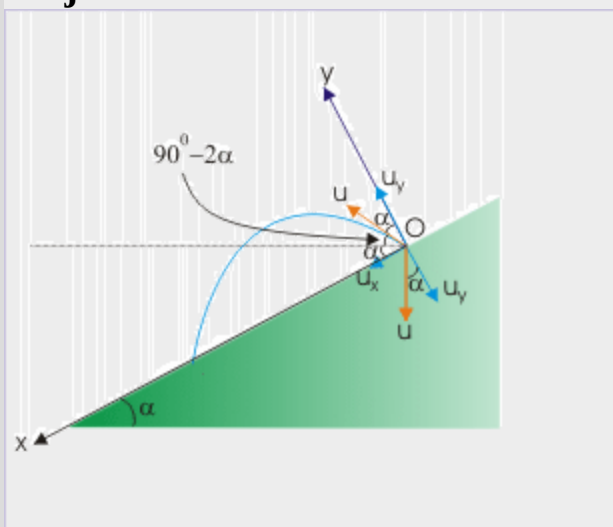
$$\Rightarrow T = \sqrt{\left(\frac{8H}{g}\right)}$$

It should be noted here that we can find the time of flight also by using standard formula of time of flight for projectile motion down the incline. The time of flight for projection down an incline is given as :

$$T = \frac{2u \sin(\theta + \alpha)}{g \cos \alpha}$$

We need to be careful while appropriating angles in the above expression. It may be recalled that all angles are measured from the horizontal. We redraw the figure to denote the value of angle of projection “ θ ” from the horizon.

Projectile motion on an incline



The ball impacts incline

elastically.

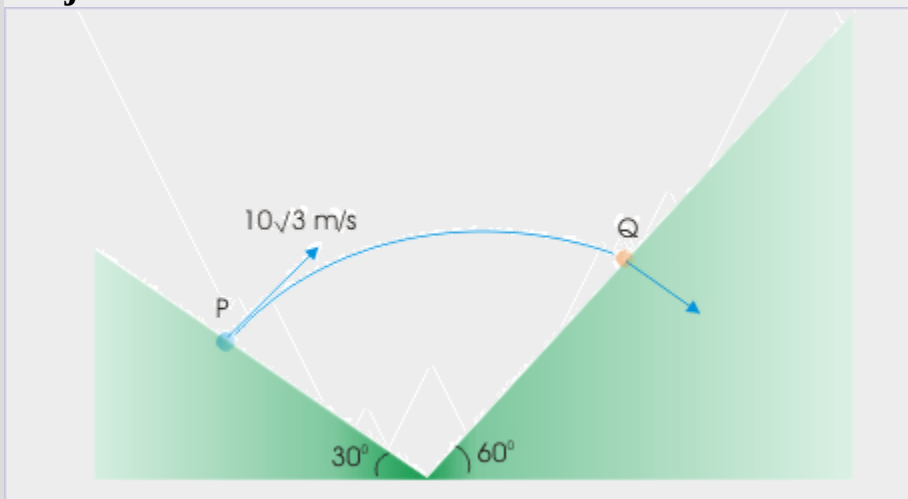
$$\Rightarrow T = \frac{2u \sin(90^\circ - 2\alpha + \alpha)}{g \cos \alpha} = \frac{2u \sin(90^\circ - \alpha)}{g \cos \alpha}$$
$$\Rightarrow T = \frac{2u \cos \alpha}{g \cos \alpha} = \frac{2u}{g} = \sqrt{\left(\frac{8H}{g}\right)}$$

Projectile motion on two inclines

Example:

Problem : Two incline plane of angles 30° and 60° are placed touching each other at the base as shown in the figure. A projectile is projected at right angle with a speed of $10\sqrt{3}$ m/s from point “P” and hits the other incline at point “Q” normally. Find the linear distance between PQ.

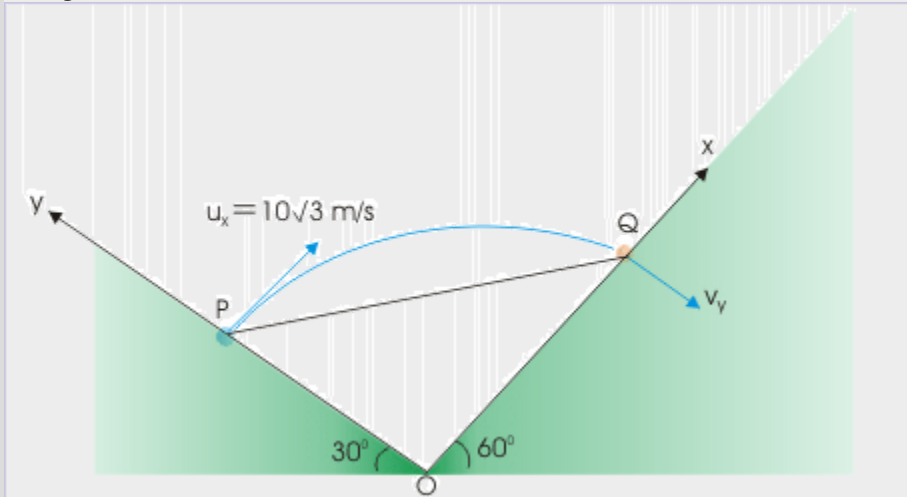
Projectile motion on two inclines



A projectile projected at right angle from an incline hits another incline at right angle.

Solution : We notice here OPQ forms a right angle triangle at “O”. The linear distance, “PQ” is related as :

Projectile motion on two inclines



A projectile projected at right angle from an incline hits another incline at right angle.

$$PQ^2 = OP^2 + OQ^2$$

In order to find “PQ”, we need to know “OP” and “OQ”. We can find “OP”, considering motion in y-direction.

$$y = OP = u_y T + \frac{1}{2} a_y T^2 = 0 + \frac{1}{2} a_y T^2 = \frac{1}{2} a_y T^2$$

Similarly,

$$x = OQ = u_x T + \frac{1}{2} a_x T^2 = 10\sqrt{3}T + \frac{1}{2} a_x T^2$$

In order to evaluate these two relations, we need to find components of accelerations and time of flight.

Considering first incline, we have :

$$a_x = -g \cos 30^\circ = -10 \times \frac{\sqrt{3}}{2} = -5\sqrt{3} \text{ m/s}^2$$

$$a_y = -g \sin 30^\circ = -10 \times \frac{1}{2} = -5 \text{ m/s}^2$$

In order to find the time of flight, we can further use the fact that the component of velocity in x-direction i.e. along the second incline is zero. This, in turn, suggests that we can analyze motion in x-direction to obtain time of flight.

In x-direction,

$$v_x = u_x + a_x T$$

$$\Rightarrow 0 = u_x + a_x T$$

$$\Rightarrow T = -\frac{u_x}{a_x}$$

Putting values in the equation and solving, we have :

$$\Rightarrow T = -\frac{10\sqrt{3}}{-5\sqrt{3}} = 2s$$

Now, we can evaluate “x” and “y” displacements as :

$$\Rightarrow y = OP = \frac{1}{2} a_y T^2 = -\frac{1}{2} \times 5 \times 2^2 = -10 \text{ m}$$

$$\Rightarrow x = OQ = 10\sqrt{3} \times 2 + \frac{1}{2} (-5\sqrt{3}) \times 2^2$$

$$x = OQ = 10\sqrt{3}$$

Considering positive values, the linear distance, “PQ” is given as :

$$PQ = \sqrt{(OP^2 + OQ^2)} = \sqrt{\left\{ (10)^2 + (10\sqrt{3})^2 \right\}} = 20m$$

Relative motion of projectiles

In this module, we shall apply the concept of relative velocity and relative acceleration to the projectile motion. The description here is essentially same as the analysis of relative motion in two dimensions, which was described earlier in the course except that there is emphasis on projectile motion. Besides, we shall extend the concept of relative motion to analyze the possibility of collision between projectiles.

We shall maintain the convention of subscript designation for relative quantities for the sake of continuity. The first letter of the subscript determines the “object”, whereas the second letter determines the “other object” with respect to which measurement is carried out. Some expansion of meaning is given here to quickly recapitulate uses of subscripted terms :

$\mathbf{v}_{AB}$: Relative velocity of object “A” with respect to object “B”

v_{ABx} : Component of relative velocity of object “A” with respect to object “B” in x-direction

For two dimensional case, the relative velocity is denoted with bold type vector symbol. We shall , however, favor use of component scalar symbol with appropriate sign to represent velocity vector in two dimensions like in the component direction along the axes of the coordinate system. The generic expression for two dimensional relative velocity are :

In vector notation :

Equation:

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

In component scalar form :

Equation:

$$v_{ABx} = v_{Ax} - v_{Bx}$$

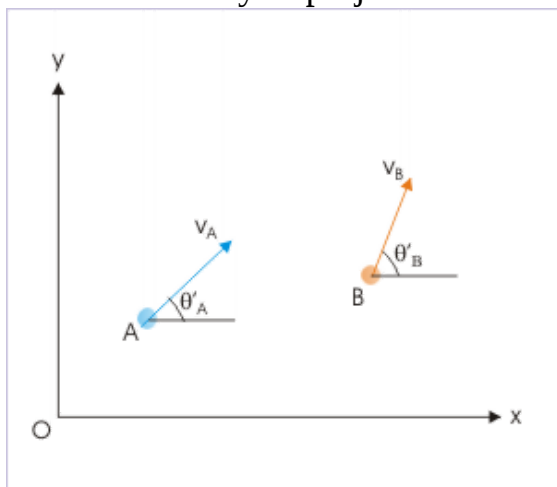
Equation:

$$v_{ABy} = v_{Ay} - v_{By}$$

Relative velocity of projectiles

The relative velocity of projectiles can be found out, if we have the expressions of velocities of the two projectiles at a given time. Let “ $\mathbf{v}_A$ ” and “ $\mathbf{v}_B$ ” denote velocities of two projectiles respectively at a given instant “t”. Then :

Relative velocity of projectiles



Velocities of projectiles.

$$\mathbf{v}_A = v_{Ax}\mathbf{i} + v_{Ay}\mathbf{j}$$

$$\mathbf{v}_B = v_{Bx}\mathbf{i} + v_{By}\mathbf{j}$$

Hence, relative velocity of projectile “A” with respect to projectile “B” is :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B = v_{Ax}\mathbf{i} + v_{Ay}\mathbf{j} - v_{Bx}\mathbf{i} - v_{By}\mathbf{j}$$

Equation:

$$\mathbf{v}_{AB} = (v_{Ax} - v_{Bx})\mathbf{i} + (v_{Ay} - v_{By})\mathbf{j}$$

We can interpret this expression of relative velocity as equivalent to consideration of relative velocity in component directions. In the nutshell, it means that we can determine relative velocity in two mutually perpendicular directions and then combine them as vector sum to obtain the resultant relative velocity.

Mathematically,

Equation:

$$\mathbf{v}_{AB} = v_{ABx}\mathbf{i} + v_{ABy}\mathbf{j}$$

where,

$$v_{ABx} = v_{Ax} - v_{Bx}$$

$$v_{ABy} = v_{Ay} - v_{By}$$

This is a significant analysis simplification as study of relative motion in one dimension can be done with scalar representation with appropriate sign.

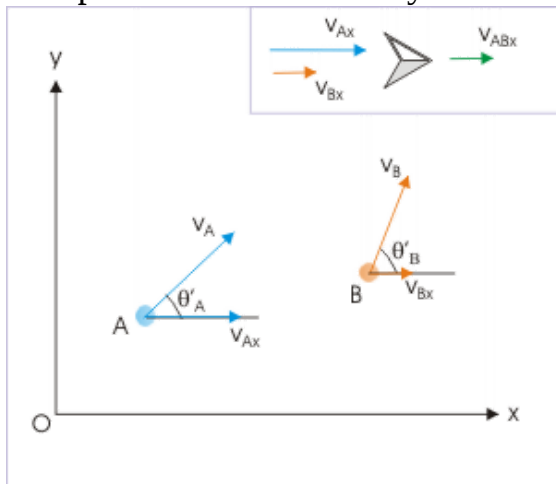
Interpretation of relative velocity of projectiles

The interpretation is best understood in terms of component relative motions. We consider motion in both horizontal and vertical directions.

Relative velocity in horizontal direction

The interpretation is best understood in terms of component relative motion. In horizontal direction, the motion is uniform for both projectiles. It follows then that relative velocity in horizontal x-direction is also a uniform velocity i.e. motion without acceleration.

Component relative velocity



Component relative velocity in
x-direction.

The relative velocity in x-direction is :

$$v_{ABx} = v_{Ax} - v_{Bx}$$

As horizontal component of velocity of projectile does not change with time, we can re-write the equation of component relative velocity as :

Equation:

$$v_{ABx} = u_{Ax} - u_{Bx}$$

Significantly, relative velocity in x-direction is determined by the initial velocities or initial conditions of the projection of two projectiles. It is expected also as components of velocities in x-direction do not change with time. The initial velocity, on the other hand, is fixed for a given projectile motion. As such, horizontal component of relative velocity is constant. A plot of relative velocity in x - direction .vs. time will be a straight line parallel to time axis.

The separation between two objects in x-direction at a given time "t" depends on two factors : (i) the initial separation of two objects in x-direction and (ii) relative velocity in x - direction. The separation in x - direction is given as :

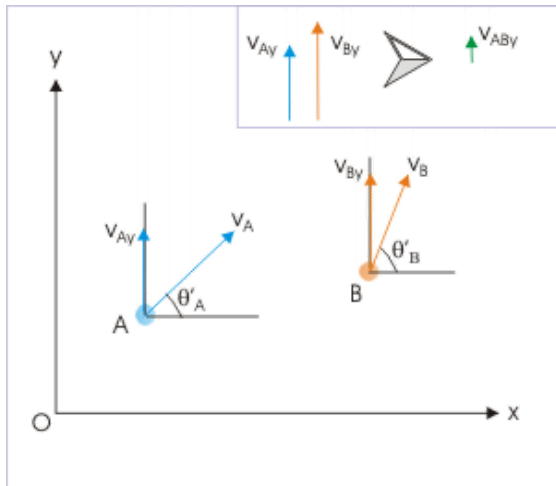
$$\Delta x = x_A - x_B = x_0 + v_{ABx}t$$

where x_0 is the initial separation between two projectiles in x-direction. Clearly, the separation in horizontal direction .vs. time plot would be a straight line with a constant slope. In physical terms, the separation between two projectiles in horizontal direction keeps increasing at a constant rate, which is equal to the magnitude of the component of relative velocity in that direction.

Relative velocity in vertical direction

The motion in the vertical direction, however, is subject to acceleration due to gravity, which always acts in vertically downward direction. The relative velocity in y-direction is :

Component relative velocity



Component relative velocity in y-direction.

$$v_{ABy} = v_{Ay} - v_{By}$$

As vertical component of motion is not a uniform motion, we can use equation of motion to determine velocity at a given time “t” as,

$$v_{Ay} = u_{Ay} - gt$$

$$v_{By} = u_{By} - gt$$

Putting in the expression of relative velocity in y-direction, we have :

$$v_{ABy} = v_{Ay} - v_{By} = u_{Ay} - gt - u_{By} + gt$$

Equation:

$$\Rightarrow v_{ABy} = u_{Ay} - u_{By}$$

The important aspect of the relative velocity in vertical y-direction is that acceleration due to gravity has not made any difference. The component relative velocity in y-direction is equal to simple difference of components of initial velocities of two projectiles in vertical direction. It is clearly due to the fact the acceleration of two projectiles in y-direction are same i.e. acceleration due to gravity and hence relative acceleration between two projectiles in vertical

direction is zero. It means that the nature of relative velocity in vertical direction is same as that in the horizontal direction. A plot of relative velocity in y -direction .vs. time will be a straight line parallel to time axis.

The separation between two objects in y-direction at a given time "t" depends on two factors : (i) the initial separation of two objects in y-direction and (ii) relative velocity in y - direction. The separation in y - direction is given as :

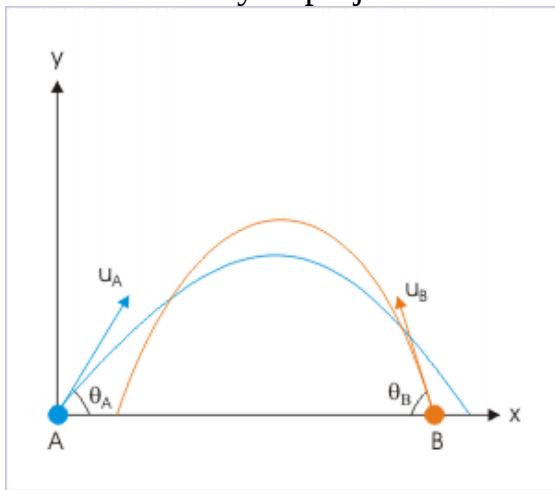
$$\Delta y = y_A - y_B = y_0 + v_{ABy}t$$

where y_0 is the initial separation between two projectiles in y-direction. Note that acceleration term has not appeared in the expression of relative velocity, because they cancel out. Clearly, the separation in vertical direction .vs. time plot would be a straight line with a constant slope. In physical terms, the separation between two projectiles in vertical direction keeps increasing at a constant rate, which is equal to the magnitude of the component of relative velocity in that direction.

Resultant relative motion

The component relative velocities in two mutually perpendicular directions have been derived in the previous section as :

Relative velocity of projectiles



Relative velocity of projectiles depends on initial velocities of projectiles.

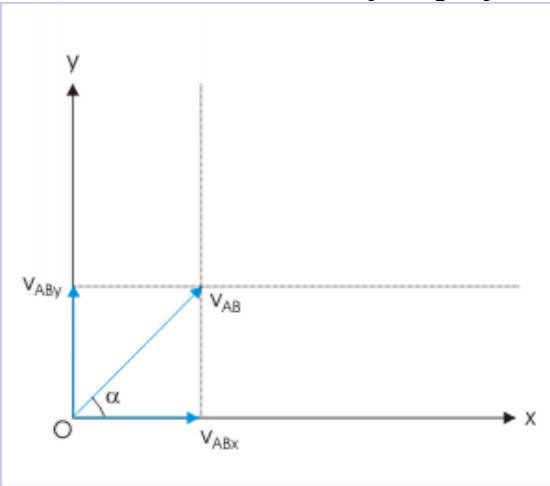
$$v_{ABx} = u_{Ax} - u_{Bx}$$

$$v_{ABy} = u_{Ay} - u_{By}$$

These equations are very important results. It means that relative velocity between projectiles is exclusively determined by initial velocities of the two projectiles i.e. by the initial conditions of the two projectiles as shown in the figure below. The component relative velocities do not depend on the subsequent motion i.e. velocities. The resultant relative velocity is vector sum of component relative velocities :

$$\mathbf{v}_{AB} = v_{ABx}\mathbf{i} + v_{ABy}\mathbf{j}$$

Resultant relative velocity of projectiles



Resultant relative velocity of
projectiles is constant.

Since the component relative velocities do not depend on the subsequent motion, the resultant relative velocity also does not depend on the subsequent motion. The magnitude of resultant relative velocity is given by :

Equation:

$$\Rightarrow v_{AB} = \sqrt{v_{ABx}^2 + v_{ABy}^2}$$

The slope of the relative velocity of “A” with respect to “B” from x-direction is given as :

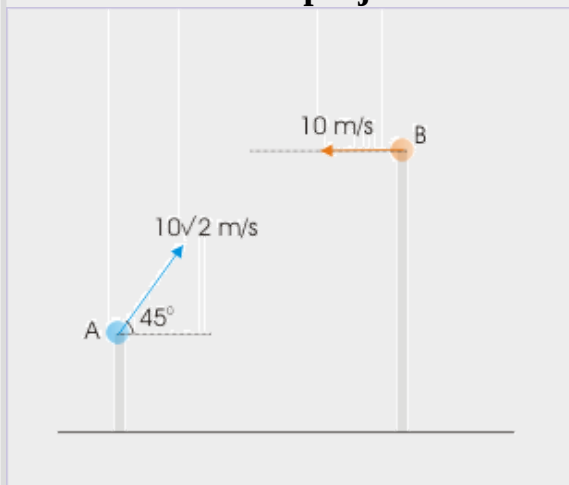
Equation:

$$\Rightarrow \tan \alpha = \frac{v_{ABy}}{v_{ABx}}$$

Example:

Problem : Two projectiles are projected simultaneously from two towers as shown in the figure. Find the magnitude of relative velocity, v_{AB} , and the angle that relative velocity makes with horizontal direction

Relative motion of projectiles



Relative motion of projectiles

Solution : We shall first calculate relative velocity in horizontal and vertical directions and then combine them to find the resultant relative velocity. Let "x" and "y" axes be in horizontal and perpendicular directions. In x-direction,

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\sqrt{2} \cos 45^\circ - (-10) = 10\sqrt{2} \times \frac{1}{\sqrt{2}} + 10 = 20 \text{ m/s}$$

In y-direction,

$$v_{AB_y} = u_{A_y} - u_{B_y} = 10\sqrt{2} \cos 45^\circ - 0 = 10\sqrt{2} \times \frac{1}{\sqrt{2}} = 10 \text{ m/s}$$

The magnitude of relative velocity is :

$$\Rightarrow v_{AB} = \sqrt{(v_{AB_x}^2 + v_{AB_y}^2)} = \sqrt{(20^2 + 10^2)} = 10\sqrt{5} \text{ m/s}$$

The angle that relative velocity makes with horizontal is :

$$\Rightarrow \tan \theta = \frac{v_{AB_y}}{v_{AB_x}} = \frac{10}{20} = \frac{1}{2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

Physical interpretation of relative velocity of projectiles

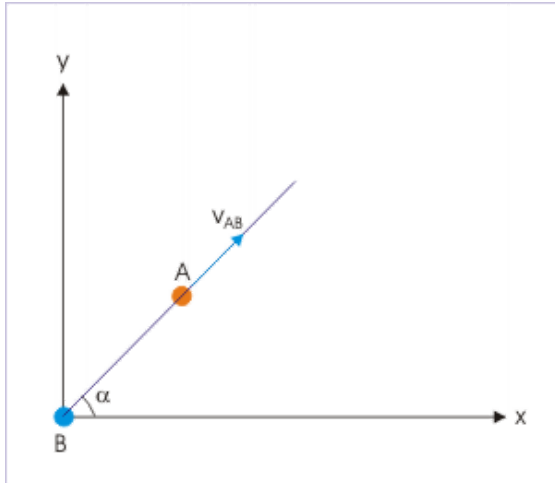
The physical interpretation of the results obtained in the previous section will help us to understand relative motion between two projectiles. We recall that relative velocity can be interpreted by assuming that the reference object is stationary. Consider the expression, for example,

$$v_{AB_x} = u_{A_x} - u_{B_x}$$

What it means that relative velocity “ v_{AB_x} ” of object “A” with respect to object “B” in x-direction is the velocity of the object “A” in x-direction as seen by the stationary object “B”. This interpretation helps us in understanding the nature of relative velocity of projectiles.

Extending the reasoning, we can say that object “A” is moving with uniform motion in “x” and “y” directions as seen by the stationary object “B”. The resultant motion of “A”, therefore, is along a straight line with a constant slope. This result may be a bit surprising as we might have expected that two projectiles see (if they could) each other moving along some curve - not a straight line.

Relative velocity of projectiles



Relative velocity of projectiles
is constant.

Special cases

There are two interesting cases. What if horizontal component of velocities of the two projectiles are same? In this case, relative velocity of projectiles in horizontal direction is zero. Also, it is imperative that there is no change in the initial separation between two projectiles in x-direction. Mathematically,

$$v_{ABx} = u_{Ax} - u_{Bx} = 0$$

There is no relative motion between two projectiles in horizontal direction. They may, however, move with respect to each other in y-direction. The relative velocity in y-direction is given by :

$$v_{ABy} = u_{Ay} - u_{By}$$

Since relative velocity in x-direction is zero, the relative velocity in y-direction is also the net relative velocity between two projectiles.

In the second case, components of velocities in y-direction are equal. In this case, there is no relative velocity in y-direction. The projectiles may, however, have relative velocity in x-direction. As such the relative velocity in x-direction is also the net relative velocity between two projectiles.

Exercises

Exercise:

Problem:

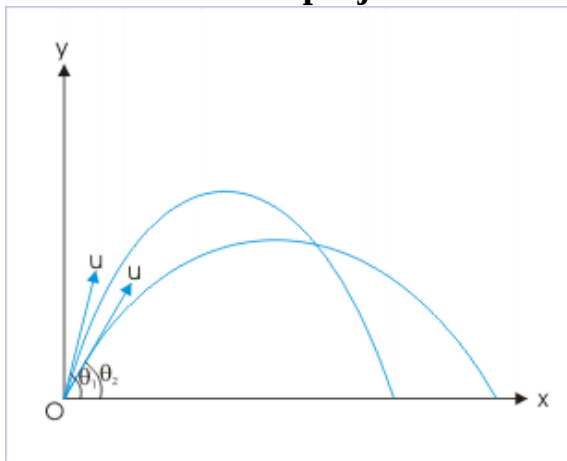
Two projectiles are projected simultaneously at same speeds, but with different angles of projections from a common point in a given vertical plane. The path of one projectile, as viewed from other projectile is :

- (a) a straight line parallel to horizontal
- (b) a straight line parallel to vertical
- (c) a circle
- (d) None of above

Solution:

The component relative velocities in horizontal and vertical directions are, defined in terms of initial velocities, which are constant for the given pair of projectiles. Therefore, the relative velocities of two projectiles in horizontal and vertical directions are constants. Let " u_A " and " u_B " be the initial velocities of two projectiles, then component relative velocities in "x" and "y" directions are :

Relative motion of projectiles



Relative motion of projectiles

$$v_{ABx} = u_{Ax} - u_{Bx}$$

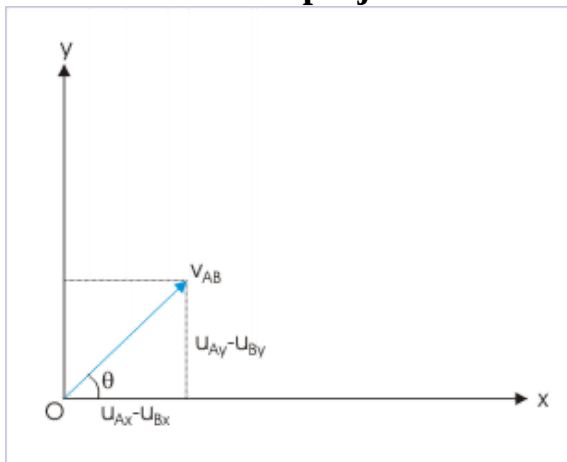
$$v_{ABy} = u_{Ay} - u_{By}$$

The resultant relative velocity is :

$$\mathbf{u}_{AB} = (u_{Ax} - u_{Bx})\mathbf{i} + (u_{Ay} - u_{By})\mathbf{j}$$

As given in the question, the initial speeds of the projectiles are same, but angles of projections are different. Since sine and cosine of two different angles are different, it follows that component velocities of two projectiles are different in either direction. This is ensured as speeds of two projectiles are same. It implies that components (horizontal or vertical) of the relative velocity are non-zero and finite constant. The resultant relative velocity is, thus, constant, making an angle " θ " with horizontal (x-axis) such that :

Relative motion of projectiles



Relative motion of projectiles

$$\tan \theta = \frac{(u_{Ay} - u_{By})}{(u_{Ax} - u_{Bx})}$$

Thus, one projectile (B) sees other projectile (A) moving in a straight line with constant velocity, which makes a constant angle with the horizontal. Hence, option (d) is correct.

Exercise:

Problem:

Two projectiles are projected simultaneously at different velocities from a common point in a given vertical plane. If the components of initial velocities of two projectiles in horizontal direction are equal, then the path of one projectile as viewed from other projectile is :

- (a) a straight line parallel to horizontal
 - (b) a straight line parallel to vertical
 - (c) a parabola
 - (d) None of above
-

Solution:

The component relative velocities in horizontal and vertical directions are, defined in terms of initial velocities, which are constant for the given projectiles. Let " u_A " and " u_B " be the initial velocities of two projectiles, then component relative velocities in "x" and "y" directions are :

$$v_{ABx} = u_{Ax} - u_{Bx}$$

$$v_{AB y} = u_{Ay} - u_{By}$$

According to the question, components of initial velocities of two projectiles in horizontal direction are equal :

$$u_{Ax} - u_{Bx} = 0$$

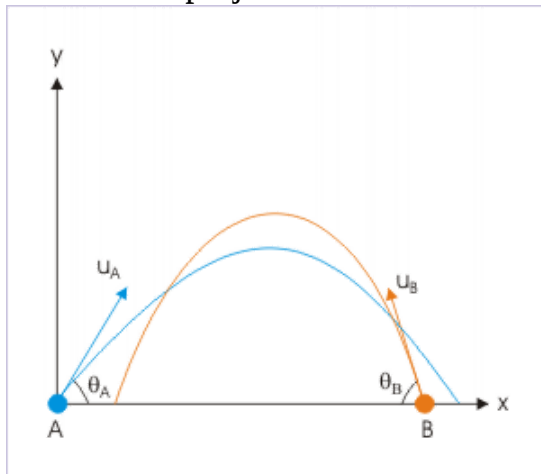
It is given that velocities of projections are different. As horizontal components of velocities in horizontal directions are equal, the components of velocities in vertical directions are different. As such, above expression evaluates to a constant vector. Thus, one projectile sees other projectile moving in a straight line parallel to vertical (i.e. direction of unit vector $\mathbf{j}$).

Hence, option (b) is correct.

Collision of projectiles

Collision between two projectiles is a rare eventuality. The precise requirement of collision is that the two projectiles are at the same position (having same x and y coordinates) at a given time instant. This is the condition for two point objects to collide or for a collision to occur. This is possible only rarely. Consider projections of two projectiles as shown in the figure.

Collision of projectiles



The position of plane with respect to the earth keeps changing with time.

We see that paths of two projectiles cross twice. Thus, there are two positions (or corresponding time instants) when it is possible that projectiles occupy same position and thus may collide with each other. But, there are infinite possibilities that they would not. Projectile “A” may rise to the required height but “B” may be at lower or higher position. Similarly, even if two projectiles are at same height they may be horizontally separated. To top these, the projectiles may have time difference at the start of their motion. However, if two projectiles collide then we can make lot many simplifying assumptions resulting from the requirement of collision that two projectiles are at the same position at the same time. In this module, we shall examine these simplifying aspects of projectile motion under collision.

Analysis of motion

Two projectiles need to approach towards each other for collision to take place. If we look at this requirement in component form, then projectiles should approach towards each other both vertically and horizontally. It is possible that projectiles have different projection times. We can, however, extend the analysis even when projectiles are projected at different times by accounting motion for the additional time available to one of projectiles. For the sake of simplicity however, we consider that two projectiles are initiated at the same time instant.

The most important aspect of analysis of projectile motion involving collision is that we can interpret condition of collision in terms of relative velocity. We actually use the fact that components of relative velocity in either x or y direction is uniform motion - not accelerated one. This simplifies the analysis a great deal.

Relative motion in x-direction

If “ x_0 ” is the initial separation between projectiles, then for collision they should cover this separation with x-component of relative velocity. Since two projectiles are initiated at the same time instant, the time when collision occurs is given by :

$$t = \frac{x_0}{v_{ABx}}$$

where v_{ABx} is the relative speed of approach of A with respect B. Note that time expression evaluates to same value whether we compute it with v_{ABx} or v_{BAx} . On the other hand, if there is no initial separation in x – direction, the projectiles should cover same horizontal distance for all time intervals. It is so because projectiles have to reach same horizontal i.e x-position at the point of collision in two dimensional space. This means that relative velocity of projectiles in x-direction is zero for the condition of collision. Mathematically,

$$v_{ABx} = v_{Ax} - v_{Bx} = 0$$

Relative motion in y-direction

If “ y_0 ” is the initial separation between projectiles, then for collision they should cover this separation with y-component of relative velocity. Since two projectiles are initiated at the same time instant, the time when collision occurs is given by :

$$t = \frac{y_0}{v_{ABy}}$$

where v_{ABy} is the relative speed of approach of A with respect B. Note that time expression evaluates to same value whether we compute it with v_{ABy} or v_{BAy} . On the other hand, if there is no initial separation in y – direction, the projectiles should cover same vertical distance for all time intervals. It is so because projectiles have to reach same vertical i.e y-position at the point of collision in two dimensional space. This means that relative velocity of projectiles in y-direction is zero for the condition of collision. Mathematically,

$$v_{ABy} = v_{Ay} - v_{By} = 0$$

Collision of projectiles initiated without vertical separation

In this case, there is no initial vertical separation. Such is the case, when projectiles are projected from same horizontal level. Both projectiles should rise to same height for all time. Clearly, relative velocity in vertical i.e y-direction is zero :

$$v_{ABy} = v_{Ay} - v_{By} = 0$$

On the other hand, time of collision is obtained by considering relative motion in x-direction :

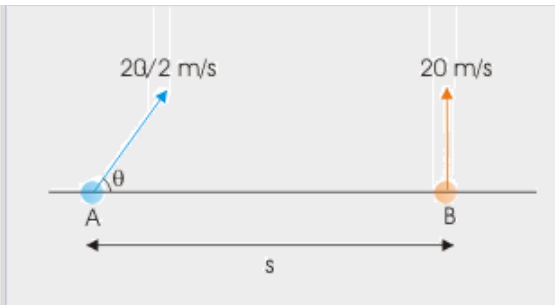
$$t = \frac{x_0}{v_{ABx}}$$

There are different cases for projection from the same horizontal level. Some important cases are : (i) one projectile is projected at certain angle to the horizontal while the other projectile is projected vertically and (ii) Both projectiles are projected at certain angles to the horizontal. Here, we shall work out examples for each of these cases.

Example:

Problem : Two projectiles “A” and “B” are projected simultaneously as shown in the figure. If they collide after 0.5 s, then determine (i) angle of projection “ θ ” and (ii) the distance “s”.

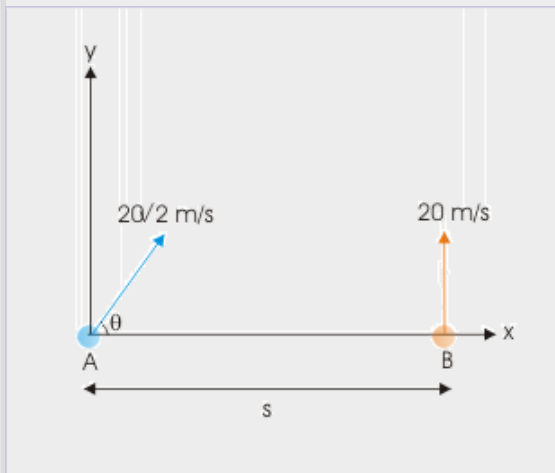
Relative motion



The projectiles collide after 0.5 s.

Solution : We see here that projectile “A” is approaching towards projectile “B” in horizontal direction. Their movement in two component directions should be synchronized so that they are at the same position at a particular given time. There is no separation in vertical direction at the start of motion. As such, relative velocity in y-direction should be zero for collision to occur.

Relative motion



The projectiles collide after 0.5 s.

$$\begin{aligned}
 v_{AB_y} &= u_{A_y} - u_{B_y} = 0 \\
 \Rightarrow 20\sqrt{2} \sin \theta &= 20 \\
 \Rightarrow \sin \theta &= \frac{1}{\sqrt{2}} = \sin 45^\circ
 \end{aligned}$$

$$\Rightarrow \theta = 45^0$$

In the x-direction, the relative velocity is :

$$v_{ABx} = u_{Ax} - u_{Bx} = 20\sqrt{2} \cos 45^0 - 0 = 20 \text{ m/s}$$

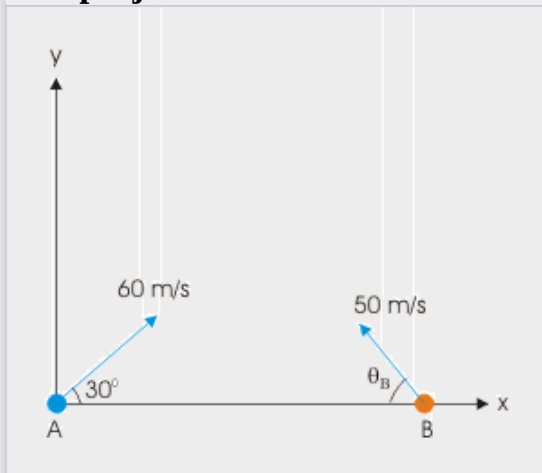
The distance “s” covered with the relative velocity in 0.5 second is :

$$s = v_{ABx} X t = 20 X 0.5 = 10m$$

Example:

Problem : Two projectiles “A” and “B” are thrown simultaneously in opposite directions as shown in the figure. If they happen to collide in the mid air, then find the time when collision takes place.

Two projectiles



Two projectiles are thrown towards each other.

Solution : There is no separation in vertical direction at the start of motion. As such, relative velocity in y-direction should be zero for collision to occur.

$$v_{ABy} = u_{Ay} - u_{By} = 0$$

$$\Rightarrow u_A \sin \theta_A = u_B \sin \theta_B$$

Putting values,

$$\Rightarrow \sin \theta_B = \frac{60}{50} \times \sin 30^\circ$$

$$\Rightarrow \sin \theta_B = \frac{3}{5}$$

The projectiles move towards each other with the relative velocity in horizontal direction. The relative velocity in x-direction is :

$$v_{ABx} = u_{Ax} - u_{Bx} = 60 \cos 30^\circ + 50 \cos \theta_B$$

In order to find relative velocity in x-direction, we need to know “ $\cos \theta_B$ ”. Using trigonometric relation, we have :

$$\cos \theta_B = \sqrt{(1 - \sin^2 \theta_B)} = \sqrt{\left\{1 - \left(\frac{3}{5}\right)^2\right\}} = \frac{4}{5}$$

Hence,

$$\Rightarrow v_{ABx} = 60 \times \frac{\sqrt{3}}{2} + 50 \times \frac{4}{5} = 30\sqrt{3} + 40 = 91.98 \approx 92 \text{ m/s}$$

We should now understand that projectiles move towards each other with a relative velocity of 92 m/s. We can interpret this as if projectile “B” is stationary and projectile “A” moves towards it with a velocity 92 m/s, covering the initial separation between two particles for collision to take place. The time of collision, therefore, is :

$$\Rightarrow t = 92/92 = 1 \text{ s}$$

Collision of projectiles initiated without horizontal separation

In this case, there is no initial horizontal separation. Projectiles are thrown from different levels. Both projectiles travel same horizontal distance for all time. Clearly, relative velocity in horizontal i.e x-direction is zero :

$$v_{ABx} = v_{Ax} - v_{Bx} = 0$$

On the other hand, time of collision is obtained by considering relative motion in y-direction :

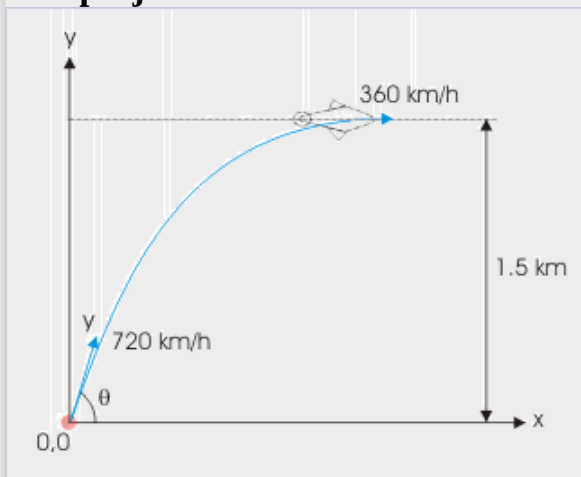
$$t = \frac{y_0}{v_{ABx}}$$

Example:

Problem : A fighter plane is flying horizontally at a speed of 360 km/hr and is exactly above an aircraft gun at a given moment. At that instant, the aircraft gun is fired to hit the plane, which is at a vertical height of 1 km. If the speed of the shell is 720 km/hr, then at what angle should the gun be aimed to hit the plane (Neglect resistance due to air)?

Solution : This is a case of collision between fighter plane and shell fired from the gun. Since plane is overhead, it is required that horizontal component of velocity of the shell be equal to that of the plane as it is flying in horizontal direction. This will ensure that shell will hit the plane whenever it rises to the height of the plane.

Two projectiles



The shell fired aircraft gun hits the fighter plane.

Now the horizontal and vertical components of shell velocity are :

$$u_x = 720 \cos \theta$$
$$\Rightarrow u_y = 720 \sin \theta$$

For collision,

$$\Rightarrow u_x = 720 \cos \theta = 360$$

$$\Rightarrow \cos \theta = \frac{1}{2} = \cos 60^\circ$$

$$\Rightarrow \theta = 60^\circ$$

However, we need to check that shell is capable to rise to the height of the plane. Here,

$$720 \text{ km/hr} = 720 \times \frac{5}{18} = 200 \text{ m/s}$$

The maximum height achieved by the shell is obtained as here :

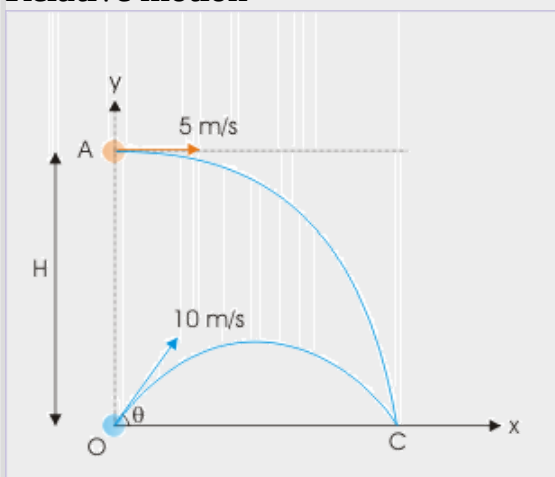
$$H = \frac{u^2 \sin^2 \theta}{2g} = \frac{200^2 \left(\frac{\sqrt{3}}{2} \right)^2}{2 \times 10} = 1500 \text{ m}$$

The shell indeed rises to the height of the plane (1000 m) and hence will hit it.

Example:

Problem : Two projectiles are projected simultaneously from a point on the ground “O” and an elevated position “A” respectively as shown in the figure. If collision occurs at the point of return of two projectiles on the horizontal surface, then find the height of “A” above the ground and the angle at which the projectile "O" at the ground should be projected.

Relative motion



The projectiles collide in the mid

The projectiles collide in the mid air.

Solution : There is no initial separation between two projectiles in x-direction. For collision to occur, the relative motion in x-direction should be zero. In other words, the component velocities in x-direction should be equal so that two projectiles cover equal horizontal distance at any given time. Hence,

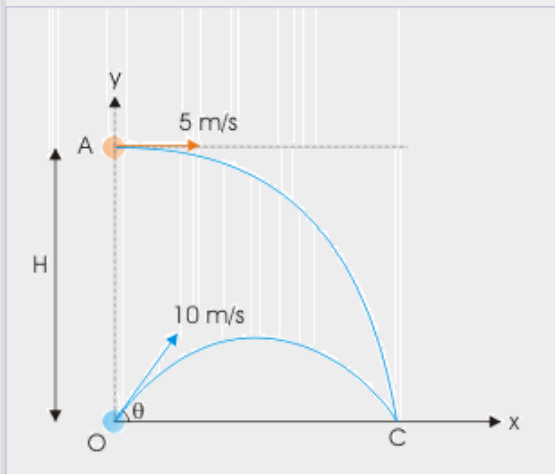
$$u_{Ox} = u_{Ax}$$

$$\Rightarrow u_O \cos \theta = u_A$$

$$\Rightarrow \cos \theta = \frac{u_A}{u_O} = \frac{5}{10} = \frac{1}{2} = \cos 60^\circ$$

$$\Rightarrow \theta = 60^\circ$$

Relative motion



The projectiles collide in the mid air.

We should ensure that collision does occur at the point of return. It means that by the time projectiles travel horizontal distances required, they should also cover vertical distances so that both projectiles are at "C" at the same time. In the nutshell, their times of flight should be equal. For projectile from "O",

$$T = \frac{2u_O \sin \theta}{g}$$

For projectile from "A",

$$T = \sqrt{\left(\frac{2H}{g}\right)}$$

For projectile from "A",

$$T = \frac{2u_O \sin \theta}{g} = \sqrt{\left(\frac{2H}{g}\right)}$$

Squaring both sides and putting values,

$$\begin{aligned}\Rightarrow H &= \frac{4u_O^2 \sin^2 \theta}{2g} \\ \Rightarrow H &= \frac{4 \times 10^2 \sin^2 60^\circ}{2 \times 10} \\ H &= 20 \left(\frac{\sqrt{3}}{2}\right)^2 = 15m\end{aligned}$$

We have deliberately worked out this problem taking advantage of the fact that projectiles are colliding at the end of their flights and hence their times of flight should be equal. We can, however, proceed to analyze in typical manner, using concept of relative velocity. The initial separation between two projectiles in the vertical direction is "H". This separation is covered with the component of relative velocity in vertical direction.

$$\Rightarrow v_{OAy} = u_{Oy} - u_{Ay} = u_O \sin 60^\circ - 0 = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \text{ m/s}$$

Now, time of flight of projectile from ground is :

$$T = \frac{2u_O \sin \theta}{g} = \frac{2 \times 10 \times \sin 60^\circ}{10} = \sqrt{3}$$

Hence, the vertical displacement of projectile from "A" before collision is :

$$\Rightarrow H = v_{OAy} \times T = 5\sqrt{3} \times \sqrt{3} = 15m$$

Collision of projectiles initiated from different horizontal and vertical levels

In this case, there are finite initial horizontal and vertical separations. For collision to occur, projectiles need to cover these separations simultaneously. Clearly, component relative velocity in x and y directions are finite. We can obtain time of collision by consideration of relative motion in either direction,

$$t = \frac{x_0}{v_{ABx}}$$

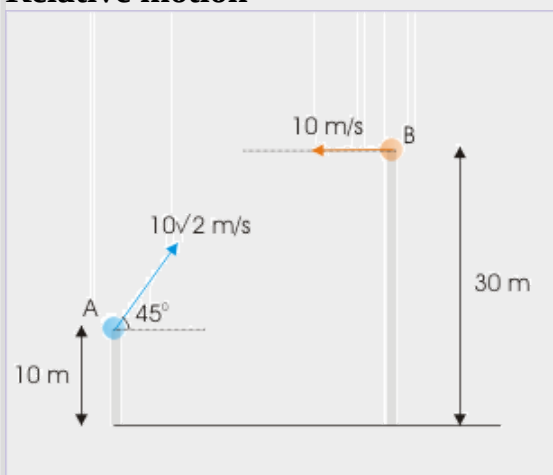
or

$$\Rightarrow t = \frac{y_0}{v_{ABx}}$$

Example:

Problem : Two projectiles are projected simultaneously from two towers as shown in the figure. If the projectiles collide in the air, then find the distance “s” between the towers.

Relative motion

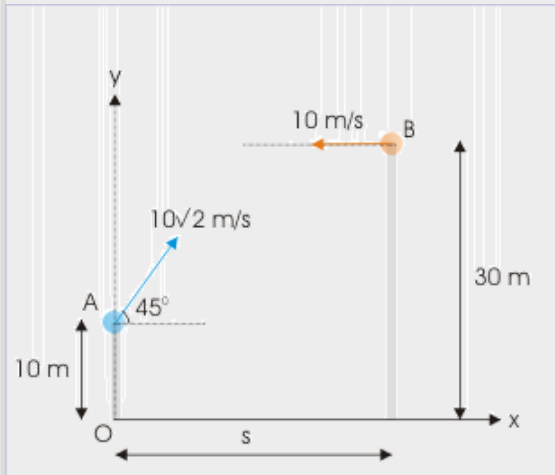


The projectiles collide in the mid air.

Solution : We see here that projectiles are approaching both horizontally and vertically. Their movement in two component directions should be synchronized so that they are at the same position at a particular given time. For collision, the necessary requirement is that relative velocity and displacement should be in the same direction.

It is given that collision does occur. It means that two projectiles should cover the displacement with relative velocity in each of the component directions.

Relative motion



The projectiles collide in the mid air.

In x-direction,

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\sqrt{2} \cos 45^\circ - (-10) = 10\sqrt{2} \times \frac{1}{\sqrt{2}} + 10 = 20 \text{ m/s}$$

If “t” is time after which collision occurs, then

$$\Rightarrow s = v_{ABx} t = 20t$$

Clearly, we need to know “t” to find “s”. The component of relative velocity in y-direction is :

$$v_{ABy} = u_{Ay} - u_{By}$$

$$\Rightarrow v_{ABy} = u \sin 45^\circ - 0 = 10\sqrt{2} \times \frac{1}{\sqrt{2}} = 10 \text{ m/s}$$

The initial vertical distance between points of projection is $30 - 10 = 20$ m. This vertical distance is covered with component of relative velocity in vertical direction. Hence, time taken to collide, “t”, is :

$$\Rightarrow t = \frac{20}{10} = 2$$

Putting this value in the earlier equation for “s”, we have :

$$\Rightarrow s = 20t = 20 \times 2 = 40 \text{ m}$$

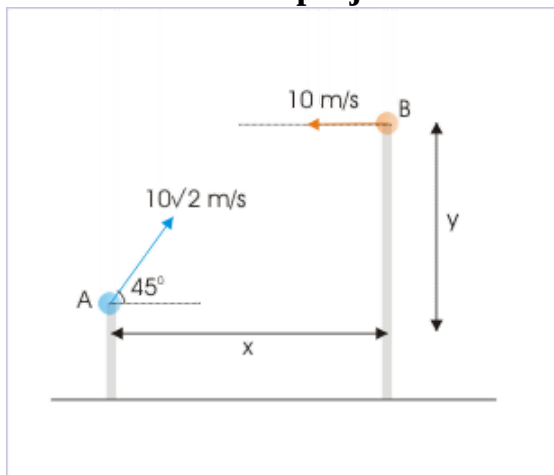
Exercises

Exercise:

Problem:

Two projectiles are projected simultaneously from two towers as shown in the figure. If collision takes place in the air, then what should be the ratio “x/y” :

Relative motion of projectiles



Relative motion of projectiles

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) $\sqrt{3}$

Solution:

According to question, collision occurs in the mid air. The necessary condition for collision is that direction of relative velocity and initial displacement between projectiles should be same.

In x-direction,

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\sqrt{2} \cos 45^\circ - (-10) = 10\sqrt{2} \times \frac{1}{\sqrt{2}} + 10 = 20 \text{ m/s}$$

In y-direction,

$$v_{ABy} = u_{Ay} - u_{By} = 10\sqrt{2} \sin 45^\circ - 0 = 10\sqrt{2} \times \frac{1}{\sqrt{2}} = 10 \text{ m/s}$$

The angle that relative velocity makes with horizontal is :

$$\Rightarrow \tan \theta = \frac{v_{ABy}}{v_{ABx}} = \frac{10}{20} = \frac{1}{2}$$

This is also the slope of displacement AB,

$$\Rightarrow \tan \theta = \frac{1}{2} = \frac{y}{x}$$

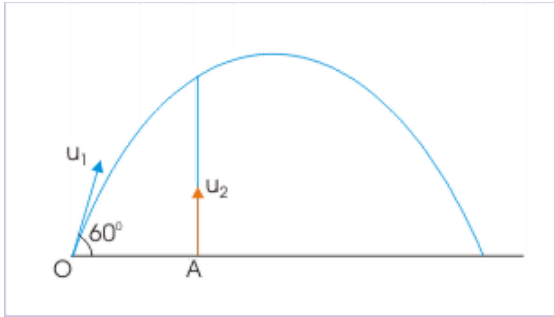
$$\Rightarrow \frac{x}{y} = 2$$

Hence, option (c) is correct.

Exercise:**Problem:**

Two balls are projected simultaneously with speeds " u_1 " and " u_2 " from two points "O" and "A" respectively as shown in the figure. If the balls collide, then find the ratio " $\frac{u_1}{u_2}$ " (consider $g = 10 \text{ m/s}^2$) :

Relative motion of projectiles



Relative motion of projectiles

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $1/\sqrt{3}$ (d) $\sqrt{3}$

Solution:

It is given that the projectiles collide. Now the initial separation in vertical direction is zero. It follows then that component of relative velocity between two projectiles in vertical direction should be zero for collision to take place. This is possible if the vertical component of velocity of projectile from "O" is equal to the speed of projectile from "A" (Remember that acceleration due to gravity applies to both projectiles and relative acceleration in vertical direction is zero).

$$u_1 \sin 60^\circ = u_2$$

$$\Rightarrow \frac{u_1}{u_2} = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}}$$

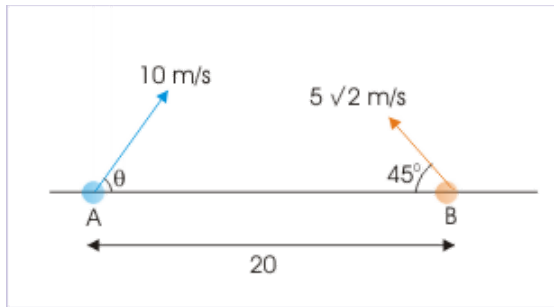
Hence, option (a) is correct.

Exercise:

Problem:

Two projectiles "A" and "B" are projected simultaneously towards each other as shown in the figure. Determine if they collide.

Relative motion

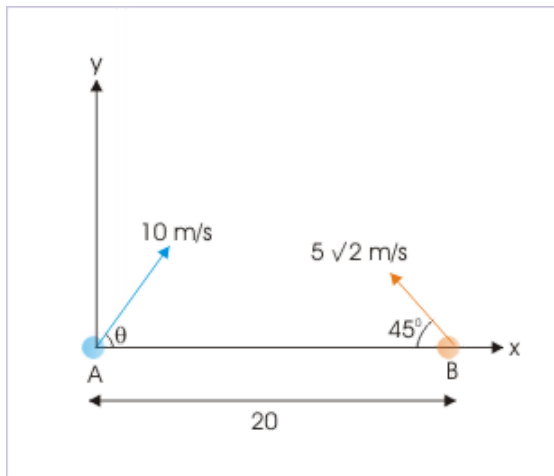


The projectiles collide in the mid air.

Solution:

There is no separation in vertical direction at the start of motion. As such, relative velocity in y-direction should be zero for collision to occur.

Relative motion



The projectiles collide in the mid air.

$$v_{AB_y} = u_{Ay} - u_{By} = 0$$

$$10 \sin \theta = 5\sqrt{2} \sin 45^\circ = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$$

$$\sin \theta = \frac{1}{2} = \sin 30^0$$

$$\theta = 30^0$$

In the x-direction, the initial separation is 15 m and the relative velocity is :

$$v_{ABx} = u_{Ax} - u_{Bx} = 10 \cos 30^0 + 5\sqrt{2} \cos 45^0$$

$$v_{ABx} = 10X\sqrt{3}\frac{1}{2} + 5\sqrt{2}X\frac{1}{\sqrt{2}} = 5\sqrt{3} + 5 = 5(\sqrt{3} + 1)$$

The time after which collision may occur is :

$$\Rightarrow t = \frac{20}{5(\sqrt{3} + 1)} = \frac{4}{(\sqrt{3} + 1)} = 1.464s$$

We should, however, check whether projectiles stay that long in the air in the first place? Now, we have seen that the projectiles should have same vertical component of velocity for collision to occur. As time of flight is a function of vertical component of velocity, the time of the flight of the projectiles are equal. For projectile "A" :

$$T = \frac{2u_A \sin 30^0}{g} = \frac{2X10X1}{10X2} = 1s$$

This means that projectiles would not stay that long in the air even though the necessary conditions (but not sufficient conditions) for the collision are fulfilled. Clearly, collision does not occur in this case. We should clearly understand that "analysis of motion with respect to collision" and "requirements of collision" are not same. In this module, we have studied the "consequence of collision" - not the "conditions for collision".

Uniform circular motion

Uniform circular motion (UCM) is the basic unit of rotational kinematics just like the uniform linear motion is the basic unit of translational kinematics.

Uniform circular motion denotes motion of a particle along a circular arc or a circle with constant speed. This statement, as a matter of fact, can be construed as the definition of uniform circular motion.

Requirement of uniform circular motion

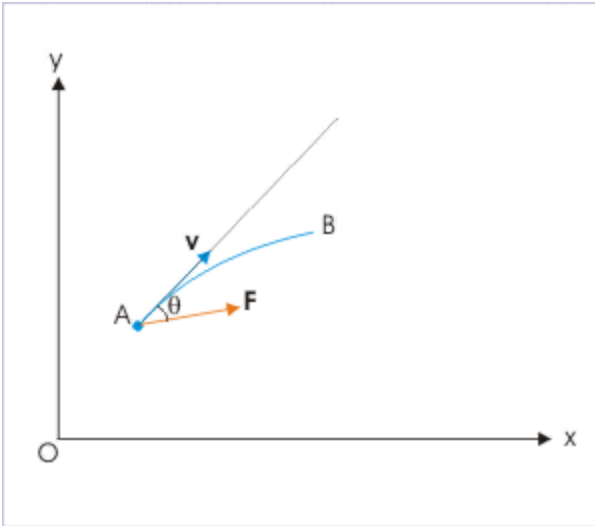
The uniform circular motion represents the basic form of rotational motion in the same manner as uniform linear motion represents the basic form of translational motion. They, however, are different with respect to the requirement of force to maintain motion.

Uniform linear motion is the reflection of the inherent natural tendency of all natural bodies. This motion by itself is the statement of Newton's first law of motion : an object keeps moving with its velocity unless there is net external force. Thus, uniform linear motion indicates "absence" of force.

On the other hand, uniform circular motion involves continuous change in the direction of velocity without any change in its magnitude (v). A change in the direction of velocity is a change in velocity ($\mathbf{v}$). It means that an uniform circular motion is associated with an acceleration and hence force. Thus, uniform circular motion indicates "presence" of force.

Let us now investigate the nature of force required to maintain uniform circular motion. We know that a force acting in the direction of motion changes only the magnitude of velocity. A change in the direction of motion, therefore, requires that velocity of the particle and force acting on it should be at an angle. However, such a force, at an angle with the direction of motion, would have a component along the direction of velocity as well and that would change the magnitude of the motion.

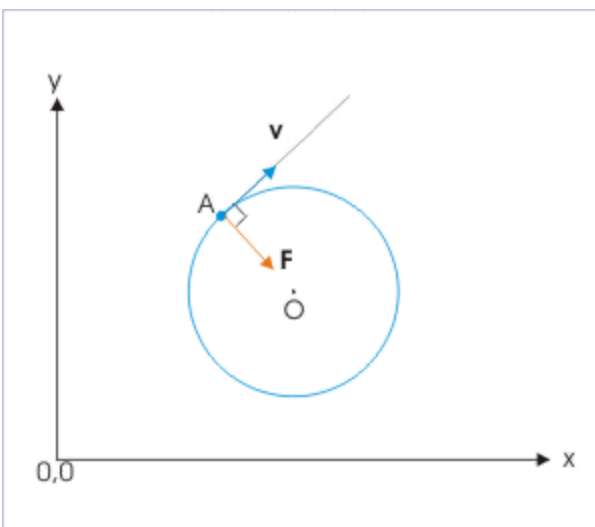
Change of direction



A change in the direction of motion requires that velocity of the particle and force should be at an angle.

In order that there is no change in the magnitude of velocity, the force should have zero component along the direction of velocity. It is possible only if the force be perpendicular to the direction of velocity such that its component in the direction of velocity is zero ($F\cos 90^\circ = 0$). Precisely, this is the requirement for a motion to be uniform circular motion.

Uniform circular motion



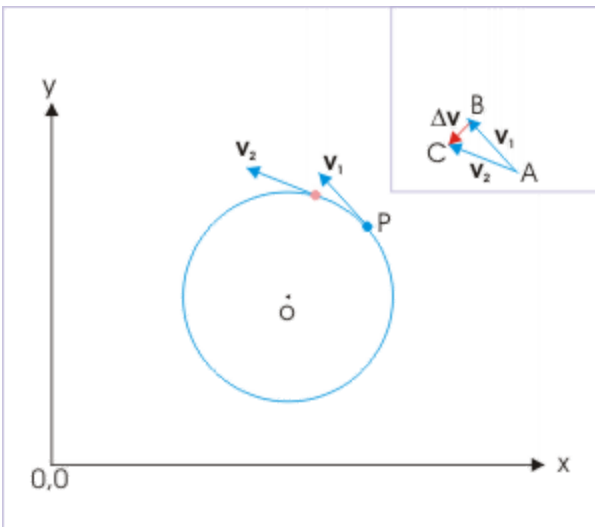
Force is perpendicular to the direction of velocity.

In plain words, uniform circular motion (UCM) needs a force, which is always perpendicular to the direction of velocity. Since the direction of velocity is continuously changing, the direction of force, being perpendicular to velocity, should also change continuously.

The direction of velocity along the circular trajectory is tangential. The perpendicular direction to the circular trajectory is, therefore, radial direction. It implies that force (and hence acceleration) in uniform direction motion is radial. For this reason, acceleration in UCM is recognized to seek center i.e. centripetal (seeking center).

This fact is also validated by the fact that the difference of velocity vectors, whose time rate gives acceleration, at two instants ($\Delta \mathbf{v}$) is radial.

Uniform circular motion



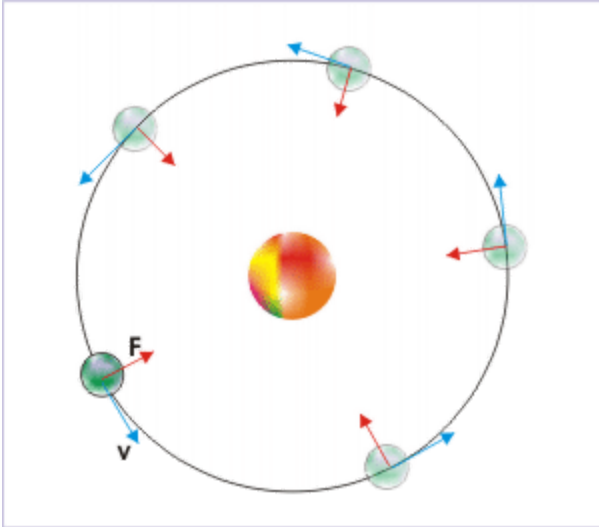
Force is centripetal.

The yet another important aspect of the UCM is that the centripetal force is radial and hence does not constitute a torque as the force is passing through the axis of rotation. A torque is force multiplied by the perpendicular distance of the line of action of the force from the axis of rotation. It must be clearly understood that the requirement of centripetal force is essentially additional or different to the force, which is required in non-uniform circular motion to accelerate the particle tangentially. The centripetal force is required to accelerate particle in radial direction and is different to one required to accelerate particle tangentially.

Irrespective of whether circular motion is uniform (constant speed) or non-uniform (varying speed), the circular motion inherently associates a radial acceleration to ensure that the direction of motion is continuously changed – at all instants. We shall learn about the magnitude of radial acceleration soon, but let us be emphatic to differentiate radial acceleration (accounting change in direction that arises from radial force) with tangential acceleration (accounting change in the speed that arises from tangential force or equivalently a torque).

Motion of natural bodies and sub-atomic particles are always under certain force system. Absence of force in the observable neighborhood is rare. Thus, uniform linear motion is rare, while uniform circular motion abounds in nature as there is availability of external force that continuously changes direction of the motion of the bodies or particles. Consider the electrostatic force between nucleus and an electron in an atom. The force keeps changing direction as electron moves. So is the case of gravitational force (indicated by red arrow in the figure), which keeps changing its direction as the planet moves around sun.

Approximated circular motion of Earth around Sun



Gravitational force is radial,
whereas velocity is tangential.

It may sometimes be perceived that a force like centripetal force should have caused the particle or body to move towards center ultimately. The important point to understand here is that a force determines initial direction of motion only when the particle is stationary. However, if the particle is already in motion, then force modifies the direction in accordance with initial inclination between velocity and force such that the resulting acceleration (change in vector velocity) is in the direction of force. We shall soon see that this is exactly the case in uniform circular motion.

Let us summarize the discussion of uniform circular motion so far :

- The trajectory of uniform circular motion is circular arc or a circle and hence planar or two dimensional.
- The speed (v) of the particle in UCM is a constant.
- The velocity ($\mathbf{v}$) of the particle in UCM is variable and is tangential or circumferential to the path of motion.
- Centripetal force ($\mathbf{F}$) is required to maintain uniform circular motion against the natural tendency of the bodies to move linearly.
- Centripetal force ($\mathbf{F}$) is variable and is radial in direction.

- Centripetal force ($\mathbf{F}$) is not a torque and does not cause acceleration in tangential direction.
- Centripetal acceleration ($\mathbf{a_R}$) is variable and radial in direction.

Analysis of uniform circular motion

It has been pointed out that any motion, that changes directions, requires more than one dimension for representation. Circular motion by the geometry of the trajectory is two dimensional motion.

In the case of circular motion, it is a matter of convenience to locate origin of the two dimensional coordinate system at the center of circle. This choice ensures the symmetry of the circular motion about the origin of the reference system.

Coordinates of the particle

The coordinates of the particle is given by the "x" and "y" coordinate pair as :

Equation:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

The angle " θ " is measured anti-clockwise from x-axis.

Position of the particle

The position vector of the position of the particle, $\mathbf{r}$, is represented in terms of unit vectors as :

Equation:

$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + y\mathbf{j} \\ \Rightarrow \mathbf{r} &= r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} \\ \Rightarrow \mathbf{r} &= r(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})\end{aligned}$$

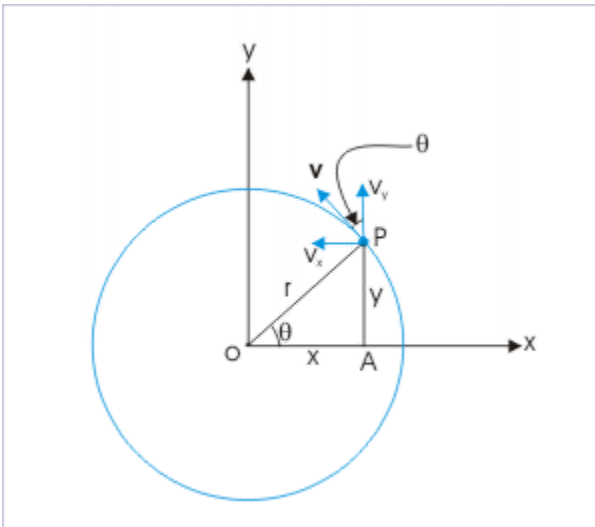
Velocity of the particle

The magnitude of velocity of the particle (v) is constant by the definition of uniform circular motion. In component form, the velocity (refer to the figure) is :

Equation:

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$$

From the geometry as shown in the figure,
Uniform circular motion



Component velocities in UCM.

Equation:

$$\begin{aligned}
v_x &= -v \sin \theta \\
v_y &= v \cos \theta \\
\Rightarrow \mathbf{v} &= -v \sin \theta \mathbf{i} + v \cos \theta \mathbf{j}
\end{aligned}$$

We may emphasize here that it is always easy to find the sign of the component of a vector like velocity. Decide the sign by the direction of component (projection) with respect to positive direction of reference axis. Note from the figure that component along x-direction is opposite to the positive reference direction and hence negative.

From the ΔOAP ,

$$\begin{aligned}
\sin \theta &= \frac{y}{r} \\
\cos \theta &= \frac{x}{r}
\end{aligned}$$

Putting these values in the expression for velocity, we have :

Equation:

$$\mathbf{v} = -\frac{vy}{r} \mathbf{i} + \frac{vx}{r} \mathbf{j}$$

Centripetal acceleration

We are now sufficiently equipped and familiarized with the nature of uniform circular motion and the associated centripetal (center seeking) acceleration. Now, we seek to determine this acceleration.

Knowing that speed, "v" and radius of circle, "r" are constants, we differentiate the expression of velocity with respect to time to obtain expression for centripetal acceleration as :

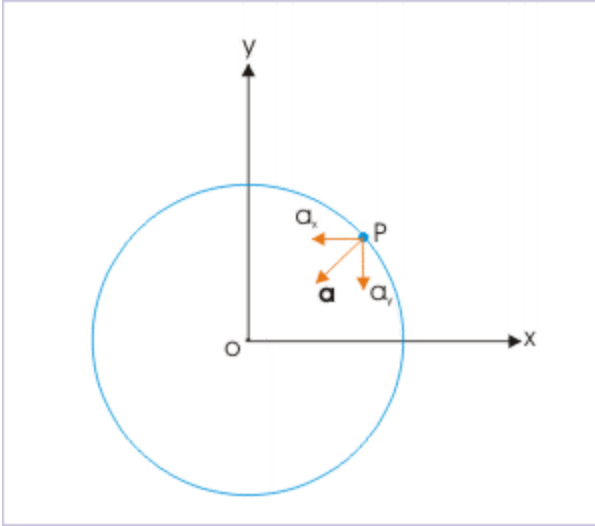
$$\begin{aligned}
\mathbf{a} &= -\frac{v}{r} \left(\frac{dy}{dt} \mathbf{i} - \frac{dx}{dt} \mathbf{j} \right) \\
\Rightarrow \mathbf{a} &= -\frac{v}{r} (v_y \mathbf{i} - v_x \mathbf{j})
\end{aligned}$$

Putting values of component velocities in terms of angle,

Equation:

$$\Rightarrow \mathbf{a} = -\frac{v}{r} (v \cos \theta \mathbf{i} + v \sin \theta \mathbf{j}) = a_x \mathbf{i} + a_y \mathbf{j}$$

Uniform circular motion



Components of radial
accelerations

where

$$a_x = -\frac{v^2}{r} \cos \theta$$

$$a_y = -\frac{v^2}{r} \sin \theta$$

It is evident from the equation of acceleration that it varies as the angle with horizontal, " θ " changes. The magnitude of acceleration is :

Equation:

$$a = |\mathbf{a}| = \sqrt{(a_x^2 + a_y^2)}$$

$$\Rightarrow a = |\mathbf{a}| = \frac{v}{r} \sqrt{\{v^2(\cos^2 \theta + \sin^2 \theta)\}}$$

$$\Rightarrow a = \frac{v^2}{r}$$

The radius of the circle is constant. The magnitude of velocity i.e. speed, "v" is constant for UCM. It, then, follows that though velocity changes with the motion (angle from reference direction), but the speed of the particle in UCM is a constant.

Example:

Problem : A cyclist negotiates the curvature of 20 m with a speed of 20 m/s. What is the magnitude of his acceleration?

Solution : The speed of the cyclist is constant. The acceleration of cyclist, therefore, is the centripetal acceleration required to move the cyclist along a circular path i.e. the acceleration resulting from the change in the direction of motion along the circular path.

Here, $v = 20 \text{ m/s}$ and $r = 20 \text{ m}$

$$\Rightarrow a = \frac{v^2}{r} = \frac{20^2}{20} = 20 \text{ m/s}^2$$

This example points to an interesting aspect of circular motion. The centripetal acceleration of the cyclist is actually two (2) times that of acceleration due to gravity ($g = 10 \text{ m/s}^2$). This fact is used to create large acceleration in small space with appropriate values of "v" and "r" as per the requirement in hand. The large acceleration so produced finds application in particle physics and for equipments designed to segregate material on the basis of difference in density. This is also used to simulate large acceleration in a centrifuge for astronauts, who experience large acceleration at the time of take off or during entry on the return.

Generation of high magnitude of acceleration during uniform circular motion also points to a potential danger to pilots, while maneuvering circular trajectory at high speed. Since a pilot is inclined with the head leaning towards the center of motion, the blood circulation in the brain is low. If his body part, including brain, is subjected to high acceleration

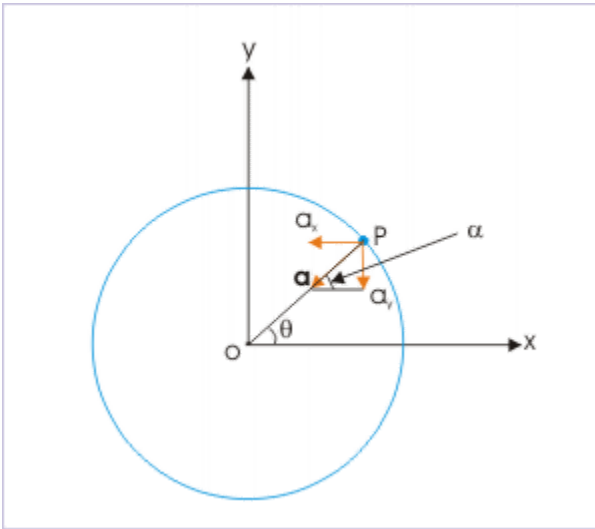
(multiple of acceleration due to gravity), then it is likely that the pilot experiences dizziness or sometimes even loses consciousness.

Circular motion has many interesting applications in real world and provides explanations for many natural events. In this module, however, we restrict ourselves till we study the dynamics of the circular motion also in subsequent modules.

Direction of centripetal acceleration

Here, we set out to evaluate the angle “ α ” as shown in the figure. Clearly, if this angle “ α ” is equal to “ θ ”, then we can conclude that acceleration is directed in radial direction.

Uniform circular motion



Radial acceleration

Now,

$$\tan \alpha = \frac{a_y}{a_x} = \frac{-\frac{v^2}{r} \sin \theta}{-\frac{v^2}{r} \cos \theta} = \tan \theta$$

$$\alpha = \theta$$

This proves that centripetal acceleration is indeed radial (i.e acting along radial direction).

Time period of uniform circular motion

A particle under UCM covers a constant distance in completing circular trajectory in one revolution, which is equal to the perimeter of the circle.

$$s = 2\pi r$$

Further the particle covers the perimeter with constant speed. It means that the particle travels the circular trajectory in a constant time given by its time period as :

Equation:

$$T = \frac{2\pi r}{v}$$

Exercise:

Problem:

An astronaut, executing uniform circular motion in a centrifuge of radius 10 m, is subjected to a radial acceleration of 4g. The time period of the centrifuge (in seconds) is :

(a) π (b) 2π (c) 3π (d) 4π

Solution:

From the inputs given in the question,

$$\begin{aligned} a &= \frac{v^2}{r} \\ \Rightarrow v^2 &= ar = 4 \times 10 \times 10 = 400 \\ \Rightarrow v &= 20 \text{ m/s} \end{aligned}$$

The time period of the motion is :

$$T = \frac{2\pi r}{v} = \frac{2\pi \times 10}{20} = \pi \text{ seconds}$$

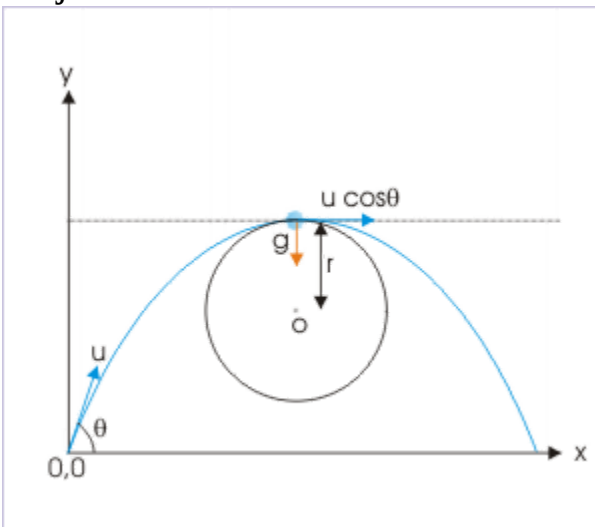
Hence, option (a) is correct.

Uniform circular motion and projectile motion

Both these motions are two dimensional motions. They are alike in the sense that motion in each case is subjected to continuous change of the direction of motion. At the same time, they are different in other details. The most important difference is that projectile motion has a constant acceleration, whereas uniform circular motion has a constant magnitude of acceleration.

Significantly, a projectile motion completely resembles uniform circular motion at one particular instant. The projectile has only horizontal component of velocity, when it is at the maximum height. At that instant, the force of gravity, which is always directed downward, is perpendicular to the direction of velocity. Thus, projectile at that moment executes an uniform circular motion.

Projectile motion



Projectile meets the condition of
UCM at maximum height

Let radius of curvature of the projectile trajectory at maximum height be “r”, then

$$a = g = \frac{v_x^2}{r} = \frac{v^2 \cos^2 \theta}{r}$$
$$\Rightarrow r = \frac{v^2 \cos^2 \theta}{g}$$

Note: The radius of curvature is not equal to maximum height – though expression appears to be same. But, they are actually different. The numerator here consists of cosine function, whereas it is sine function in the expression of maximum height. The difference is also visible from the figure.

Exercise

Exercise:

Problem:

A particle moves along a circle of radius “r” meter with a time period of “T”. The acceleration of the particle is :

(a) $\frac{4\pi^2 r}{T^2}$ (b) $\frac{4\pi^2 r^2}{T^2}$ (c) $4\pi r T$ (d) $\frac{4\pi r}{T}$

Solution:

In order to determine centripetal acceleration, we need to find the speed of the particle. We can determine the speed of the particle, using expression of time period :

$$T = \frac{2\pi r}{v}$$

$$\Rightarrow v = \frac{2\pi r}{T}$$

The acceleration of the particle is :

$$a = \frac{v^2}{r} = \frac{(2\pi r)^2}{T^2 r} = \frac{4\pi^2 r}{T^2}$$

Hence, option (a) is correct.

Exercise:

Problem:

Which of the following quantities are constant in uniform circular motion :

- (a) velocity and acceleration
 - (b) magnitude of acceleration and radius of the circular path
 - (c) radius of the circular path and speed
 - (d) speed and acceleration
-

Solution:

Three quantities, namely, radius of the circular path, magnitude of acceleration and speed are constant in uniform circular motion.

Hence, options (b) and (c) are correct.

Exercise:

Problem:

Which of these is/are correct for a particle in uniform circular motion :

- (a) the acceleration of the particle is tangential to the path
- (b) the acceleration of the particle in the direction of velocity is zero
- (c) the time period of the motion is inversely proportional to the radius

(d) for a given acceleration, velocity of the particle is proportional to the radius of circle

Solution:

The options (a), (c) and (d) are wrong.

Hence, option (b) is correct.

Exercise:

Problem:

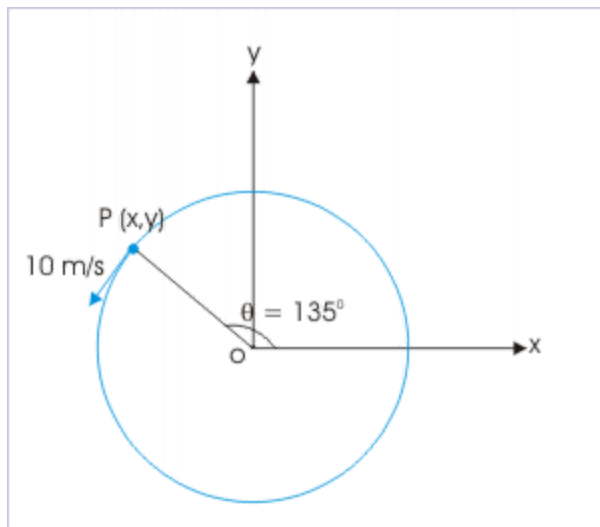
A particle moves with a speed of 10 m/s along a horizontal circle of radius 10 m in anti-clockwise direction. The x and y coordinates of the particle is 0, r at $t = 0$. The velocity of the particle (in m/s), when its position makes an angle 135° with x - axis, is :

- (a) $-5\sqrt{2}\mathbf{i} - 5\sqrt{2}\mathbf{j}$ (b) $5\sqrt{2}\mathbf{i} - 5\sqrt{2}\mathbf{j}$ (c) $5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}$ (d) $-5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}$
-

Solution:

The velocity is a vector quantity. We need to know the components of the velocity when particle makes the angle 135° with x - axis.

Uniform circular motion



$$v_x = -v \sin \theta = -10 \sin 135^\circ = -10 \times \left(\frac{1}{\sqrt{2}} \right) = -5\sqrt{2}$$

$$v_y = v \cos \theta = 10 \cos 135^\circ = 10 \times \left(\frac{-1}{\sqrt{2}} \right) = -5\sqrt{2}$$

The component v_x is in the reference x - direction, whereas component v_y is in the negative reference y - direction.

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$$

$$\Rightarrow \mathbf{v} = -\left(5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j} \right) m/s$$

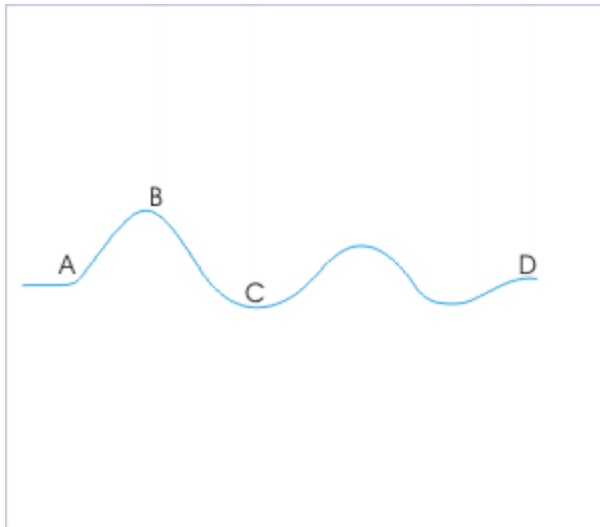
Hence, option (a) is correct.

Exercise:

Problem:

A car moves along a path as shown in the figure at a constant speed. Let a_A , a_B , a_C and a_D be the centripetal accelerations at locations A, B, C and D respectively. Then

Uniform circular motion

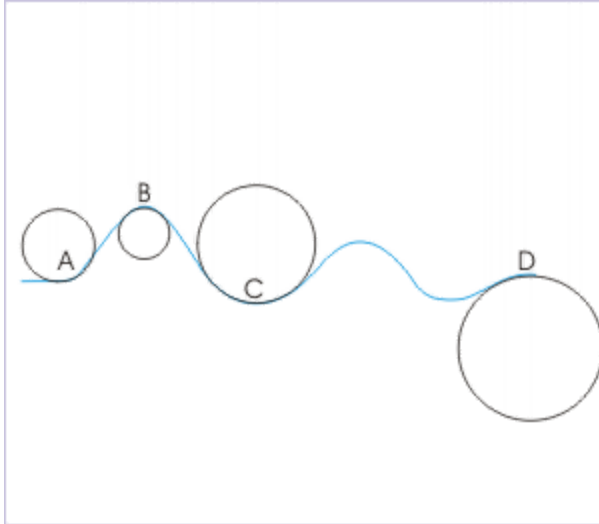


- (a) $a_A < a_C$ (b) $a_A > a_D$ (c) $a_B < a_C$ (d) $a_B > a_D$

Solution:

The curvatures of path at four points are indicated with circles drawn so that they align exactly with the curved path at these points. The radii of these circles at points A and B are smaller than that at points C and D.

Uniform circular motion



Now, the centripetal acceleration is given by :

$$a = \frac{v^2}{r}$$

The smaller the radius, greater is centripetal acceleration. Thus,

$$a_A > a_D$$

and

$$a_B > a_D$$

Hence, options (b) and (d) are correct.

Exercise:

Problem:

A particle executes uniform circular motion with a speed “v” in xy plane, starting from position (r,0), where “r” denotes the radius of the circle. Then

- (a) the component of velocity along x - axis is constant
 - (b) the component of acceleration along x - axis is constant
 - (c) the angle swept by the line joining center and particle in equal time interval is constant
 - (d) the velocity is parallel to external force on the particle
-

Solution:

The component of velocity along x - direction is “ $-v \sin\theta$ ” and is not a constant. The component of acceleration along x - direction is “ $a_x = -\frac{v^2}{r} \cos \theta$ ” and is not a constant. The external force in uniform circular motion is equal to centripetal force, which is directed towards the center. Thus, the velocity is perpendicular to external force.

On the other hand, the particle covers equal circular arc in equal time interval. Hence, the angle subtended by the arc at the center is also equal in equal time interval.

Hence, option (c) is correct.

Exercise:

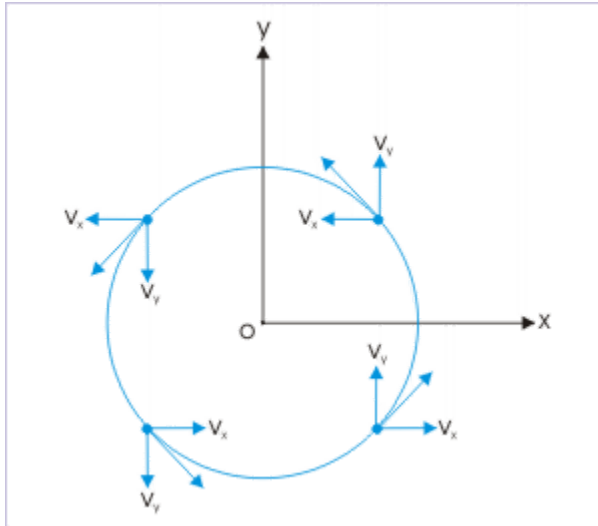
Problem:

A particle executes uniform circular motion with a speed “ v ” in anticlockwise direction, starting from position $(r,0)$, where “ r ” denotes the radius of the circle. Then, both components of the velocities are positive in :

- (a) first quadrant of the motion
 - (b) second quadrant of the motion
 - (c) third quadrant of the motion
 - (d) fourth quadrant of the motion
-

Solution:

From the figure, it is clear that the components of velocity in the fourth quadrant are both positive.

Uniform circular motion

Hence, option (d) is correct.

Exercise:**Problem:**

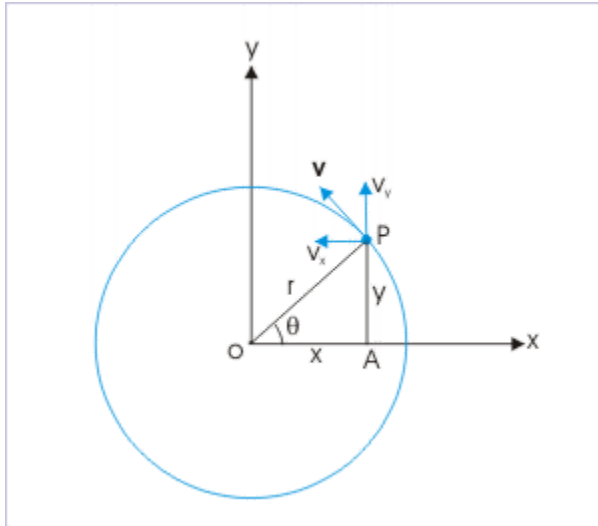
A particle executes uniform circular motion with a constant speed of 1 m/s in xy - plane. The particle starts moving with constant speed from position $(r,0)$, where "r" denotes the radius of the circle. If center of circle is the origin of the coordinate system, then what is the velocity in "m/s" of the particle at the position specified by the coordinates $(3m, -4m)$?

- (a) $\frac{1}{5}(\mathbf{i} + \mathbf{j})$ (b) $(\mathbf{i} - \mathbf{j})$ (c) $\frac{1}{5}(\mathbf{i} - \mathbf{j})$ (d) $\frac{1}{5}(\mathbf{i} + \mathbf{j})$

Solution:

Here, the inputs are given in terms of coordinates. Thus, we shall use the expression of velocity, which involves coordinates. Velocity of the particle in circular motion is given by :

Uniform circular motion



$$\mathbf{v} = -\frac{vy}{r}\mathbf{i} + \frac{vx}{r}\mathbf{j}$$

Here, radius of the circle is :

$$r = \sqrt{(3^2 + 4^2)} = 5 \text{ m/s}$$

$$\Rightarrow \mathbf{v} = -\frac{1 \times (-4)}{5}\mathbf{i} + \frac{1 \times 3}{5}\mathbf{j}$$

$$\Rightarrow \mathbf{v} = \frac{1}{5}(4\mathbf{i} + 3\mathbf{j})$$

Hence, option (a) is correct.

Uniform circular motion (application)

Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the uniform circular motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Direction of velocity
- Direction of position vector
- Velocity
- Relative speed
- Nature of UCM

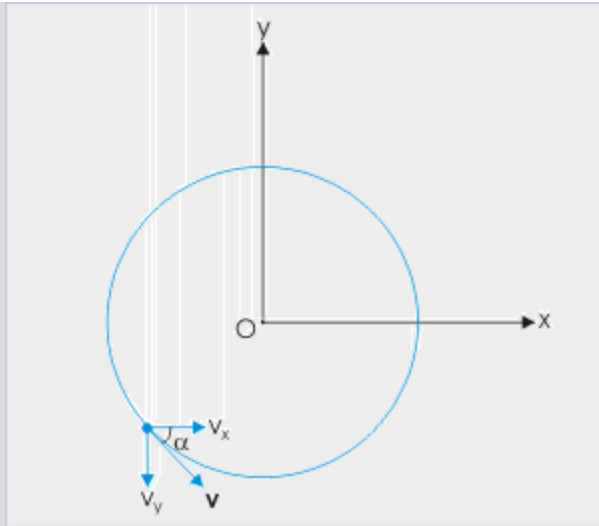
Direction of velocity

Example:

Problem : A particle moves in xy-plane along a circle of radius "r". The particle moves at a constant speed in anti-clockwise direction with center of circle as the origin of the coordinate system. At a certain instant, the velocity of the particle is $\mathbf{i} - \sqrt{3}\mathbf{j}$. Determine the angle that velocity makes with x-direction.

Solution : The sign of y-component of velocity is negative, whereas that of x-component of velocity is positive. It means that the particle is in the third quadrant of the circle as shown in the figure.

Top view of uniform circular motion in xy-plane



The acute angle formed by the velocity with x-axis is obtained by considering the magnitude of components (without sign) as :

$$\tan \alpha = \frac{v_y}{v_x} = \frac{\sqrt{3}}{1} = \sqrt{3} = \tan 60^\circ$$

$$\Rightarrow \alpha = 60^\circ$$

This is the required angle as measured in clockwise direction from x-axis. If the angle is measured in anti-clockwise direction from positive direction of x-axis, then

$$\Rightarrow \alpha = 360^\circ - 60^\circ = 300^\circ$$

Direction of position vector

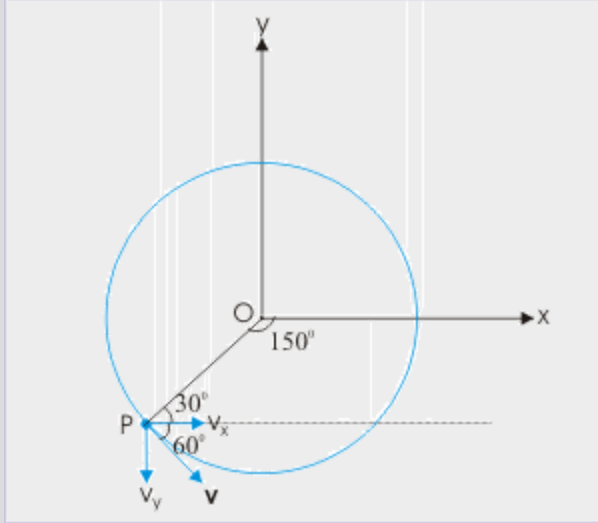
Example:

Problem : A particle moves in xy-plane along a circle of radius “r”. The particle moves at a constant speed in anti-clockwise direction with center of circle as the origin of the coordinate system. At a certain instant, the

velocity of the particle is $\mathbf{i} - \sqrt{3}\mathbf{j}$. Determine the angle that position vector makes with x-direction.

Solution : The sign of y-component of velocity is negative, whereas that of x-component of velocity is positive. It means that the particle is in the third quadrant of the circle as shown in the figure.

Top view of uniform circular motion in xy-plane



The acute angle formed by the velocity with x-axis is obtained by considering the magnitude of components (without sign) as :

$$\tan \alpha = \frac{v_y}{v_x} = \frac{\sqrt{3}}{1} = \sqrt{3} = \tan 60^\circ$$
$$\Rightarrow \alpha = 60^\circ$$

But, we know that position vector is perpendicular to velocity vector. By geometry,

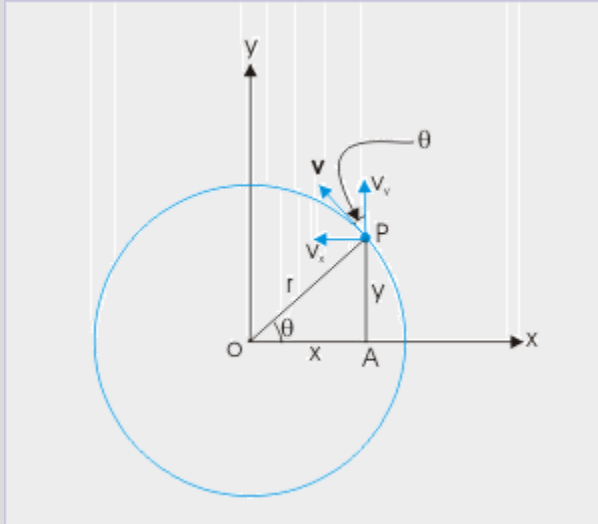
$$\theta = 180^\circ - 30^\circ = 150^\circ$$

This is the angle as measured in clockwise direction from x-axis. If the angle is measured in anti-clockwise direction from positive direction of x-axis, then

$$\Rightarrow \alpha' = 360^\circ - 150^\circ = 210^\circ$$

Note : Recall the derivation of the expression of velocity vector in the previous module. We had denoted “ θ ” as the angle that position vector makes with x-axis (not the velocity vector). See the figure that we had used to derive the velocity expression.

Top view of uniform circular motion in xy-plane



As a matter of fact “ θ ” is the angle that velocity vector makes with y-axis (not x-axis). We can determine the angle “ θ ” by considering the sign while evaluating $\tan \theta$,

$$\tan \theta = \frac{v_x}{v_y} = \frac{1}{-\sqrt{3}} = \tan 150^\circ$$

$$\theta = 150^\circ$$

Velocity

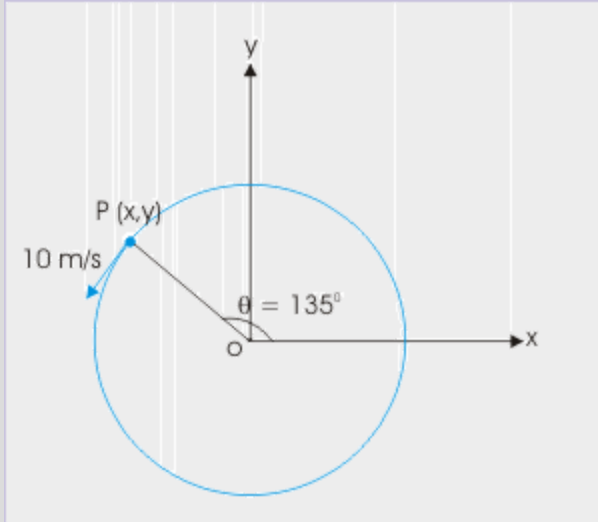
Example:

Problem : A particle moves with a speed 10 m/s in xy-plane along a circle of radius 10 m in anti-clockwise direction. The particle starts moving with constant speed from position $(r,0)$, where "r" denotes the radius of the

circle. Find the velocity of the particle (in m/s), when its position makes an angle 135° with x – axis.

Solution : The velocity of the particle making an angle " θ " with x – axis is given as :

Uniform circular motion



$$v = v_x + v_y = -v \sin \theta + v \cos \theta$$

Here,

$$v_x = -v \sin \theta = -10 \sin 135^\circ = -10 \times \frac{1}{\sqrt{2}} = -5\sqrt{2}$$

$$v_y = v \cos \theta = 10 \cos 135^\circ = 10 \times -\frac{1}{\sqrt{2}} = -5\sqrt{2}$$

Here, both the components are negative.

$$= v_x + v_y$$

$$\Rightarrow = -5\sqrt{2} + 5\sqrt{2} \text{ m/s}$$

Relative speed

Example:

Problem : Two particles tracing a circle of radius 10 m begin their journey simultaneously from a point on the circle in opposite directions. If their speeds are 2.0 m/s and 1.14 m/s respectively, then find the time after which they collide.

Solution : The particles approach each other with a relative speed, which is equal to the sum of their speeds.

$$v_{rel} = 2.0 + 1.14 = 3.14 \text{ m/s}$$

For collision to take place, the particles need to cover the initial separation with the relative speed as measured above. The time for collision is, thus, obtained as :

$$t = \frac{2\pi r}{v_{rel}} = \frac{2 \times 3.14 \times 10}{3.14} = 20 \text{ s}$$

Nature of UCM

Example:

Problem : Two particles “A” and “B” are moving along circles of radii " r_A " and " r_B " respectively at constant speeds. If the particles complete one revolution in same time, then prove that speed of the particle is directly proportional to radius of the circular path.

Solution : As the time period of the UCM is same,

$$T = \frac{2\pi r_A}{v_A} = \frac{2\pi r_B}{v_B}$$

$$\Rightarrow \frac{v_A}{r_A} = \frac{v_B}{r_B}$$

$$\Rightarrow \frac{v_A}{v_B} = \frac{r_A}{r_B}$$

Hence, speed of the particle is directly proportional to the radius of the circle.

Example:

Problem : Two particles “A” and “B” are moving along circles of radii " r_A " and " r_B " respectively at constant speeds. If the particles have same acceleration, then prove that speed of the particle is directly proportional to square root of the radius of the circular path.

Solution : As the acceleration of the UCM is same,

$$\begin{aligned}\frac{v_A^2}{r_A} &= \frac{v_B^2}{r_B} \\ \Rightarrow \frac{v_A^2}{v_B^2} &= \frac{r_A}{r_B} \\ \Rightarrow \frac{v_A}{v_B} &= \frac{\sqrt{r_A}}{\sqrt{r_B}}\end{aligned}$$

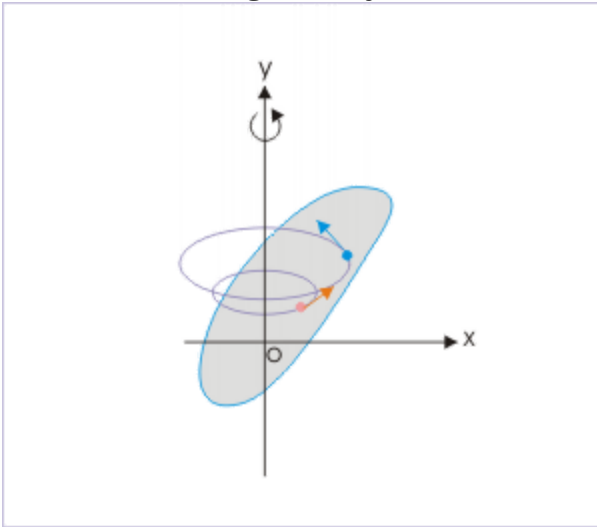
Hence, speed of the particle is directly proportional to square root of the radius of the circular path.

Circular motion and rotational kinematics

Each part of a rigid body under pure rotational motion describes a circular motion about a fixed axis.

In pure rotational motion, the constituent particles of a rigid body rotate about a fixed axis in a circular trajectory. The particles, composing the rigid body, are always at a constant perpendicular distance from the axis of rotation as their internal distances within the rigid body is locked. Farther the particle from the axis of rotation, greater is the speed of rotation of the particle. Clearly, rotation of a rigid body comprises of circular motion of individual particles.

Rotation of a rigid body about a fixed axis



Each particle constituting the body executes an uniform circular motion about the fixed axis.

We shall study these and other details about the rotational motion of rigid bodies at a later stage. For now, we confine ourselves to the aspects of rotational motion, which are connected to the circular motion as executed by a particle. In this background, we can say that uniform circular motion (UCM) represents the basic form of circular motion and circular motion, in turn, constitutes rotational motion of a rigid body.

The description of a circular and hence that of rotational motion is best suited to corresponding angular quantities as against linear quantities that we have so far used to describe translational motion. In this module, we shall introduce these angular quantities and prepare the ground work to enable us apply the concepts of angular quantities to “circular motion” in general and “uniform circular motion” in particular.

Most important aspect of angular description as against linear description is that there exists one to one correspondence of quantities describing motion : angular displacement (linear displacement), angular velocity (linear velocity) and angular acceleration (linear acceleration).

Angular quantities

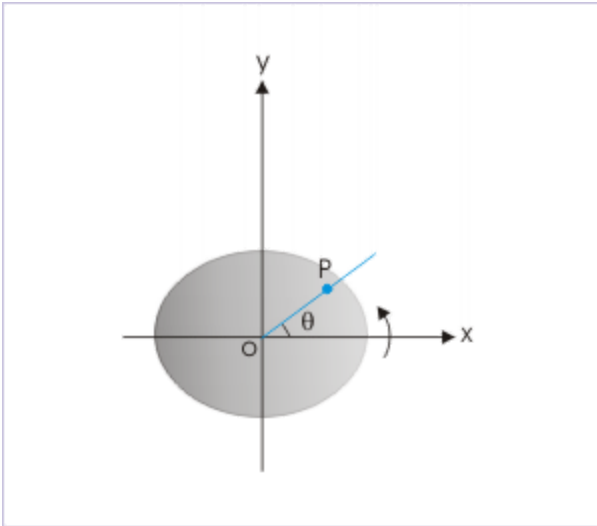
In this section, we discuss some of the defining quantities, which are used to study uniform circular motion of a particle and rotational motion of rigid bodies. These quantities are angular position, angular displacement and angular velocity. They possess directional properties. Their measurement in counter clockwise direction is considered positive, whereas quantities measured in clockwise direction is considered negative. This gives us a simplified scheme to represent an angular vector by a simple variable, whose sign indicates its direction.

Notably, we shall not discuss angular acceleration in this module. It will be discussed as a part of non-uniform circular motion in a separate module.

Angular position (θ)

We need two straight lines to measure an angle. In rotational motion, one of them represents fixed direction, while another represents the rotating arm containing the particle. Both these lines are perpendicular to the rotating axis. The rotating arm, additionally, passes through the position of the particle.

Angular position (θ)

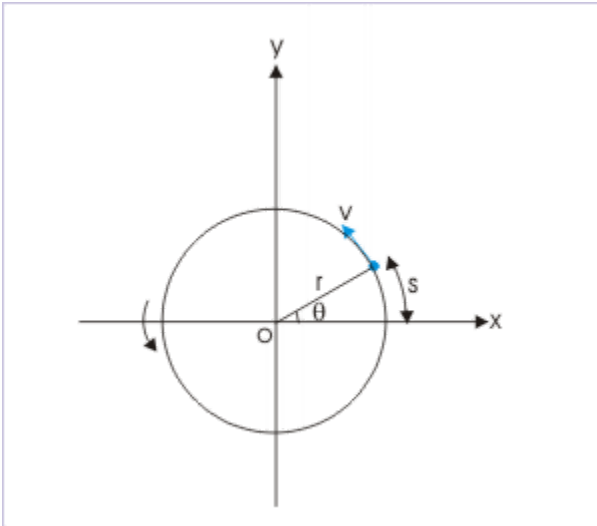


Angular position is the angle between reference direction and rotating arm.

For convenience, the reference direction like x – axis of the coordinate system serves to represent fixed direction. The angle between reference direction and rotating arm (OP) at any instant is the angular position of the particle (θ).

It must be clearly understood that angular position (θ) is an angle and does not represent the position of the particle by itself. It requires to be paired with radius of the circle (r) along which particle moves in order to specify the position of the particle. Thus, a specification of a position in the reference system will require both “ r ” and “ θ ” to be specified.

Relation between distance (s) and angle (θ)



By geometry,

Equation:

$$\theta = \frac{s}{r}$$

$$\Rightarrow s = \theta r$$

where "s" is the length of the arc subtending angle "θ" at the origin and "r" is the radius of the circle containing the position of the particle. The angular position is measured in "radian", which has no dimension, being ratio of two lengths. One revolution contains 2π radians. The unit of radian is related to other angle measuring units "degree" and "revolution" as :

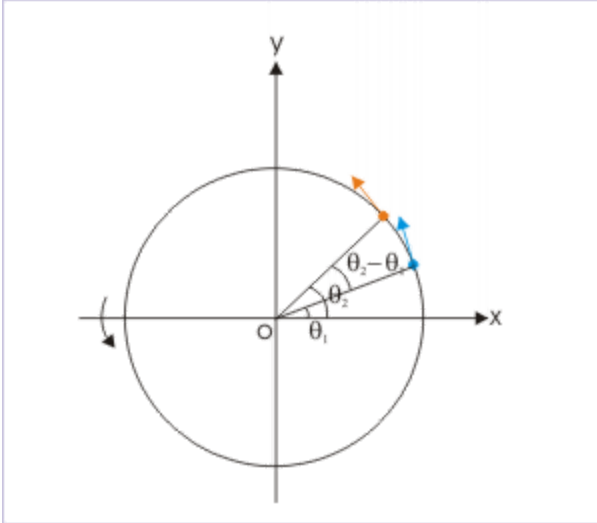
$$1 \text{ revolution} = 360^\circ = 2\pi \text{ radian}$$

Note: The quantities related to angular motion are expressed in terms of angular position. It must be ensured that values of angular position wherever it appears in the expression be substituted in radians only. If the given value is in some other unit, then we first need to change the value into radian. It is so because, radian is a unit derived from the definition of the angle. The defining relation $\theta = s/r$ will not hold unless "θ" is in radian.

Angular displacement ($\Delta\theta$)

Angular displacement is equal to the difference of angular positions at two instants of rotational motion.

Angular displacement ($\Delta\theta$)



Angular displacement is equal to the difference of angular positions at two positions.

Equation:

$$\Delta\theta = \theta_2 - \theta_1$$

The angular displacement is also measured in “radian” like angular position. In case our measurement of angular position coincides with the reference direction, we can make substitution as given here :

$$\theta_1 = 0$$

$$\theta_2 = \theta$$

With these substitution, we can simply express angular displacement in terms of angle as :

$$\Rightarrow \Delta\theta = \theta_2 - \theta_1 = \theta - 0 = \theta$$

Angular velocity (ω)

Angular speed is the ratio of the magnitude of angular displacement and time interval.

Equation:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

This ratio is called average angular velocity, when it is evaluated for finite time interval; and instantaneous angular velocity, when it is evaluated for infinitesimally small period ($\Delta \rightarrow 0$).

Equation:

$$\omega = \frac{d\theta}{dt}$$

The angular velocity is measured in “rad/s”.

Description of circular motion

Circular motion is completely described when angular position of a particle is given as a function of time like :

$$\theta = f(t)$$

For example, $\theta = 2t^2 - 3t + 1$ tells us the position of the particle with the progress of time. The attributes of circular motion such as angular velocity and acceleration are first and second time derivatives of this function in time. Similarity to pure translational motion is quite obvious here. In pure translational motion, each particle constituting a rigid body follows parallel linear paths. The position of a particle is a function of time, whereby :

$$x = f(t)$$

Example:

Problem : The angular position (in radian) of a particle under circular motion about a perpendicular axis with respect to reference direction is given by the function in time (seconds) as :

$$\theta = t^2 - 0.2t + 1$$

Find (i) angular position when angular velocity is zero and (ii) determine whether rotation is clock-wise or anti-clockwise.

Solution : The angular velocity is equal to first derivative of angular position,

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt} (t^2 - 0.2t + 1) = 2t - 0.2$$

For $\omega = 0$, we have :

$$\begin{aligned} 2t - 0.2 &= 0 \\ \Rightarrow t &= 0.1 \text{ s} \end{aligned}$$

The angular position at $t = 0.1$ s,

$$\begin{aligned} \theta &= (0.1)^2 - 0.2 \times 0.1 + 1 = 0.99 \text{ rad} \\ \Rightarrow \theta &= \frac{0.99 \times 360}{2\pi} = \frac{0.99 \times 360 \times 7}{2 \times 22} = 56.7^\circ \end{aligned}$$

As the particle makes a positive angle with respect to reference direction, we conclude that the particle is moving in anti-clockwise direction (We shall discuss the convention regarding direction of angular quantities in detail subsequently).

Relationship between linear (v) and angular speed (ω)

In order to understand the relation, let us consider two uniform circular motions with equal time period (T) along two circular trajectories of radii r_1 and r_2 ($r_2 > r_1$). It is evident that particle along the outer circle is moving at a greater speed as it has to cover greater perimeter or distance. On the

other hand angular speeds of the two particles are equal as they transverse equal angles in a given time.

This observation is key to understand the relation between linear and angular speed. Now, we know that :

$$s = r\theta$$

Differentiating with respect to time, we have :

$$\frac{ds}{dt} = \frac{dr}{dt}\theta + r\frac{d\theta}{dt}$$

Since, “r” is constant for a given circular motion, $\frac{dr}{dt} = 0$.

$$\frac{ds}{dt} = \frac{d\theta}{dt}r = \omega r$$

Now, $\frac{ds}{dt}$ is equal to linear speed, v. Hence,

Equation:

$$v = \omega r$$

This is the relation between angular and linear speeds. Though it is apparent, but it is emphasized here for clarity that angular and linear speeds do not represent two separate individual speeds. Remember that a particle can have only one speed at a particular point of time. They are, as a matter of fact, equivalent representation of the same change of position with respect to time. They represent same speed – but in different language or notation.

Example:

Problem : The angular position (in radian) of a particle with respect to reference direction, along a circle of radius 0.5 m is given by the function in time (seconds) as :

$$\theta = t^2 - 0.2t$$

Find linear velocity of the particle at $t = 0$ second.

Solution : The angular velocity is given by :

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt} (t^2 - 0.2t) = 2t - 0.2$$

For $t = 0$, the angular velocity is :

$$\Rightarrow \omega = 2 \times 0 - 0.2 = 0.2 \text{ rad/s}$$

The linear velocity at this instant is :

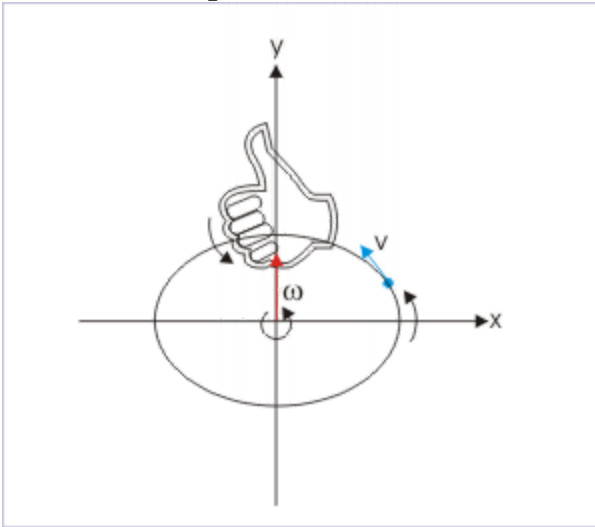
$$\Rightarrow v = \omega r = 0.2 \times 0.5 = 0.1 \text{ m/s}$$

Vector representation of angular quantities

The angular quantities (displacement, velocity and acceleration) are also vector quantities like their linear counter parts and follow vector rules of addition and multiplication, with the notable exception of angular displacement. Angular displacement does not follow the rule of vector addition strictly. In particular, it can be shown that addition of angular displacement depends on the order in which they are added. This is contrary to the property of vector addition. Order of addition should not affect the result. We intend here to skip the details of this exception to focus on the subject matter in hand. Besides, we should know that we may completely ignore this exception if the angles involved have small values.

The vector angular quantities like angular velocity ($\vec{\omega}$) or ω (as scalar representation of angular vector) is represented by a vector, whose direction is obtained by applying “Right hand rule”. We just hold the axis of rotation with right hand in such a manner that the direction of the curl of fingers is along the direction of the rotation. The direction of extended thumb (along y-axis in the figure below) then represents the direction of angular velocity ($\vec{\omega}$).

Vector cross product



Right Hand Rule (RHR)

The important aspect of angular vector representation is that the angular vector is essentially a straight line of certain magnitude represented on certain scale with an arrow showing direction (shown in the figure as a red line with arrow) – not a curl as some might have expected. Further, the angular vector quantities are axial in nature. This means that they apply along the axis of rotation. Now, there are only two possible directions along the axis of rotation. Thus, we can work with sign (positive or negative) to indicate directional attribute of angular quantities. The angular quantities measured in counter clockwise direction is considered positive, whereas quantities measured in clockwise direction is considered negative.

This simplicity resulting from fixed axis of rotation is very useful. We can take the liberty to represent angular vector quantities in terms of signed scalar quantities as done in the case of linear quantities. The sign of the angular quantity represents the relative direction of the angular quantity with respect to a reference direction.

Analysis of angular motion involves working interchangeably between linear and angular quantities. We must understand here that the relationships essentially involve both axial (angular) and polar (linear)

vectors. In this context, it is recommended that we know the relationship between linear and angular quantities in vector forms as vector relation provide complete information about the quantities involved.

Linear and angular velocity relation in vector form

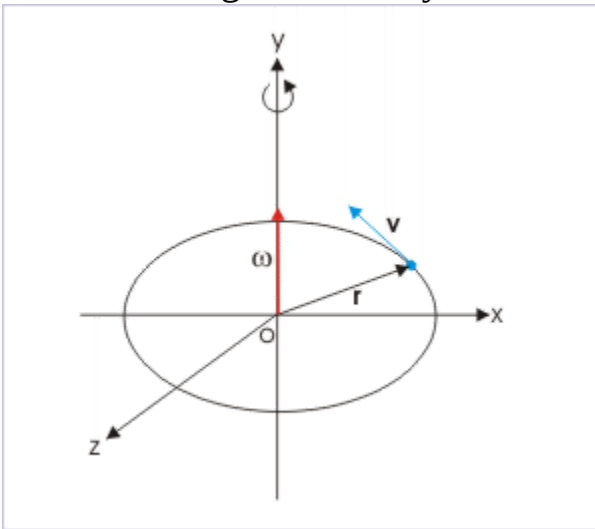
If we want to write the relation for velocities (as against the one derived for speed, $v = \omega r$), then we need to write the relation as vector cross product :

Equation:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

The order of quantities in vector product is important. A change in the order of cross product like $(\mathbf{r} \times \boldsymbol{\omega})$ represents the product vector in opposite direction. The directional relationship between these vector quantities are shown in the figure. The vectors “ $\mathbf{v}$ ” and “ $\mathbf{r}$ ” are in the plane of “xz” plane, whereas angular velocity ($\boldsymbol{\omega}$), is in y-direction.

Linear and angular velocity

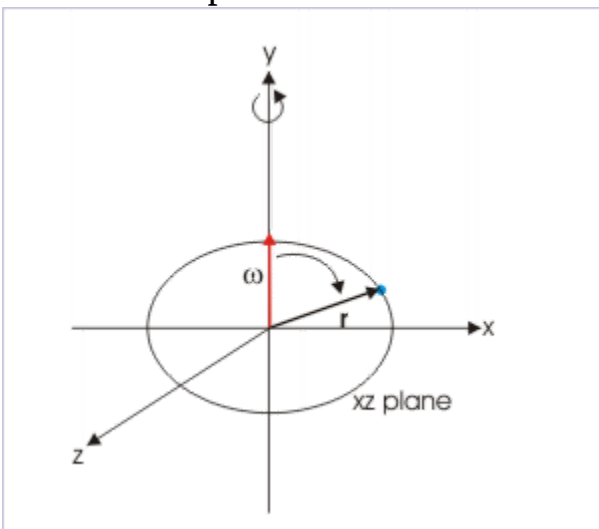


Directional relation between
linear and angular velocity

Here, we shall demonstrate the usefulness of vector notation. Let us do a bit of interpretation here to establish the directional relationship among the quantities from the vector notation. It is expected from the equation ($\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$) that the vector product of angular velocity ($\boldsymbol{\omega}$) and radius vector ($\mathbf{r}$) should yield the direction of velocity ($\mathbf{v}$).

Remember that a vector cross product is evaluated by Right Hand Rule (RHR). We move from first vector ($\boldsymbol{\omega}$) to the second vector ($\mathbf{r}$) of the vector product in an arc as shown in the figure.

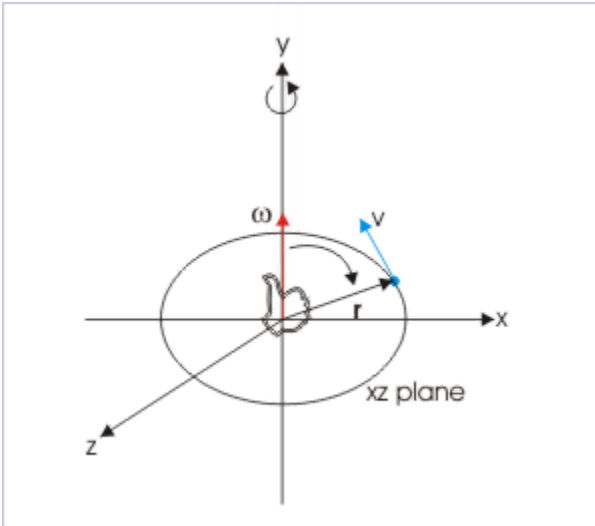
Vector cross product



Determining direction of vector
cross product

We place our right hand such that the curl of fingers follows the direction of arc. The extended thumb, then, represents the direction of cross product ($\mathbf{v}$), which is perpendicular (this fact lets us draw the exact direction) to each of the vectors and the plane containing two vectors ($\boldsymbol{\omega}$ and $\mathbf{r}$) whose products is being evaluated. In the case of circular motion, vectors $\boldsymbol{\omega}$ and $\mathbf{r}$ are perpendicular to each other and vector $\mathbf{v}$ is perpendicular to the plane defined by vectors $\boldsymbol{\omega}$ and $\mathbf{r}$.

Vector cross product



Determining direction of vector cross product

Thus, we see that the interpretation of cross products completely defines the directions of quantities involved at the expense of developing skill to interpret vector product (we may require to do a bit of practice).

Also, we can evaluate magnitude (speed) as :

Equation:

$$v = |\mathbf{v}| = \omega r \sin \theta$$

where θ is the angle between two vectors ω and r . In the case of circular motion, $\theta = 90^\circ$, Hence,

$$\Rightarrow v = |\mathbf{v}| = \omega r$$

Thus, we have every detail of directional quantities involved in the equation by remembering vector form of equation.

Uniform circular motion

In the case of the uniform circular motion, the speed (v) of the particle is constant (by definition). This implies that angular velocity ($\omega = v/r$) in uniform circular motion is also constant.

$$\Rightarrow \omega = \frac{v}{r} = \text{constant}$$

Also, the time period of the uniform circular motion is :

Equation:

$$\Rightarrow T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

Linear .vs. angular quantity

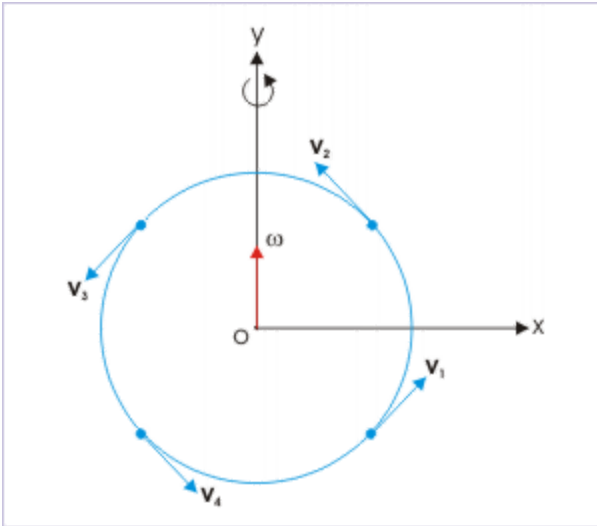
The description of circular motion is described better in terms of angular quantity than its linear counter part.

The reasons are easy to understand. For example, consider the case of uniform circular motion. Here, the velocity of particle is changing - though the motion is “uniform”. The two concepts do not go together. The general connotation of the term “uniform” indicates “constant”, but the velocity is actually changing all the time.

When we describe the same uniform circular motion in terms of angular velocity, there is no contradiction. The velocity (i.e. angular velocity) is indeed constant. This is the first advantage of describing uniform circular motion in terms of angular velocity.

In other words, the vector manipulation or analysis of linear velocity along the circular path is complicated as its direction is specific to a particular point on the circular path and is basically multi-directional. On the other hand, direction of angular velocity is limited to be bi-directional at the most, along the fixed axis of rotation.

Linear and angular velocity



Second advantage is that angular velocity conveys the physical sense of the rotation of the particle as against linear velocity, which indicates translational motion. Alternatively, angular description emphasizes the distinction between two types of motion (translational and rotational).

Finally, angular quantities allow to write equations of motion as available for translational motion with constant acceleration. For illustration purpose, we can refer to equation of motion connecting initial and final angular velocities for a motion with constant angular acceleration “ α ” as :

$$\omega_2 = \omega_1 + \alpha t$$

We shall study detailed aspect of circular motion under constant angular acceleration in a separate module.

Exercises

Exercise:

Problem:

A flywheel of a car is rotating at 300 revolutions per minute. Find its angular speed in radian per second.

$$(a) 10 \quad (b) 10\pi \quad (c) \frac{10}{\pi} \quad (d) \frac{20}{\pi}$$

Solution:

Flywheel covers angular displacement of " 2π " in one revolution.
Therefore, the angular speed in "radian/second" is :

$$\omega = \frac{2\pi \times 300}{60} = 10\pi \text{ rad/s}$$

Hence, option (b) is correct.

Exercise:**Problem:**

A particle is rotating along a circle of radius 0.1 m at an angular speed of π rad/s. The time period of the rotation is :

$$(a) 1 s \quad (b) 2 s \quad (c) 3 s \quad (d) 4 s$$

Solution:

The time period of rotation is :

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$
$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2 s$$

Note : The input of radius is superfluous here.

Hence, option (b) is correct.

Exercise:

Problem:

A particle is rotating at constant speed along a circle of radius 0.1 m, having a time period of 1 second. Then, angular speed in "revolution/s" is :

- (a)1 (b)2 (c)3 (d)4

Solution:

The time period of rotation is :

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

Rearranging, the angular speed is :

$$\Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

Converting to in the unit of revolution per second, the angular speed is :

$$\Rightarrow \omega = 2\pi \text{ rad /s} = \frac{2\pi}{2\pi} = 1 \text{ revolution/s}$$

Hence, option (a) is correct.

Exercise:**Problem:**

A particle is moving with a constant angular speed " ω " along a circle of radius " r " about a perpendicular axis passing through the center of the circle. If " $\mathbf{n}$ " be the unit vector in the positive direction of axis of rotation, then linear velocity is given by :

- (a) $-\mathbf{r} \times \omega$ (b) $r\omega\mathbf{n}$ (c) $-\omega \times \mathbf{r}$ (d) $\omega \times \mathbf{r}$

Solution:

The linear velocity is given by the following vector cross product,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

We know that a change in the order of operands in cross product reverses the resulting vector. Hence,

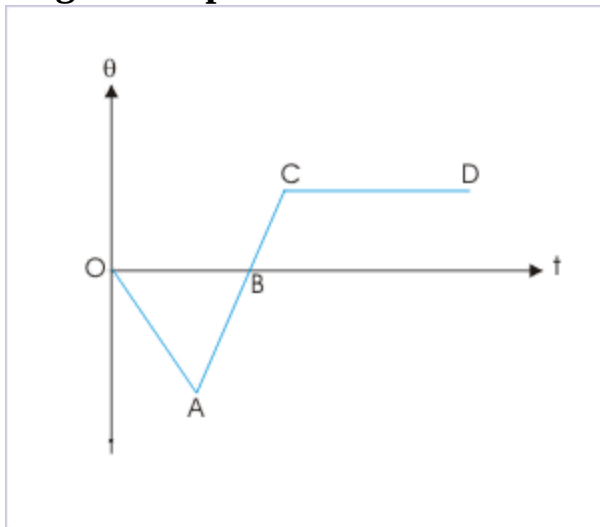
$$\mathbf{v} = -\mathbf{r} \times \boldsymbol{\omega}$$

Further, it is given in the question that positive axial direction has the unit vector "n". But, the plane of motion is perpendicular to axis of rotation. Hence, velocity is not aligned in the direction of unit vector **n**.

Hence, options (a) and (b) are correct.

Exercise:**Problem:**

The figure shows the plot of angular displacement and time of a rotating disc. Corresponding to the segments marked on the plot, the direction of rotation is as :

Angular displacement

Angular displacement

- (a) disk rotates in clockwise direction in the segments OA and AB.
 - (b) disk rotates in clockwise direction in the segment OA, but in anti-clockwise direction in the segment AB.
 - (c) disk rotates in anti-clockwise direction in the segment BC.
 - (d) disk rotates in anti-clockwise direction in the segment CD.
-

Solution:

The direction of rotation is determined by the sign of angular velocity. In turn, the sign of angular velocity is determined by the sign of the slope on angular displacement - time plot. The sign of slope is negative for line OA, positive for line AC and zero for line CD.

The positive angular velocity indicates anti-clockwise rotation and negative angular velocity indicates clockwise rotation. The disk is stationary when angular velocity is zero.

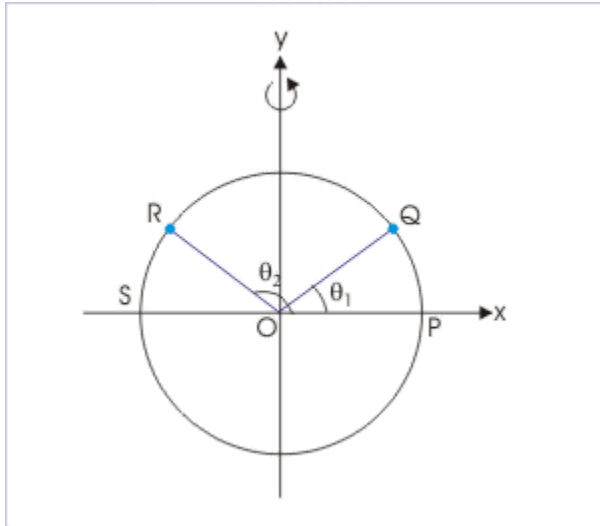
Hence, options (b) and (c) are correct.

Exercise:

Problem:

A particle moves along a circle in xy-plane with center of the circle as origin. It moves from position $Q(1, \sqrt{3})$ to $R(-1, \sqrt{3})$ as shown in the figure. The angular displacement is :

Angular displacement



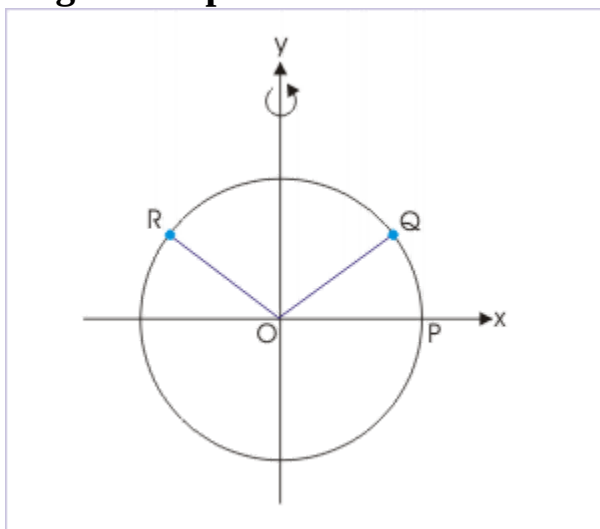
Angular displacement

(a) 30° (b) 45° (c) 60° (d) 75°

Solution:

In order to find angular displacement, we need to find initial and final angular positions. From the geometry,

Angular displacement



Angular displacement

$$\tan \theta_1 = \frac{\sqrt{3}}{1} = \sqrt{3} = \tan 60^\circ$$
$$\Rightarrow \theta_1 = 60^\circ$$

Similarly,

$$\tan \theta_2 = -\frac{\sqrt{3}}{1} = -\sqrt{3} = \tan 120^\circ$$
$$\Rightarrow \theta_2 = 120^\circ$$

Thus, angular displacement is :

$$\Rightarrow \Delta\theta = \theta_2 - \theta_1 = 120^\circ - 60^\circ = 60^\circ$$

Hence, option (c) is correct.

Exercise:

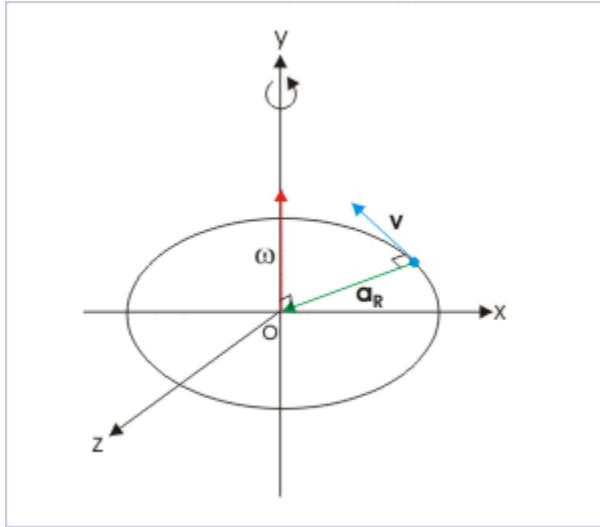
Problem: Select the correct statement(s) :

- (a) The direction of angular velocity is tangential to the circular path.
- (b) The direction of angular velocity and centripetal acceleration are radial towards the center of the circle.
- (c) The linear velocity, angular velocity and centripetal accelerations are mutually perpendicular to each other.
- (d) The direction of angular velocity is axial.

Solution:

The direction of linear velocity is normal to radial direction. The direction of centripetal acceleration is radial. The direction of angular velocity is axial. See the figure.

Angular motion



Angular motion

Clearly, the three directions are mutually perpendicular to each other.

Hence, options (c) and (d) are correct.

Exercise:

Problem: The magnitude of centripetal acceleration is given by :

$$(a) \frac{v^2}{r} \quad (b) \frac{\omega^2}{r} \quad (c) \omega r \quad (d) |\omega \times \mathbf{r}|$$

Solution:

The centripetal acceleration is given by :

$$a_R = \frac{v^2}{r}$$

The linear speed "v" is related to angular speed by the relation,

$$\Rightarrow v = \omega r$$

Substituting this in the expression of centripetal acceleration, we have :

$$\Rightarrow a_R = \frac{\omega^2 r^2}{r} = \omega^2 r$$

We need to evaluate fourth cross product expression to know whether its modulus is equal to the magnitude of centripetal acceleration or not. Let the cross product be equal to a vector "A". The magnitude of vector "A" is :

$$A = |\boldsymbol{\omega} \times \mathbf{v}| = \omega v \sin \theta$$

For circular motion, the angle between linear and angular velocity is 90° . Hence,

$$\Rightarrow A = |\boldsymbol{\omega} \times \mathbf{v}| = \omega v \sin 90^\circ = \omega v$$

Substituting for " ω ", we have :

$$\Rightarrow A = \omega v = \frac{v^2}{r}$$

The modulus of the expression, therefore, is equal to the magnitude of centripetal acceleration.

Note : The vector cross product " $|\boldsymbol{\omega} \times \mathbf{v}|$ " is the vector expression of centripetal force.

Hence, options (a) and (d) are correct.

Circular motion and rotational kinematics (application)

Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the circular motion and rotational kinematics. The questions are categorized in terms of the characterizing features of the subject matter :

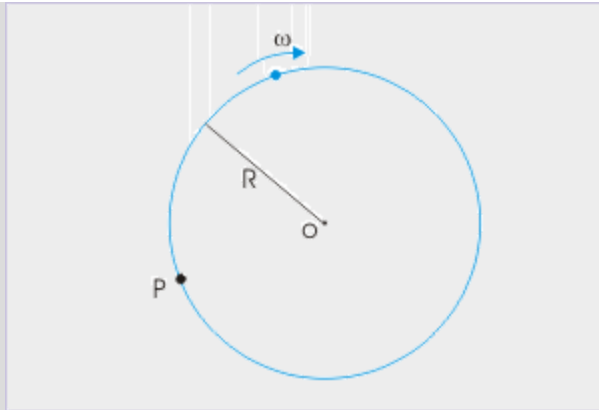
- Measurement of angular displacement
- Angular speeds
- Centripetal acceleration

Measurement of angular displacement

Example:

Problem : A particle in uniform circular motion about the center has angular velocity " ω ". What is its angular velocity with respect to a point "P" on the circumference of the circle ?

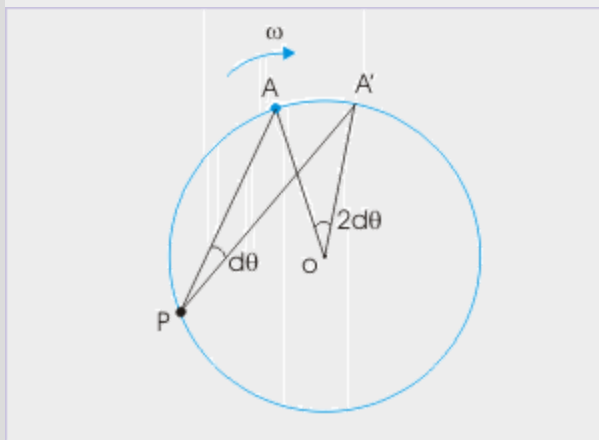
Uniform circular motion



Solution : Angular velocity is a measure of angle in unit time. In the question, measurement of angular velocity about the center of circle is given. It is, therefore, imperative that we seek a relation of angles formed by the motion of the particle at two points of references.

We consider a small arc AA' as shown in the figure, which is covered by the particle in time "dt". By geometry, if the arc subtends an angle "dθ" at "P", then the arc subtends an angle "2dθ" at the center.

Uniform circular motion



Let ω_P be the angular velocity of the particle with respect to point "P", then

$$\frac{d\theta}{dt}$$

From the relation between angles as obtained earlier, the angular velocity of the particle with respect to center is :

$$\frac{d\theta}{dt} = 2 \frac{d\theta}{dt}$$

—

Angular speeds

Example:

Problem : Let ω_1 and ω_2 respectively be the angular speeds of the hour hand of a watch and that of the earth around its own axis. Compare the angular speeds of earth and hour hand of a watch.

Solution : The time periods of hour hand and that of the earth are 12 hours and 24 hours respectively. Now,

—

and

—

Example:

Problem : A particle is kept on a uniformly rotating turn table at a radial distance of 2 m from the axis of rotation. The speed of the particle is “v”. The particle is then shifted to a radial distance of 1 m from the axis of rotation. Find the speed of the particle at the new position.

Solution : The angular speed of rotation is constant. Now, linear velocities of the particle at the two positions are :

—

Centripetal acceleration

Example:

Problem : Find the centripetal acceleration (in m/s^2) at a point on the equator of the earth (consider earth as a sphere of radius = 6400 km).

Solution : The centripetal acceleration in terms of angular speed is given by :

Now, angular speed, in terms of time period is :

—

Combining two equations, we have :

—

—————

Accelerated motion in two dimensions

Motion in two dimensions with one dimensional acceleration (projectile) is analyzed with component motions in coordinate system, whereas motion in two dimensions with two dimensional acceleration (circular motion) is analyzed with the help of component accelerations - tangential and normal accelerations.

We have already studied two dimensional motions such as projectile and uniform circular motion. These motions are the most celebrated examples of two dimensional motion, but it is easy to realize that they are specific instances with simplifying assumptions. The motions that we investigate in our surrounding mostly occur in two or three dimensions in a non-specific manner. The stage is, therefore, set to study two-dimensional motion in non-specific manner i.e. in a very general manner. This requires clear understanding of both linear and non-linear motion. As we have already studied circular motion - an instance of non-linear motion, we can develop an analysis model for a general case involving non-linear motion.

The study of two dimensional motion without any simplifying assumptions, provides us with an insight into the actual relationship among the various motional attributes, which is generally concealed in the consideration of specific two dimensional motions like projectile or uniform circular motion. We need to develop an analysis frame work, which is not limited by any consideration. In two dimensional motion, the first and foremost consideration is that acceleration denotes a change in velocity that reflects a change in the velocity due to any of the following combinations :

- change in the magnitude of velocity i.e. speed
- change in the direction of velocity
- change in both magnitude and direction of velocity

In one dimensional motion, we mostly deal with change in magnitude and change in direction limited to reversal of motion. Such limitations do not exist in two or three dimensional motion. A vector like velocity can change by virtue of even direction only as in the case of uniform circular motion. Further, a circular motion may also involve variable speed i.e. a motion in which velocity changes in both direction and magnitude.

Most importantly, the generalized consideration here will resolve the subtle differences that arises in interpreting vector quantities like displacement, velocity etc. We have noted that there are certain subtle differences in interpreting terms such as $\Delta \mathbf{r}$ and $|\Delta \mathbf{r}|$; $d\mathbf{r}/dt$ and $|d\mathbf{r}/dt|$; $d\mathbf{v}/dt$ and $|d\mathbf{v}/dt|$ etc. In words, we have seen that time rate of change in the magnitude of velocity (speed) is not equal to the magnitude of time rate of change in velocity. This is a subtle, but significant difference that we should account for. In this module, we shall find that time rate of change in the magnitude of velocity (speed), as a matter of fact, represents the magnitude of a component of acceleration known as "tangential acceleration".

Characteristics of two dimensional motion

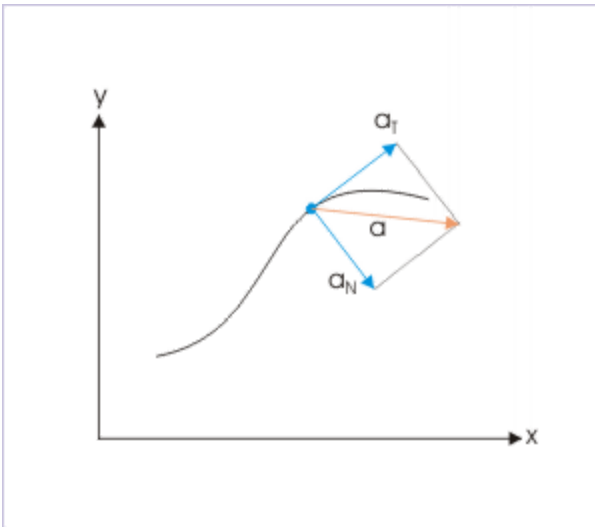
Let us have a look at two dimensional motions that we have so far studied. We observe that projectile motion is characterized by a constant acceleration, “g”, i.e. acceleration due to gravity. What it means that though the motion itself is two dimensional, but acceleration is one dimensional. Therefore, this motion presents the most simplified two dimensional motion after rectilinear motion, which can be studied with the help of consideration of motion in two component directions.

Uniform circular motion, on the other hand, involves an acceleration, which is not one dimensional. It is constant in magnitude, but keeps changing direction along the line connecting the center of the circle and the particle. The main point is that acceleration in uniform circular motion is two dimensional unlike projectile motion in which acceleration (due to gravity) is one dimensional. As a more generalized case, we can think of circular motion in which both magnitude and direction of acceleration is changing. Such would be the case when particle moves with varying speed along the circular path.

In the nutshell, we can conclude that two dimensional motion types (circular motion, elliptical motion and other non-linear motion) involve varying acceleration in two dimensions. In order to facilitate study of general class of motion in two dimensions, we introduce the concept of components of acceleration in two specific directions. Notably, these directions are not same as the coordinate directions (“x” and “y”). One of

the component acceleration is called “tangential acceleration”, which is directed along the tangent to the path of motion and the other is called “normal acceleration”, which is perpendicular to the tangent to the path of motion. Two accelerations are perpendicular to each other. The acceleration (sometimes also referred as total acceleration) is the vector sum of two mutually perpendicular component accelerations,

Two dimensional acceleration



There are tangential and normal components of acceleration.

Equation:

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_N$$

The normal acceleration is also known as radial or centripetal acceleration, $\mathbf{a}_R$, particularly in reference of circular motion.

Tangential acceleration

Tangential acceleration is directed tangentially to the path of motion. Since velocity is also tangential to the path of motion, it is imperative that

tangential acceleration is directed in the direction of velocity. This leads to an important meaning. We recall that it is only the component of force in the direction of velocity that changes the magnitude of velocity. This means that component of acceleration in the tangential direction represents the change in the magnitude of velocity (read speed). In non-uniform circular motion, the tangential acceleration accounts for the change in the speed (we shall study non-uniform circular motion in detail in a separate module).

By logical extension, we can define that tangential acceleration is time rate of change of "speed". The speed is highlighted here to underscore the character of tangential acceleration. Mathematically,

Equation:

$$a_T = \frac{dv}{dt}$$

This insight into the motion should resolve the differences that we had highlighted earlier, emphasizing that rate of change in the magnitude of velocity (dv/dt) is not equal to the magnitude of rate of change of velocity ($|d\mathbf{v}/dt|$). What we see now that rate of change in the magnitude of velocity (dv/dt) is actually just a component of total acceleration ($d\mathbf{v}/dt$).

It is easy to realize that tangential acceleration comes into picture only when there is change in the magnitude of velocity. For example, uniform circular motion does not involve change in the magnitude of velocity (i.e. speed is constant). There is, therefore, no tangential acceleration involved in uniform circular motion.

Normal acceleration

Normal (radial) acceleration acts in the direction perpendicular to tangential direction. We have seen that the normal acceleration, known as centripetal acceleration in the case of uniform circular motion, is given by :

Equation:

$$a_N = \frac{v^2}{r}$$

where “r” is the radius of the circular path. We can extend the expression of centripetal acceleration to all such trajectories of two dimensional motion, which involve radius of curvature. It is so because, radius of the circle is the radius of curvature of the circular path of motion.

In the case of tangential acceleration, we have argued that the motion should involve a change in the magnitude of velocity. Is there any such inference about normal (radial) acceleration? If motion is along a straight line without any change of direction, then there is no normal or radial acceleration involved. The radial acceleration comes into being only when motion involves a change in direction. We can, therefore, say that two components of accelerations are linked with two elements of velocity (magnitude and direction). A time rate of change in magnitude represents tangential acceleration, whereas a time rate of change of direction represents radial (normal) acceleration.

The above deduction has important implication for uniform circular motion. The uniform circular motion is characterized by constant speed, but continuously changing velocity. The velocity changes exclusively due to change in direction. Clearly, tangential acceleration is zero and radial acceleration is finite and acting towards the center of rotation.

Total acceleration

Total acceleration is defined in terms of velocity as :

Equation:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

In terms of component accelerations, we can write total accelerations in the following manner :

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_N$$

The magnitude of total acceleration is given as :

Equation:

$$a = |\mathbf{a}| = \left| \frac{d\mathbf{v}}{dt} \right| = \sqrt{a_T^2 + a_N^2}$$

where

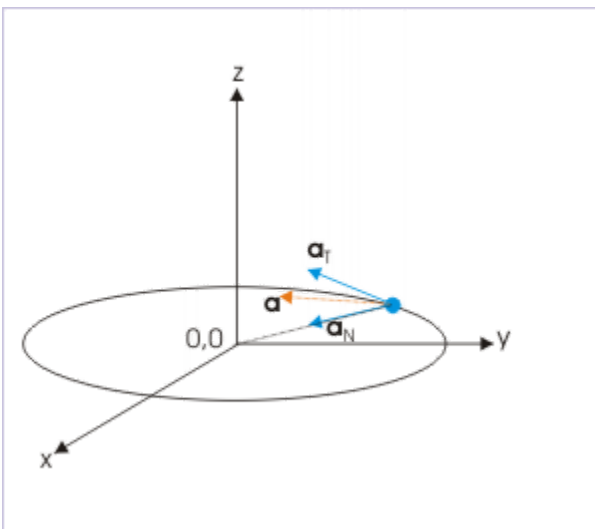
$$a_T = \frac{dv}{dt}$$

In the nutshell, we see that time rate of change in the speed represents a component of acceleration in tangential direction. On the other hand, magnitude of time rate of change in velocity represents the magnitude of total acceleration. Vector difference of total and tangential acceleration is equal to normal acceleration in general. In case of circular motion or motion with curvature, radial acceleration is normal acceleration.

Tangential and normal accelerations in circular motion

We consider motion of a particle along a circular path. As pointed out in the section above, the acceleration is given as vector sum of two acceleration components as :

Two dimensional circular motion



There are tangential and normal

components of acceleration.

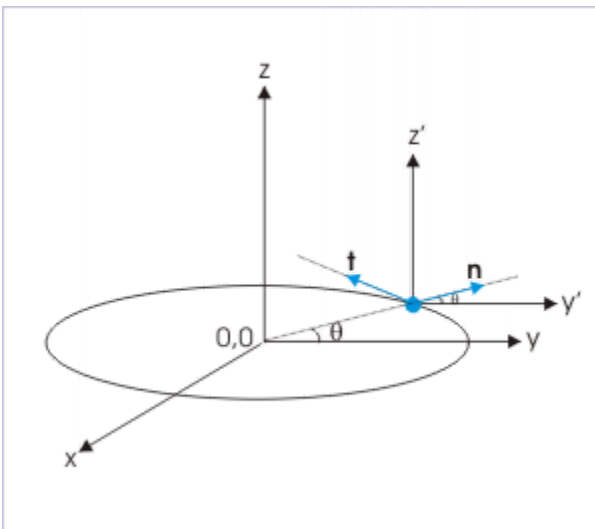
$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_N$$

Equation:

$$\mathbf{a} = a_T \mathbf{t} + a_N \mathbf{n}$$

where “ $\mathbf{t}$ ” and “ $\mathbf{n}$ ” are unit vectors in the tangential and radial directions. Note that normal direction is same as radial direction. For the motion shown in the figure, the unit vector in radial direction is :

Unit vectors



Unit vectors in tangential and normal directions.

Equation:

$$\mathbf{n} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$$

Similarly, the unit vector in tangential direction is :

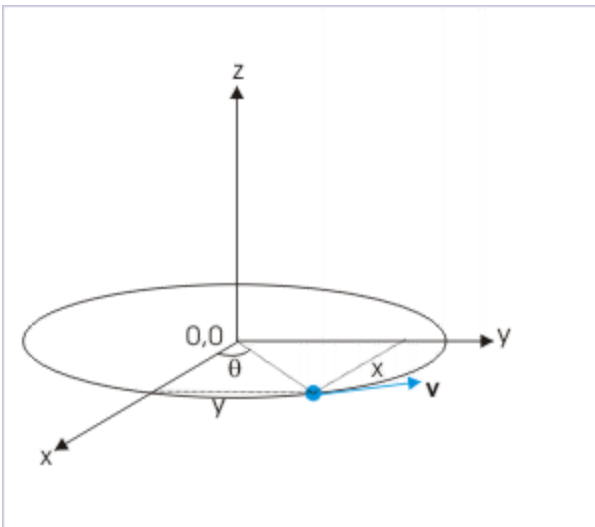
Equation:

$$\mathbf{t} = -1 \times \sin \theta \mathbf{i} + 1 \times \cos \theta \mathbf{j} = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$$

Note: There is an easy way to find the sign of component, using graphical representation. Shift the vector at the origin, if the vector in question does not start from the origin. Simply imagine the component of a vector as projection on the coordinate. If the projection is on the positive side of the coordinate, then sign of component is positive; otherwise negative.

The position vector of a particle in circular motion is given in terms of components as :

Position vector



Position vector of a particle moving along a circular path.

Equation:

$$\mathbf{r} = r\mathbf{n} = x\mathbf{i} + y\mathbf{j} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}$$

1: Velocity

The velocity of the particle, therefore, is obtained by differentiating with respect to time,:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(-r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \right) \frac{d\theta}{dt}$$

Equation:

$$\mathbf{v} = (-r\omega \sin \theta \mathbf{i} + r\omega \cos \theta \mathbf{j}) = -r\omega \mathbf{t}$$

where $\omega = d\theta/dt$ is angular velocity. Also note that velocity is directed tangentially to path. For this reason, velocity vector is expressed with the help of unit vector in tangential direction.

2: Acceleration

The acceleration of the particle is obtained by differentiating the above expression of velocity with respect to time. However, as the radius of the circle is a constant, we take the same out of the differentiation,

Equation:

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \left\{ r\omega \frac{d}{dt} \left(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \right) \right\} + \left\{ r \left(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \right) \frac{d\omega}{dt} \right\} \\ \Rightarrow \mathbf{a} &= \left\{ r\omega \left(-\cos \theta \mathbf{i} - \sin \theta \mathbf{j} \right) \frac{d\theta}{dt} \right\} + \left\{ r \left(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \right) \frac{d\omega}{dt} \right\} \\ \Rightarrow \mathbf{a} &= -r\omega^2 \mathbf{n} + r \frac{d\omega}{dt} \mathbf{t} \\ \Rightarrow \mathbf{a} &= -\frac{v^2}{r} \mathbf{n} + \frac{dv}{dt} \mathbf{t} \end{aligned}$$

Thus, we see that :

$$\begin{aligned} \Rightarrow a_T &= \frac{dv}{dt} \\ \Rightarrow a_N &= -\frac{v^2}{r} \end{aligned}$$

The above expressions, therefore, give two components of total acceleration in two specific directions. Again, we should emphasize that these directions are not the same as coordinate directions.

The derivation of acceleration components for two dimensional motion has, though, been carried out for circular motion, but the concepts of acceleration components as defined here can be applied - whenever there is curvature of path (non-linear path). In the case of rectilinear motion, normal acceleration reduces to zero as radius of curvature is infinite and as such total acceleration becomes equal to tangential acceleration.

Elliptical motion

In order to illustrate the features of two dimensional motion, we shall consider the case of elliptical motion of a particle in a plane. We shall use this motion to bring out the basic elements associated with the understanding of acceleration and its relation with other attributes of motion.

It is important that we work with the examples without any pre-notion such as “constant” acceleration etc. The treatment here is very general and intuitive of the various facets of accelerated motion in two dimensions.

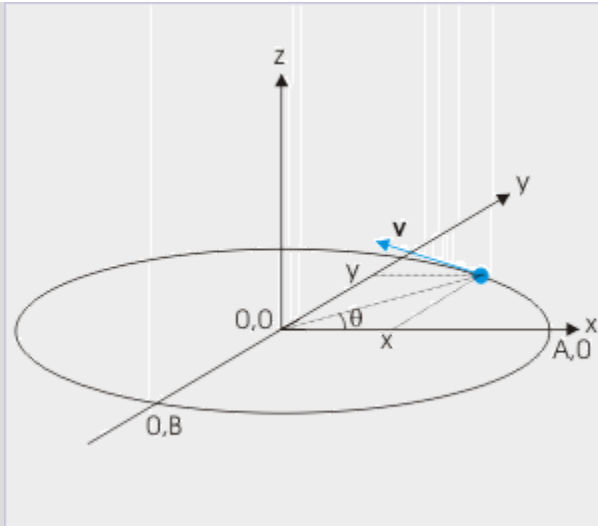
Path of motion

Example:

Problem : The coordinates of a particle moving in a plane are given by $x = A \cos(\omega t)$ and $y = B \sin(\omega t)$ where $A, B (< A)$ and ω are positive constants. Find the nature of path of motion.

Solution : We shall use the general technique to find path of motion in two dimensional case. In order to find the path motion, we need to have an equation that connects “x” and “y” coordinates of the planar coordinate system. Note that there is no third coordinate.

Elliptical motion



Motion of a particle moving along an elliptical path path.

An inspection of the expressions of “x” and “y” suggests that we can use the trigonometric identity,

$$\sin^2 \theta + \cos^2 \theta = 1$$

Here, we have :

$$\begin{aligned} x &= A \cos(\omega t) \\ \Rightarrow \cos(\omega t) &= \frac{x}{A} \end{aligned}$$

Similarly, we have :

$$\begin{aligned} y &= B \sin(\omega t) \\ \Rightarrow \sin(\omega t) &= \frac{y}{B} \end{aligned}$$

Squaring and adding two equations,

$$\begin{aligned} \sin^2(\omega t) + \cos^2(\omega t) &= 1 \\ \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} &= 1 \end{aligned}$$

This is an equation of ellipse. Hence, the particle follows an elliptical path.

Nature of velocity and acceleration

Example:

Problem : The coordinates of a particle moving in a plane are given by $x = A \cos(\omega t)$ and $y = B \sin(\omega t)$ where $A, B (< A)$ and ω are positive constants. Investigate the nature of velocity and acceleration for this motion. Also, discuss the case for $A = B$ and when " ω " is constant.

Solution : We can investigate the motion as required if we know expressions of velocity and acceleration. Therefore, we need to determine velocity and acceleration. Since components of position are given, we can find components of velocity and acceleration by differentiating the expression with respect to time.

1: Velocity

The components of velocity in "x" and "y" directions are :

$$\begin{aligned}\frac{dx}{dt} &= v_x = -A\omega \sin(\omega t) \\ \frac{dy}{dt} &= v_y = B\omega \cos(\omega t)\end{aligned}$$

The velocity of the particle is given by :

$$\Rightarrow \mathbf{v} = \omega \{-A \sin(\omega t)\mathbf{i} + B \cos(\omega t)\mathbf{j}\}$$

Evidently, magnitude and direction of the particle varies with time.

2: Acceleration

We find the components of acceleration by differentiating again, as :

$$\begin{aligned}\frac{d^2x}{dt^2} &= a_x = -A\omega^2 \cos(\omega t) \\ \frac{d^2y}{dt^2} &= a_y = -B\omega^2 \sin(\omega t)\end{aligned}$$

Both “x” and “y” components of the acceleration are trigonometric functions. This means that acceleration varies in component direction. The net or resultant acceleration is :

$$\Rightarrow \mathbf{a} = -\omega^2 \{ A \cos(\omega t) \mathbf{i} + B \sin(\omega t) \mathbf{j} \}$$

3: When $A = B$ and " ω " is constant

When $A = B$, the elliptical motion reduces to circular motion. Its path is given by the equation :

$$\begin{aligned} \frac{x^2}{A^2} + \frac{y^2}{B^2} &= 1 \\ \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{A^2} &= 1 \\ \Rightarrow x^2 + y^2 &= A^2 \end{aligned}$$

This is an equation of circle of radius “A”. The speed for this condition is given by :

$$\begin{aligned} v &= \sqrt{\{ A^2 \omega^2 \sin^2(\omega t) + A^2 \omega^2 \cos^2(\omega t) \}} \\ v &= A\omega \end{aligned}$$

Thus, speed becomes a constant for circular motion, when $\omega = \text{constant}$. The magnitude of acceleration is :

$$\begin{aligned} a &= \omega^2 \sqrt{\{ A^2 \sin^2(\omega t) + A^2 \cos^2(\omega t) \}} \\ a &= A\omega^2 \end{aligned}$$

Thus, acceleration becomes a constant for circular motion, when $\omega = \text{constant}$.

Application

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a

verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

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We discuss problems, which highlight certain aspects of the study leading to the accelerated motion in two dimensions. The questions are categorized in terms of the characterizing features of the subject matter :

- Path of motion
- Tangential and normal accelerations
- Nature of motion
- Displacement in two dimensions

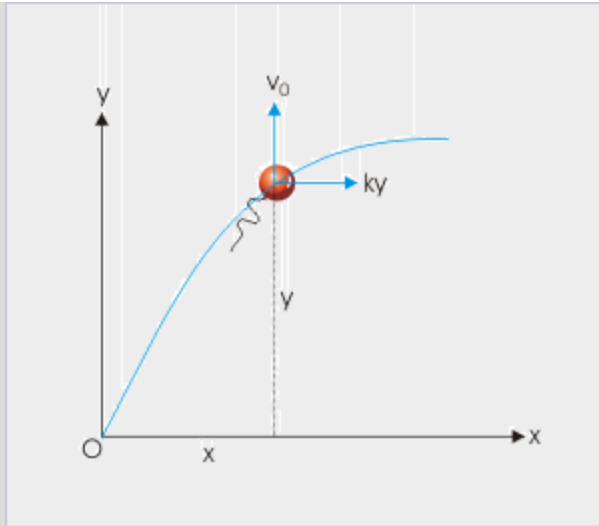
Path of motion

Example:

Problem : A balloon starts rising from the surface with a constant upward velocity, “ v_0 ”. The balloon gains a horizontal drift due to the wind. The horizontal drift velocity is given by “ ky ”, where “ k ” is a constant and “ y ” is the vertical height of the balloon from the surface. Derive an expression of path of the motion.

Solution : An inspection of the equation of drift velocity ($v = ky$) suggests that balloon drifts more with the gain in height. A suggestive x-y plot of the motion is shown here.

Motion of a balloon



The balloon moves with an acceleration in horizontal direction.

Let vertical and horizontal direction corresponds to “y” and “x” axes of the coordinate system. Here,

$$v_y = v_0$$

$$v_x = ky$$

We are required to know the relation between vertical and horizontal components of displacement from the expression of component velocities. It means that we need to know a lower order attribute from higher order attribute. Thus, we shall proceed with integration of differential equation, which defines velocity as :

$$\Rightarrow \frac{dx}{dt} = ky$$

Similarly,

$$\frac{dy}{dt} = v_0$$

$$\Rightarrow \dot{y} = v_0 \dot{t}$$

Combining two equations by eliminating “dt”,

$$dx = \frac{ky \dot{y}}{v_0}$$

Now, integrating both sides, we have :

$$x = \int \frac{ky \dot{y}}{v_0}$$

Taking out constants out of the integral,

$$\Rightarrow x = \frac{k}{v_0} \int y \dot{y}$$

$$x = \frac{ky^2}{2v_0}$$

This is the required equation of motion, which is an equation of a parabola. Thus, the suggested plot given in the beginning, as a matter of fact, was correct.

Tangential and normal accelerations

Example:

Problem : A balloon starts rising from the surface with a constant upward velocity, “ v_0 ”. The balloon gains a horizontal drift due to the wind. The horizontal drift velocity is given by “ky”, where “k” is a constant and “y” is the vertical height of the balloon from the surface. Derive expressions for the tangential and normal accelerations of the balloon.

Solution : We can proceed to find the magnitude of total acceleration by first finding the expression of velocity. Here, velocity is given as :

$$\mathbf{v} = ky\mathbf{i} + v_0\mathbf{j}$$

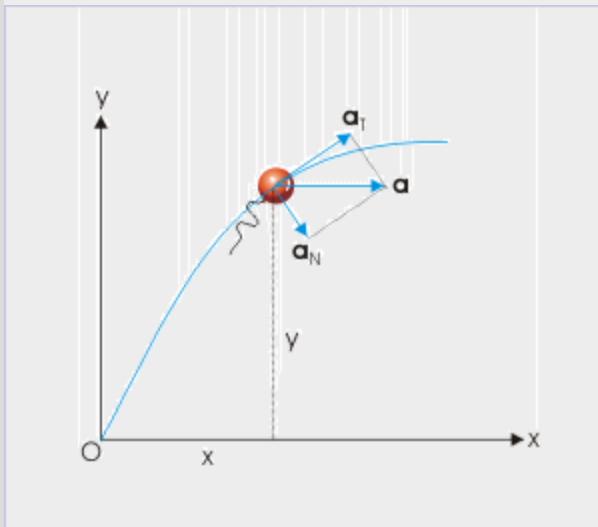
Since acceleration is higher order attribute, we obtain its expression by differentiating the expression of velocity with respect to time :

$$\Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = kv_y\mathbf{i} = kv_0\mathbf{i}$$

It is obvious that acceleration is one dimensional. It is evident from the data given also. The balloon moving with constant vertical velocity has no acceleration in y-direction. The speed of the balloon in x-direction, however, keeps changing with height (time) and as such total acceleration of the balloon is in x-direction. The magnitude of total acceleration is :

$$a = |\mathbf{a}| = kv_y = kv_0$$

Motion of a balloon



The acceleration of the balloon has two components in mutually perpendicular directions.

Thus, we see that total acceleration is not only one dimensional, but constant as well. However, this does not mean that component accelerations viz tangential and normal accelerations are also constant. We

need to investigate their expressions. We can obtain tangential acceleration as time rate of change of the magnitude of velocity i.e. the time rate of change of speed. We, therefore, need to first know an expression of the speed. Now, speed is :

$$v = \sqrt{(ky)^2 + (v_0)^2}$$

Differentiating with respect to time, we have :

$$\Rightarrow a_T = \frac{dv}{dt} = \frac{2k^2y}{2\sqrt{(k^2y^2 + v_0^2)}} \times \frac{dy}{dt}$$

$$\Rightarrow a_T = \frac{dv}{dt} = \frac{k^2yv_0}{\sqrt{(k^2y^2 + v_0^2)}}$$

In order to find the normal acceleration, we use the fact that total acceleration is vector sum of two mutually perpendicular tangential and normal accelerations.

$$a^2 = a_T^2 + a_N^2$$

$$\Rightarrow a_N^2 = a^2 - a_T^2 = k^2v_0^2 - \frac{k^4y^2v_0^2}{(k^2y^2 + v_0^2)}$$

$$\Rightarrow a_N^2 = k^2v_0^2 \left\{ 1 - \frac{k^2y^2}{(k^2y^2 + v_0^2)} \right\}$$

$$\Rightarrow a_N^2 = k^2v_0^2 \left\{ \frac{k^2y^2 + v_0^2 - k^2y^2}{(k^2y^2 + v_0^2)} \right\}$$

$$\Rightarrow a_N^2 = \frac{k^2v_0^4}{(k^2y^2 + v_0^2)}$$

$$\Rightarrow a_N = \frac{kv_0^2}{\sqrt{(k^2y^2 + v_0^2)}}$$

Nature of motion

Example:

Problem : The coordinates of a particle moving in a plane are given by $x = A \cos(\omega t)$ and $y = B \sin(\omega t)$ where $A, B (< A)$ and “ ω ” are positive constants of appropriate dimensions. Prove that the velocity and acceleration of the particle are normal to each other at $t = \pi/2\omega$.

Solution : By differentiation, the components of velocity and acceleration are as given under :

The components of velocity in “x” and “y” directions are :

$$\frac{dx}{dt} = v_x = -A\omega \sin \omega t$$

$$\frac{dy}{dt} = v_y = B\omega \cos \omega t$$

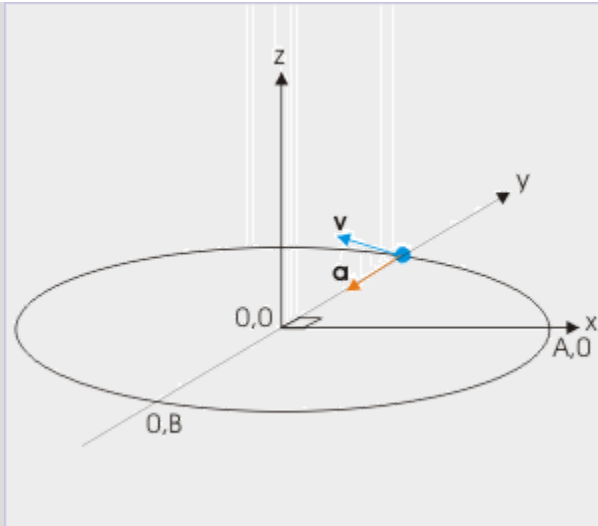
The components of acceleration in “x” and “y” directions are :

$$\frac{d^2x}{dt^2} = a_x = -A\omega^2 \cos \omega t$$

$$\frac{d^2y}{dt^2} = a_y = -B\omega^2 \sin \omega t$$

At time, $t = \frac{\pi}{2\omega}$ and $\theta = \omega t = \frac{\pi}{2}$. Putting this value in the component expressions, we have :

Motion along elliptical path



Velocity and acceleration are perpendicular at the given instant.

$$\Rightarrow v_x = -A\omega \sin \omega t = -A\omega \sin \pi/2 = -A\omega$$

$$\Rightarrow v_y = B\omega \cos \omega t = B\omega \cos \pi/2 = 0$$

$$\Rightarrow a_x = -A\omega^2 \cos \omega t = -A\omega^2 \cos \pi/2 = 0$$

$$\Rightarrow a_y = -B\omega^2 \sin \omega t = -B\omega^2 \sin \pi/2 = -B\omega^2$$

The net velocity is in negative x-direction, whereas net acceleration is in negative y-direction. Hence at $t = \frac{\pi}{2\omega}$, velocity and acceleration of the particle are normal to each other.

Example:

Problem : Position vector of a particle is :

$$\mathbf{r} = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

Show that velocity vector is perpendicular to position vector.

Solution : We shall use a different technique to prove as required. We shall use the fact that scalar (dot) product of two perpendicular vectors is zero. We, therefore, need to find the expression of velocity. We can obtain the same by differentiating the expression of position vector with respect to time as :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a \sin \omega t \mathbf{i} + a \cos \omega t \mathbf{j}$$

To check whether velocity is perpendicular to the position vector, we take the scalar product of $\mathbf{r}$ and $\mathbf{v}$ as :

$$\Rightarrow \mathbf{r} \cdot \mathbf{v} = (a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}) \cdot (-a \sin \omega t \mathbf{i} + a \cos \omega t \mathbf{j})$$

$$\Rightarrow \mathbf{r} \cdot \mathbf{v} = -a \sin \omega t \cos \omega t + a \sin \omega t \cos \omega t = 0$$

This means that the angle between position vector and velocity are at right angle to each other. Hence, velocity is perpendicular to position vector.

Displacement in two dimensions

Example:

Problem : The coordinates of a particle moving in a plane are given by $x = A \cos(\omega t)$ and $y = B \sin(\omega t)$ where $A, B (< A)$ and ω are positive constants of appropriate dimensions. Find the displacement of the particle in time interval $t = 0$ to $t = \pi/2 \omega$.

Solution : In order to find the displacement, we shall first know the positions of the particle at the start of motion and at the given time. Now, the position of the particle is given by coordinates :

$$x = A \cos \omega t$$

and

$$y = B \sin \omega t$$

At $t = 0$, the position of the particle is given by :

$$\Rightarrow x = A \cos(\omega \cdot 0) = A \cos 0 = A$$

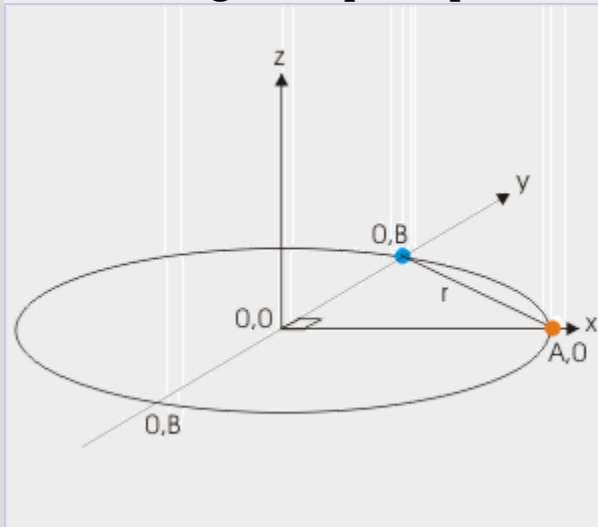
$$\Rightarrow y = B \sin(\omega \cdot 0) = B \sin 0 = 0$$

At $t = \frac{\pi}{2\omega}$, the position of the particle is given by :

$$\Rightarrow x = A \cos(\omega \cdot \frac{\pi}{2\omega}) = A \cos \pi/2 = 0$$

$$\Rightarrow y = B \sin(\omega \cdot \frac{\pi}{2\omega}) = B \sin \pi/2 = B$$

Motion along an elliptical path



The linear distance equals displacement.

Therefore , the displacement in the given time interval is :

$$r = \sqrt{(A^2 + B^2)}$$

Transformation of graphs

Transformation of graphs means changing graphs. This generally allows us to draw graphs of more complicated functions from graphs of basic or simpler functions by applying different transformation techniques. It is important to emphasize here that plotting a graph is an extremely powerful technique and method to know properties of a function such as domain, range, periodicity, polarity and other features which involve differentiability of a function. Subsequently, we shall see that plotting enables us to know these properties more elegantly and easily as compared to other analytical methods.

Graphing of a given function involves modifying graph of a core function. We modify core function and its graph, applying various mathematical operations on the core function. There are two fundamental ways in which we operate on core function and hence its graph. We can either modify input to the function or modify output of function.

Broad categories of transformation

- Transformation applied by modification to input
- Transformation applied by modification to output
- Transformation applied by modulus function
- Transformation applied by greatest integer function
- Transformation applied by fraction part function
- Transformation applied by least integer function

We shall cover first transformation in this module. Others will be taken up in other modules.

Important concepts

Graph of a function

It is a plot of values of function against independent variable x . The value of function changes in accordance with function rule as x changes. Graph depicts these changes pictorially. In the current context, both core function

and modified function graphs are plotted against same independent variable x .

Input to the function

What is input to the function? How do we change input to the function? Values are passed to the function through argument of the function. The argument itself is a function in x i.e. independent variable. The simplest form of argument is " x " like in function $f(x)$. The modified arguments are " $2x$ " in function $f(2x)$ or " $2x-1$ " in function $f(2x-1)$. This changes input to the function. Important to underline is that independent variable x remains what it is, but argument of the function changes due to mathematical operation on independent variable. Thus, we modify argument through mathematical operation on independent variable x . Basic possibilities of modifying argument i.e. input by using arithmetic operations on x are addition, subtraction, multiplication, division and negation. In notation, we write modification to the input of the function as :

$$\text{Argument/input} = bx + c; \quad b, c \in R$$

These changes are called internal or pre-composition modifications.

Output of the function

A modification in input to the graph is reflected in the values of the function. This is one way of modifying output and hence corresponding graph. Yet another approach of changing output is by applying arithmetic operations on the function itself. We shall represent such arithmetic operations on the function as :

$$af(x) + d; \quad a, d \in R$$

These changes are called external or post-composition modifications.

Arithmetic operations

Addition/subtraction operations

Addition and subtraction to independent variable x is represented as :

$$x + c; \quad c \in R$$

The notation represents addition operation when c is positive and subtraction when c is negative. In particular, we should underline that notation “ $bx+c$ ” does not represent addition to independent variable. Rather it represents addition/ subtraction to “ bx ”. We shall develop proper algorithm to handle such operations subsequently. Similarly, addition and subtraction operation on function is represented as :

$$f(x) + d; \quad d \in R$$

Again, “ $af(x) + d$ ” is addition/ subtraction to “ $af(x)$ ” not to “ $f(x)$ ”.

Product/division operations

Product and division operations are defined with a positive constant for both independent variable and function. It is because negation i.e. multiplication or division with -1 is a separate operation from the point of graphical effect. In the case of product operation, the magnitude of constants (a or b) is greater than 1 such that resulting value is greater than the original value.

$$bx; \quad |b| > 1 \quad \text{for independent variable}$$

$$af(x); \quad |a| > 1 \quad \text{for function}$$

The division operation is equivalent to product operation when value of multiplier is less than 1. In this case, magnitude of constants (a or b) is less than 1 such that resulting value is less than the original value.

$$bx; \quad 0 < |b| < 1 \quad \text{for independent variable}$$

$$af(x); \quad 0 < |a| < 1 \quad \text{for function}$$

Negation

Negation means multiplication or division by -1.

Effect of arithmetic operations

Addition/ subtraction operation on independent variable results in shifting of core graph along x-axis i.e horizontally. Similarly, product/division operations results in scaling (shrinking or stretching) of core graph horizontally. The change in graphs due to negation is reflected as mirroring (across y-axis) horizontally. Clearly, modifications resulting from modification to input modifies core graph horizontally. Another important aspect of these modification is that changes takes place opposite to that of operation on independent variable. For example, when “2” is added to independent variable, then core graph shifts left which is opposite to the direction of increasing x. A multiplication by 2 shrinks the graph horizontally by a factor 2, whereas division by 2 stretches the graph by a factor of 2.

On the other hand, modification in the output of function is reflected in change in graphs along y-axis i.e. vertically. Effects such as shifting, scaling (shrinking or stretching) or mirroring across x-axis takes place in vertical direction. Also, the effect of modification in output is in the direction of modification as against effects due to modifications to input. A multiplication of function by a positive constant greater than 1, for example, stretches the graph in y-direction as expected. These aspects will be clear as we study each of the modifications mentioned here.

Forms of representation

There is a bit of ambiguity about the nature of constants in symbolic representation of transformation. Consider the representation,

$$af(bx + c) + d; \quad a, b, c \in R$$

In this case "a", "b", "c" and "d" can be either positive or negative depending on the particular transformation. A positive "d" means that graph is shifted up. On the other hand, we can specify constants to be positive in the following representation :

$$\pm af(\pm bx \pm c) \pm d; \quad a, b, c > 0$$

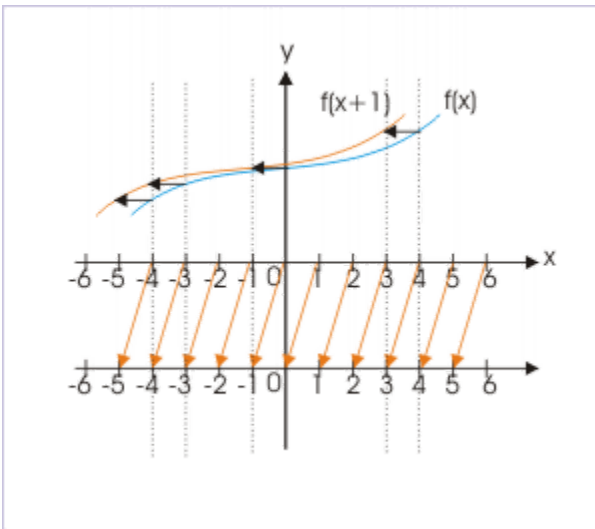
The form of representation appears to be cumbersome, but is more explicit in its intent. It delinks sign from the magnitude of constants. In this case, the signs preceding positive constants need to be interpreted for the nature of transformation. For example, a negative sign before c denotes right horizontal shift. It is, however, clear that both representations are essentially equivalent and their use depends on personal choice or context. This difference does not matter so long we understand the process of graphing.

Transformation of graph by input

Addition and subtraction to independent variable

In order to understand this type of transformation, we need to explore how output of the function changes as input to the function changes. Let us consider an example of functions $f(x)$ and $f(x+1)$. The integral values of independent variable are same as integral values on x-axis of coordinate system. Note that independent variable is plotted along x-axis as real number line. The integral $x+1$ values to the function $f(x+1)$ - such that input values are same as that of $f(x)$ - are shown on a separate line just below x-axis. The corresponding values are linked with arrow signs. Input to the function $f(x+1)$ which is same as that of $f(x)$ corresponds to x which is 1 unit smaller. It means graph of $f(x+1)$ is same as graph of $f(x)$, which has been shifted by 1 unit towards left. Else, we can say that the origin of plot (also x-axis) has shifted right by 1 unit.

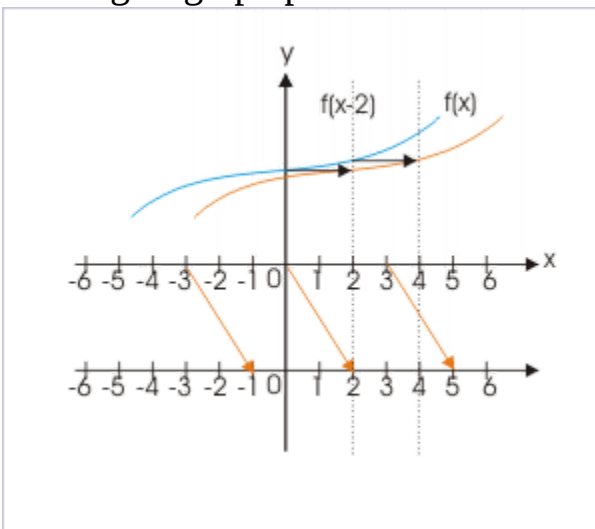
Shifting of graph parallel to x-axis



Each element of graph is shifted left by same value.

Let us now consider an example of functions $f(x)$ and $f(x-2)$. Input to the function $f(x-2)$ which is same as that of $f(x)$ now appears 2 unit later on x-axis. It means graph of $f(x-2)$ is same as graph of $f(x)$, which has been shifted by 2 units towards right. Else, we can say that the origin of plot (also x-axis) has shifted left by 2 units.

Shifting of graph parallel to x-axis



Each element of graph is shifted

right by same value.

The addition/subtraction transformation is depicted symbolically as :

$$y = f(x) \Rightarrow y = f(x \pm |a|); \quad |a| > 0$$

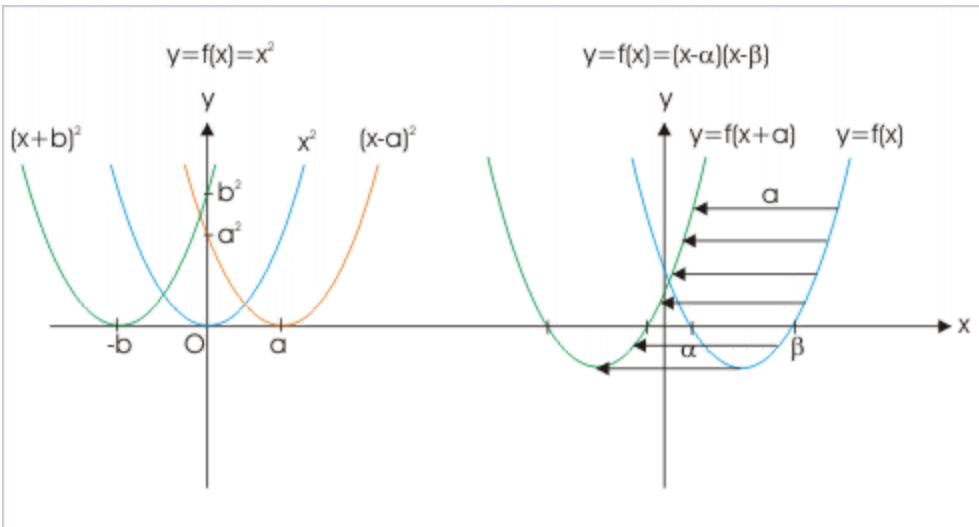
If we add a positive constant to the argument of the function, then value of y at $x=x$ in the new function $y=f(x+|a|)$ is same as that of $y=f(x)$ at $x=x-|a|$. For this reason, the graph of $f(x+|a|)$ is same as the graph of $y=f(x)$ shifted left by unit “ a ” in x -direction. Similarly, the graph of $f(x-|a|)$ is same as the graph of $y=f(x)$ shifted right by unit “ a ” in x -direction.

1 : The plot of $y=f(x+|a|)$; is the plot of $y=f(x)$ shifted left by unit “ $|a|$ ”.

2 : The plot of $y=f(x-|a|)$; is the plot of $y=f(x)$ shifted right by unit “ $|a|$ ”.

We use these facts to draw graph of transformed function $f(x\pm a)$ by shifting graph of $f(x)$ by unit “ a ” in x -direction. Each point forming the plot is shifted parallel to x -axis (see quadratic graph shown in the figure below). The graph in the center of left figure depicts monomial function $y = x^2$ with vertex at origin. It is shifted right by “ a ” units ($a>0$) and the function representing shifted graph is $y = (x - a)^2$. Note that vertex of parabola is shifted from (0,0) to (a,0). Further, the graph is shifted left by “ b ” units ($b>0$) and the function representing shifted graph is $y = (x + b)^2$. In this case, vertex of parabola is shifted from (0,0) to (-b,0).

Shifting of graph parallel to x -axis



Each element of graph is shifted by same value.

Example:

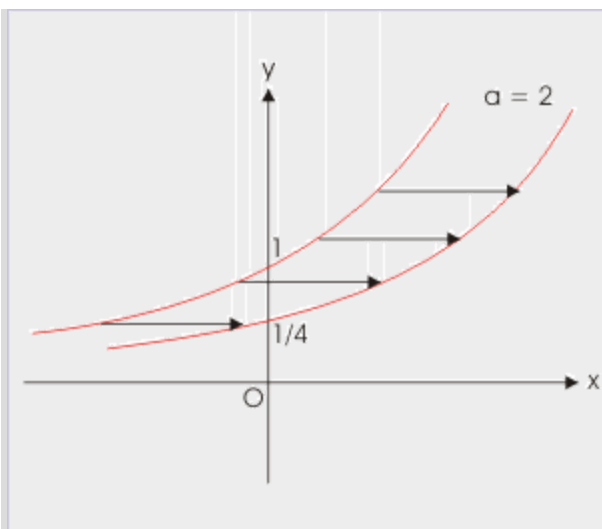
Problem : Draw graph of function $4y = 2^x$.

Solution : Given function is exponential function. On simplification, we have :

$$\Rightarrow y = 2^{-2} \times 2^x = 2^{x-2}$$

Here, core graph is $y = 2^x$. We draw its graph first and then shift the graph right by 2 units to get the graph of given function.

Shifting of exponential graph parallel to x-axis



Each element of graph is shifted
by same value.

Note that the value of function at $x=0$ for core and modified functions, respectively, are :

$$\Rightarrow y = 2^x = 2^0 = 1$$

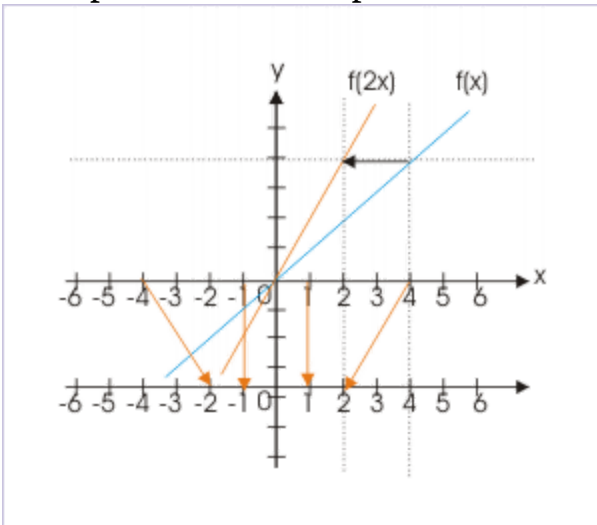
$$\Rightarrow y = 2^{x-2} = 2^{-2} = \frac{1}{4} = 0.25$$

Multiplication and division of independent variable

Let us consider an example of functions $f(x)$ and $f(2x)$. The integral values of independent variable are same as integral values on x -axis of coordinate system. Note that independent variable is plotted along x -axis as real number line. The integral $2x$ values to the function $f(2x)$ - such that input values are same as that of $f(x)$ - are shown on a separate line just below x -axis. The corresponding values are linked with arrow signs. Input to the function $f(2x)$ which is same as that of $f(x)$ now appears closer to origin by a factor of 2. It means graph of $f(2x)$ is same as graph of $f(x)$, which has

been shrunk by a factor 2 towards origin. Else, we can say that x-axis has been stretched by a factor 2.

Multiplication of independent variable

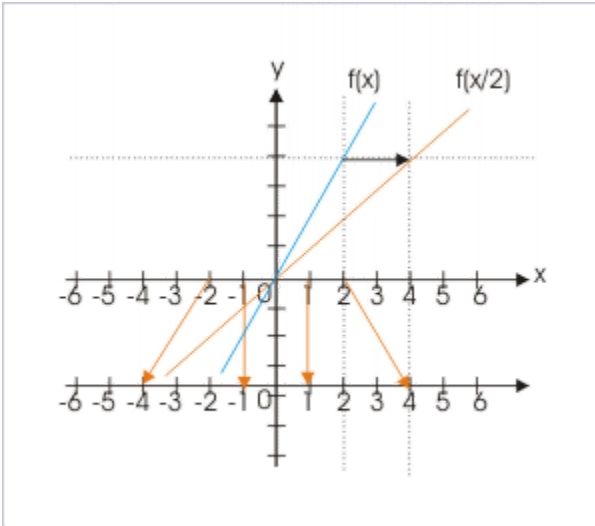


The graph shrinks towards origin.

$$y = f(x) \quad \Rightarrow \quad y = f(bx); \quad |b| > 1$$

Let us consider another example of functions $f(x)$ and $f(x/2)$. The integral values of independent variable are same as integral values on x-axis of coordinate system. Note that independent variable is plotted along x-axis as real number line. The integral $x/2$ values to the function $f(x/2)$ - such that input values are same as that of $f(x)$ - are shown on a separate line just below x-axis. The corresponding values are linked with arrow signs. Input to the function $f(x/2)$ which is same as that of $f(x)$ now appears away from origin by a factor of 2. It means graph of $f(x/2)$ is same as graph of $f(x)$, which has been stretched by a factor 2 away from origin. Else, we can say that x-axis has been shrunk by a factor 2.

Multiplication of independent variable



The graph stretches away from origin.

$$y = f(x) \quad \Rightarrow \quad y = f\left(\frac{x}{b}\right); \quad |b| > 1$$

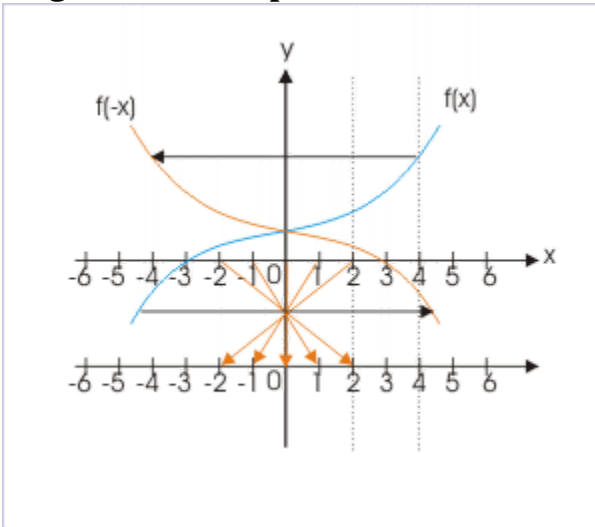
Important thing to note about horizontal scaling (shrinking or stretching) is that it takes place with respect to origin of the coordinate system and along x-axis – not about any other point and not along y-axis. What it means that behavior of graph at $x=0$ remains unchanged. In equivalent term, we can say that y-intercept of graph remains same and is not affected by scaling resulting from multiplication or division of the independent variable.

Negation of independent variable

Let us consider an example of functions $f(x)$ and $f(-x)$. The integral values of independent variable are same as integral values on x-axis of coordinate system. Note that independent variable is plotted along x-axis as real number line. The integral $-x$ values to the function $f(-x)$ - such that input values are same as that of $f(x)$ - are shown on a separate line just below x-axis. The corresponding values are linked with arrow signs. Input to the

function $f(-x)$ which is same as that of $f(x)$ now appears to be flipped across y-axis. It means graph of $f(-x)$ is same as graph of $f(x)$, which is mirror image in y-axis i.e. across y-axis.

Negation of independent variable



The graph flipped across y. - axis.

The form of transformation is depicted as :

$$y = f(x) \Rightarrow y = f(-x)$$

A graph of a function is drawn for values of x in its domain. Depending on the nature of function, we plot function values for both negative and positive values of x . When sign of the independent variable is changed, the function values for negative x become the values of function for positive x and vice-versa. It means that we need to flip the plot across y-axis. In the nutshell, the graph of $y=f(-x)$ can be obtained by taking mirror image of the graph of $y=f(x)$ in y-axis.

While using this transformation, we should know about even function. For even function. $f(x)=f(-x)$. As such, this transformation will not have any implication for even functions as they are already symmetric about y-axis. It means that two parts of the graph of even function across y-axis are

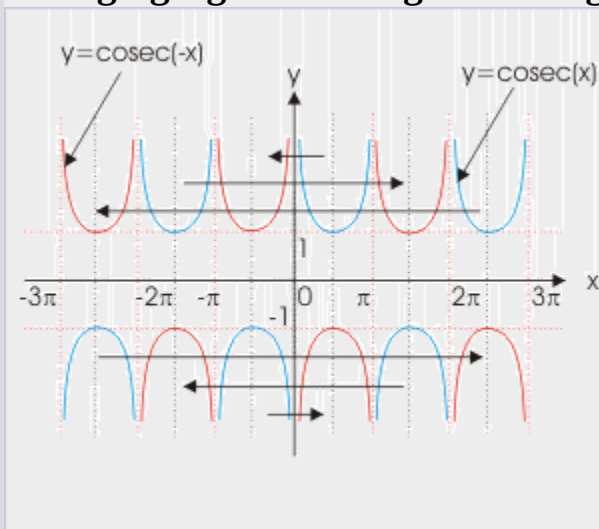
image of each other. For this reason, $y = \cos(-x) = \cos(x)$, $y = |-x| = |x|$ etc. The graphs of these even functions are not affected by change in sign of independent variable.

Example:

Problem : Draw graph of $y = \operatorname{cosec}(-x)$ function

Solution : The plot is obtained by plotting image of core graph $y = \operatorname{cosec}(x)$ in y axis.

Changing sign of the argument of graph



The transformed graph is image of core graph in y-axis.

Combined input operations

Certain function are derived from core function as a result of multiple arithmetic operations on independent variable. Consider an example :

$$f(x) = -2x - 2$$

We can consider this as a function composition which is based on identity function $f(x) = x$ as core function. From the composition, it is apparent that order of formation consists of operations as :

- (i)** $f(2x)$ i.e. multiply independent variable by 2 i.e. shrink the graph horizontally by half.
- (ii)** $f(-2x)$ i.e. negate independent variable x i.e. flip the graph across y -axis.
- (iii)** $f(-2x-2)$ i.e. subtract 2 from $-2x$.

This sequence of operation is not correct for the reason that third operation is a subtraction operation to $-2x$ not to independent variable x , whereas we have defined transformation for subtraction from independent variable. The order of operation for transformation resulting from modifications to input can, therefore, be determined using following considerations :

1 : Order of operations for transformation due to input is opposite to the order of composition.

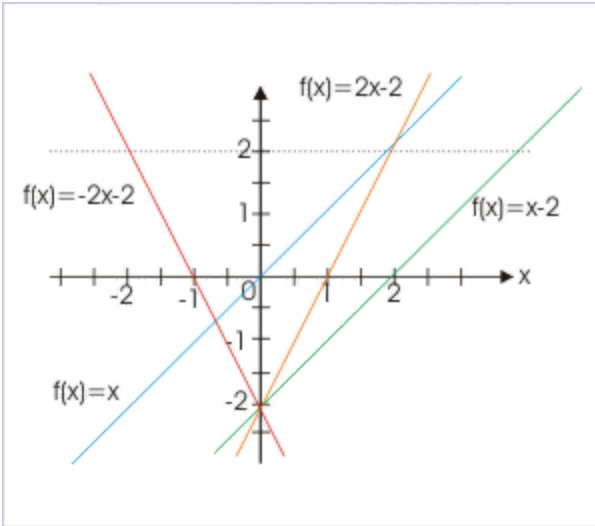
2 : Precedence of addition/subtraction is higher than that of multiplication/division.

Keeping above two rules in mind, let us rework transformation steps :

- (i)** $f(x-2)$ i.e. subtract 2 from independent variable x i.e. shift the graph right by 2 units.
- (ii)** $f(2x-2)$ i.e. multiply independent variable x by 2 i.e. shrink the graph horizontally by half.
- (iii)** $f(-2x-2)$ i.e. negate independent variable i.e. flip the graph across y -axis.

This is the correct sequence as all transformations involved are as defined. The resulting graph is shown in the figure below :

Graph of transformed function



Operations are carried in sequence.

It is important the way graph is shrunk horizontally towards origin. Important thing is to ensure that y-intercept is not changed. It can be seen that function before being shrunk is :

$$f(x) = x - 2$$

Its y-intercept is 2. When the graph is shrunk by a factor by 2, the function is :

$$f(x) = 2x - 2$$

The y-intercept is again 2. The graph moves 1 unit half of x-intercept towards origin. Further, we can verify validity of critical points like x and y intercepts to ensure that transformation steps are indeed correct. Here,

$$x = 0, \quad y = -2X0 - 2 = -2$$

$$y = 0, \quad x = -\frac{(y + 2)}{2} = -\frac{2}{2} = -1$$

We can decompose a given function in more than one ways so long transformations are valid as defined. Can we rewrite function as $y = f\{-2(x+1)\}$? Let us see :

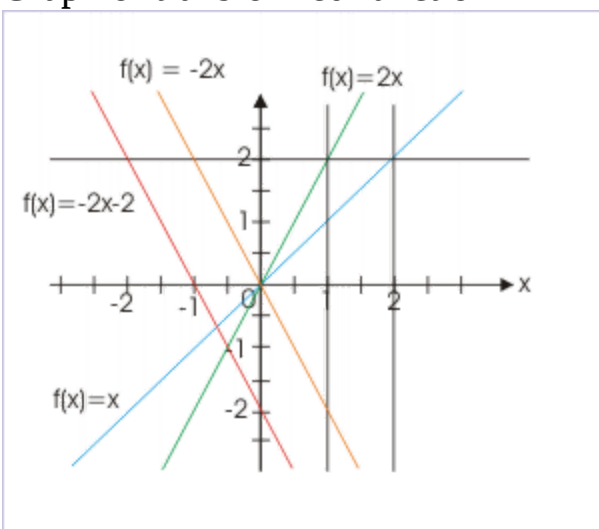
(i) $f(2x)$ i.e. multiply independent variable x by 2 i.e. i.e. shrink the graph horizontally by half.

(ii) $f(-2x)$ i.e. negate independent variable i.e. flip the graph across y -axis.

(iii) $f\{-2(x+1)\}$ i.e. add 1 to independent variable x i.e. shift the graph left by 1 unit.

This decomposition is valid as transformation steps are consistent with the transformations allowed for arithmetic operations on independent variable.

Graph of transformed function



Operations are carried in sequence.

Horizontal shift

We have discussed transformation resulting in horizontal shift. In the simple case of operation with independent variable alone, the horizontal shift is “c”. In this case, transformation is represented by $f(x+c)$. What is horizontal shift for more general case of transformation represented by $f(bx+c)$? Let us rearrange argument of the function,

$$f(bx + c) = f\left\{b\left(x + \frac{c}{b}\right)\right\}$$

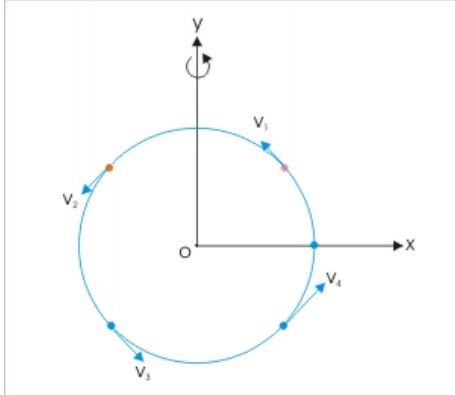
Comparing with $f(x+c)$, horizontal shift is given by :

$$\text{Horizontal shift} = \frac{c}{b}$$

Non-uniform circular motion

What do we mean by non-uniform circular motion? The answer lies in the definition of uniform circular motion, which defines it to be a circular motion with constant speed. It follows then that non-uniform circular motion denotes a change in the speed of the particle moving along the circular path as shown in the figure. Note specially the change in the velocity vector sizes, denoting change in the magnitude of velocity.

Circular motion



The speed of the particle changes with time in non-uniform circular motion.

A change in speed means that unequal length of arc (s) is covered in equal time intervals. It further means that the change in the velocity ($\mathbf{v}$) of the particle is not limited to change in direction as in the case of uniform circular motion. In other words, the magnitude of the velocity ($\mathbf{v}$) changes with time, in addition to continuous change in direction, which is inherent to the circular motion owing to the requirement of the particle to follow a non-linear circular path.

Radial (centripetal) acceleration

We have seen that change in direction is accounted by radial acceleration (centripetal acceleration), which is given by following relation,

$$a_R = \frac{v^2}{r}$$

The change in speed have implications on radial (centripetal) acceleration. There are two possibilities :

- 1: The radius of circle is constant (like in the motion along a circular rail or motor track)

A change in “ v ” shall change the magnitude of radial acceleration. This means that the centripetal acceleration is not constant as in the case of uniform circular motion. Greater the speed, greater is the radial acceleration. It can be easily visualized that a particle moving at higher speed will need a greater radial force to change direction and vice-versa, when radius of circular path is constant.

2: The radial (centripetal) force is constant (like a satellite rotating about the earth under the influence of constant force of gravity)

The circular motion adjusts its radius in response to change in speed. This means that the radius of the circular path is variable as against that in the case of uniform circular motion.

In any eventuality, the equation of centripetal acceleration in terms of “speed” and “radius” must be satisfied. The important thing to note here is that though change in speed of the particle affects radial acceleration, but the change in speed is not affected by radial or centripetal force. We need a tangential force to affect the change in the magnitude of a tangential velocity. The corresponding acceleration is called tangential acceleration.

Angular velocity

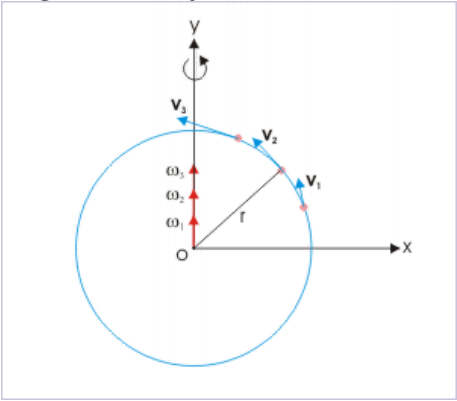
The angular velocity in non-uniform circular motion is not constant as $\omega = v/r$ and “v” is varying.

We construct a data set here to have an understanding of what is actually happening to angular speed with the passage of time. Let us consider a non-uniform circular motion of a particle in a centrifuge, whose linear speed, starting with zero, is incremented by 1 m/s at the end of every second. Let the radius of the circle be 10 m.

-----	t	v	ω	(s)	(m/s)	(rad/s)	-----
-----	0	0	0.0	1	1	0.1	-----
-----	2	2	0.2	3	3	0.3	-----
-----							-----

The above data set describes just a simplified situation for the purpose of highlighting variation in angular speed. We can visualize this change in terms of angular velocity vector with increasing magnitudes as shown in the figure here :

Angular velocity



Angular velocity changes with time in non-uniform circular motion.

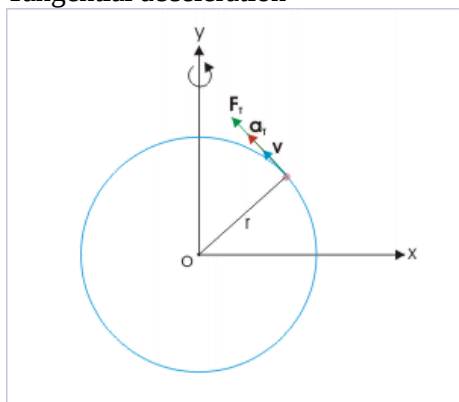
Note that magnitude of angular velocity i.e. speed changes, but not its direction in the illustrated case. However, it has been pointed out earlier that there are actually two directional possibilities i.e.

clockwise and anti-clockwise rotation. Thus, a circular motion may also involve change of direction besides a change in its magnitude.

Tangential acceleration

The non-uniform circular motion involves a change in speed. This change is accounted by the tangential acceleration, which results due to a tangential force and which acts along the direction of velocity circumferentially as shown in the figure. It is easy to realize that tangential velocity and acceleration are tangential to the path of motion and keeps changing their direction as motion progresses.

Tangential acceleration



Velocity, tangential acceleration and tangential force all act along the same direction.

We note that velocity, tangential acceleration and tangential force all act along the same direction. It must, however, be recognized that force (and hence acceleration) may also act in the opposite direction to the velocity. In that case, the speed of the particle will decrease with time.

The magnitude of tangential acceleration is equal to the time rate of change in the speed of the particle.

Equation:

$$a_T = \frac{dv}{dt}$$

Example:

Problem : A particle, starting from the position (5 m,0 m), is moving along a circular path about the origin in xy – plane. The angular position of the particle is a function of time as given here,

$$\theta = t^2 + 0.2t + 1$$

Find (i) centripetal acceleration and (ii) tangential acceleration and (iii) direction of motion at t = 0 .

Solution : From the data on initial position of the particle, it is clear that the radius of the circle is 5 m.

(i) For determining centripetal acceleration, we need to know the linear speed or angular speed at a given time. Here, we differentiate angular position function to obtain angular speed as :

$$\omega = \frac{d\theta}{dt} = 2t + 0.2$$

Angular speed is varying as it is a function of time. For $t = 0$,

$$\omega = 0.2 \text{ rad /s}$$

Now, the centripetal acceleration is :

$$a_R = \omega^2 r = (0.2)^2 \times 5 = 0.04 \times 5 = 0.2 \text{ m/s}^2$$

(ii) For determining tangential acceleration, we need to have expression of linear speed in time.

$$v = \omega r = (2t + 0.2) \times 5 = 10t + 1$$

We obtain tangential acceleration by differentiating the above function :

$$a_T = \frac{dv}{dt} = 10 \text{ m/s}^2$$

Evidently, the tangential acceleration is constant and is independent of time.

(iii) Since, the angular speed is evaluated to be positive at $t = 0$, it means that angular velocity is positive. This, in turn, means that the particle is rotating anti-clockwise at $t = 0$.

Angular acceleration

The magnitude of angular acceleration is the ratio of angular speed and time interval.

Equation:

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

If the ratio is evaluated for finite time interval, then the ratio is called average angular acceleration and If the ratio is evaluated for infinitesimally small period ($\Delta t \rightarrow 0$), then the ratio is called instantaneous angular acceleration. Mathematically, the instantaneous angular acceleration is :

Equation:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

The angular acceleration is measured in “ rad /s² ”. It is important to emphasize here that this angular acceleration is associated with the change in angular speed (ω) i.e. change in the linear speed of the particle ($v = \omega r$) - not associated with the change in the direction of the linear velocity ($\mathbf{v}$). In the case of uniform circular motion, $\omega = \text{constant}$, hence angular acceleration is zero.

Relationship between linear and angular acceleration

We can relate angular acceleration (α) with tangential acceleration (a_T) in non – uniform circular motion as :

$$a_T = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d^2}{dt^2} (r \theta) = r \frac{d^2\theta}{dt^2}$$
$$\Rightarrow a_T = \alpha r$$

We see here that angular acceleration and tangential acceleration are representation of the same aspect of motion, which is related to the change in angular speed or the equivalent linear speed. It is only the difference in the manner in which change of the magnitude of motion is described.

The existence of angular or tangential acceleration indicates the presence of a tangential force on the particle.

Note : All relations between angular quantities and their linear counterparts involve multiplication of angular quantity by the radius of circular path “r” to yield to corresponding linear equivalents. Let us revisit the relations so far arrived to appreciate this aspect of relationship :

Equation:

$$s = \theta r$$

$$v = \omega r$$

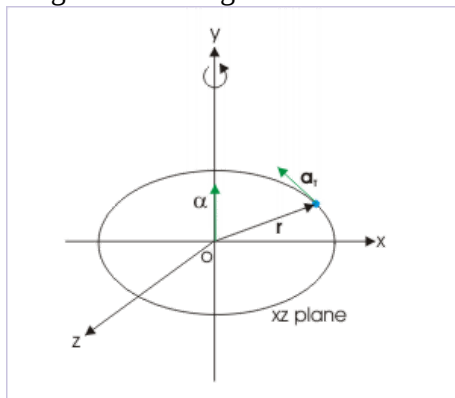
$$a_T = \alpha r$$

Linear and angular acceleration relation in vector form

We can represent the relation between angular acceleration and tangential acceleration in terms of vector cross product :

$$\mathbf{a}_T = \alpha \times \mathbf{r}$$

Tangential and angular acceleration



Angular acceleration is an axial vector.

The order of quantities in vector product is important. A change in the order of cross product like ($\mathbf{r} \times \boldsymbol{\alpha}$) represents the product vector in opposite direction. The directional relationship between these vector quantities are shown in the figure. The vectors “ $\mathbf{a}_T$ ” and “ $\mathbf{r}$ ” are in “xz” plane i.e. in the plane of motion, whereas angular acceleration ($\boldsymbol{\alpha}$) is in y-direction i.e. perpendicular to the plane of motion. We can know about tangential acceleration completely by analyzing the right hand side of vector equation. The spatial relationship among the vectors is automatically conveyed by the vector relation.

We can evaluate magnitude of tangential acceleration as :

$$\mathbf{a}_T = \boldsymbol{\alpha} \times \mathbf{r}$$

$$\Rightarrow a_T = |\mathbf{a}_T| = \alpha r \sin \theta$$

where θ is the angle between two vectors $\boldsymbol{\alpha}$ and $\mathbf{r}$. In the case of circular motion, $\theta = 90^\circ$, Hence,

$$a_T = |\boldsymbol{\alpha} \times \mathbf{r}| = \alpha r$$

Uniform circular motion

In the case of the uniform circular motion, the speed (v) of the particle in uniform circular motion is constant (by definition). This implies that tangential acceleration, a_T , is zero. Consequently, angular acceleration ($\frac{a_T}{r}$) is also zero.

$$\Rightarrow a_T = 0$$

$$\Rightarrow \alpha = 0$$

Description of circular motion using vectors

In the light of new quantities and new relationships, we can attempt analysis of the general circular motion (including both uniform and non-uniform), using vector relations. We have seen that :

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

A close scrutiny of the quantities on the right hand of the expression of velocity indicate two possible changes :

- change in angular velocity ($\boldsymbol{\omega}$) and
- change in position vector ($\mathbf{r}$)

The angular velocity ($\boldsymbol{\omega}$) can change either in its direction (clockwise or anti-clockwise) or can change in its magnitude. There is no change in the direction of axis of rotation, however, which is fixed. As far as position vector ($\mathbf{r}$) is concerned, there is no change in its magnitude i.e. $|\mathbf{r}|$ or r is constant, but its direction keeps changing with time. So there is only change of direction involved with vector “ $\mathbf{r}$ ”.

Now differentiating the vector equation, we have

$$\frac{d\mathbf{v}}{dt} = \frac{d\omega}{dt} \mathbf{X}\mathbf{r} + \omega \mathbf{X} \frac{d\mathbf{r}}{dt}$$

We must understand the meaning of each of the acceleration defined by the differentials in the above equation :

- The term " $\frac{d\mathbf{v}}{dt}$ " represents total acceleration ($\mathbf{a}$) i.e. the resultant of radial ($\mathbf{a}_R$) and tangential acceleration ($\mathbf{a}_T$).
- The term $\frac{d\omega}{dt}$ represents angular acceleration (α)
- The term $\frac{d\mathbf{r}}{dt}$ represents velocity of the particle ($\mathbf{v}$)

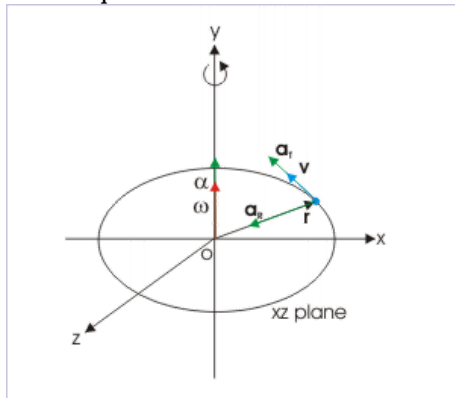
$$\Rightarrow \mathbf{a} = \alpha \mathbf{X}\mathbf{r} + \omega \mathbf{X}\mathbf{v}$$

$$\Rightarrow \mathbf{a} = \mathbf{a}_T + \mathbf{a}_R$$

where, $\mathbf{a}_T = \alpha \mathbf{X}\mathbf{r}$ is tangential acceleration and is measure of the time rate change of the magnitude of the velocity of the particle in the tangential direction and $\mathbf{a}_R = \omega \mathbf{X}\mathbf{v}$ is the radial acceleration also known as centripetal acceleration, which is measure of time rate change of the velocity of the particle in radial direction.

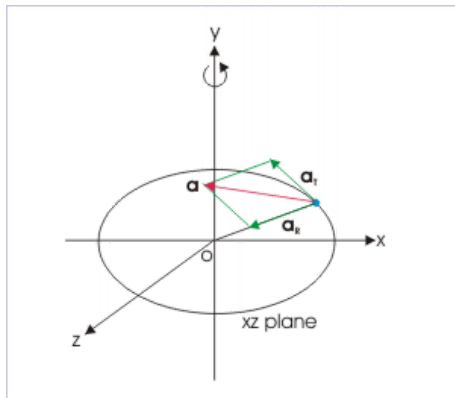
Various vector quantities involved in the equation are shown graphically with respect to the plane of motion (xz plane) :

Vector quantities



The magnitude of total acceleration in general circular motion is given by :

Resultant acceleration



$$a = |\mathbf{a}| = \sqrt{(a_T^2 + a_R^2)}$$

Example:

Problem : At a particular instant, a particle is moving with a speed of 10 m/s on a circular path of radius 100 m. Its speed is increasing at the rate of 1 m/s^2 . What is the acceleration of the particle ?

Solution : The acceleration of the particle is the vector sum of mutually perpendicular radial and tangential accelerations. The magnitude of tangential acceleration is given here to be 1 m/s^2 . Now, the radial acceleration at the particular instant is :

$$a_R = \frac{v^2}{r} = \frac{10^2}{100} = 1 \text{ m/s}^2$$

Hence, the magnitude of the acceleration of the particle is :

$$a = |\mathbf{a}| = \sqrt{(a_T^2 + a_R^2)} = \sqrt{(1^2 + 1^2)} = \sqrt{2} \text{ m/s}^2$$

Exercises

Exercise:

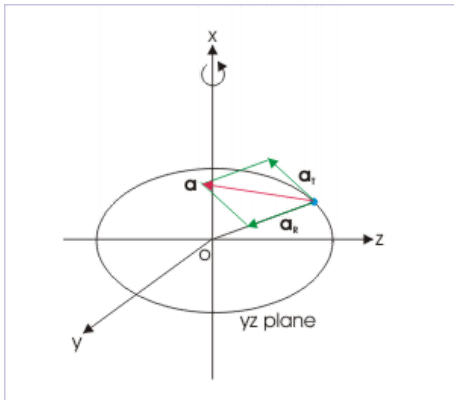
Problem: A particle is moving along a circle in yz - plane with varying linear speed. Then

- (a) acceleration of the particle is in x – direction
- (b) acceleration of the particle lies in xy – plane
- (c) acceleration of the particle lies in xz – plane
- (d) acceleration of the particle lies in yz – plane

Solution:

The figure here shows the acceleration of the particle as the resultant of radial and tangential accelerations. The resultant acceleration lies in the plane of motion i.e yz – plane.

Circular motion



Hence, option (d) is correct.

Exercise:

Problem:

A particle is moving along a circle of radius “r”. The linear and angular velocities at an instant during the motion are “v” and “ω” respectively. Then, the product $v\omega$ represents :

- (a) centripetal acceleration
- (b) tangential acceleration
- (c) angular acceleration divided by radius
- (d) None of the above

Solution:

The given product expands as :

$$v\omega = v \times \frac{v}{r} = \frac{v^2}{r}$$

This is the expression of centripetal acceleration.

Hence, option (a) is correct.

Exercise:

Problem:

Which of the following expression represents the magnitude of centripetal acceleration :

- (a) $\left| \frac{d^2\mathbf{r}}{dt^2} \right|$ (b) $\left| \frac{d\mathbf{v}}{dt} \right|$ (c) $r \frac{d\theta}{dt}$ (d) None of these

Solution:

The expression $\left| \frac{d\mathbf{v}}{dt} \right|$ represents the magnitude of total or resultant acceleration. The differential $\frac{d\theta}{dt}$ represents the magnitude of angular velocity. The expression $r \frac{d\theta}{dt}$ represents the magnitude of tangential velocity. The expression $\frac{d^2\mathbf{r}}{dt^2}$ is second order differentiation of position vector ($\mathbf{r}$). This is actually the expression of acceleration of a particle under motion. Hence, the expression $\left| \frac{d^2\mathbf{r}}{dt^2} \right|$ represents the magnitude of total or resultant acceleration.

Hence, option (d) is correct.

Exercise:

Problem:

A particle is circling about origin in xy-plane with an angular speed of 0.2 rad/s. What is the linear speed (in m/s) of the particle at a point specified by the coordinate (3m,4m) ?

(a) 1 (b) 2 (c) 3 (d) 4

Solution:

The linear speed “v” is given by :

$$v = \omega r$$

Now radius of the circle is obtained from the position data. Here, x = 3 m and y = 4 m. Hence,

$$r = \sqrt{(3^2 + 4^2)} = 5 \text{ m}$$

$$\Rightarrow v = 0.2 \times 5 = 1 \text{ m/s}$$

Hence, option (a) is correct.

Exercise:

Problem:

A particle is executing circular motion. The velocity of the particle changes from zero to $(0.3\mathbf{i} + 0.4\mathbf{j})$ m/s in a period of 1 second. The magnitude of average tangential acceleration is :

(a) 0.1 m/s^2 (b) 0.2 m/s^2 (c) 0.3 m/s^2 (d) 0.5 m/s^2

Solution:

The magnitude of average tangential acceleration is the ratio of the change in speed and time :

$$a_T = \frac{\Delta v}{\Delta t}$$

Now,

$$\Delta v = \sqrt{(0.3^2 + 0.4^2)} = \sqrt{0.25} = 0.5 \text{ m/s}$$

$$a_T = 0.5 \text{ m/s}^2$$

Hence, option (d) is correct.

Exercise:

Problem:

The radial and tangential accelerations of a particle in motions are a_T and a_R respectively. The motion can be circular if :

- (a) $a_R \neq 0, a_T \neq 0$ (b) $a_R = 0, a_T = 0$ (c) $a_R = 0, a_T \neq 0$ (d) $a_R \neq 0, a_T = 0$

Solution:

Centripetal acceleration is a requirement for circular motion and as such it should be non-zero. On the other hand, tangential acceleration is zero for uniform circular acceleration and non-zero for non-uniform circular motion. Clearly, the motion can be circular motion if centripetal acceleration is non-zero.

Hence, options (a) and (d) are correct.

Exercise:

Problem:

Which of the following pair of vector quantities is/are parallel to each other in direction ?

- (a) angular velocity and linear velocity
(b) angular acceleration and tangential acceleration
(c) centripetal acceleration and tangential acceleration
(d) angular velocity and angular acceleration

Solution:

The option (d) is correct.

Exercise:

Problem:

A particle is moving along a circle in a plane with axis of rotation passing through the origin of circle. Which of the following pairs of vector quantities are perpendicular to each other :

- (a) tangential acceleration and angular velocity
(b) centripetal acceleration and angular velocity
(c) position vector and angular velocity
(d) angular velocity and linear velocity

Solution:

Clearly, vector attributes in each given pairs are perpendicular to each other.

Hence, options (a), (b), (c) and (d) are correct.

Exercise:

Problem:

A particle is executing circular motion along a circle of diameter 2 m, with a tangential speed given by $v = 2t$.

- (a) Tangential acceleration directly varies with time.
 - (b) Tangential acceleration inversely varies with time.
 - (c) Centripetal acceleration directly varies with time.
 - (d) Centripetal acceleration directly varies with square of time.
-

Solution:

Tangential acceleration is found out by differentiating the expression of speed :

$$a_T = \frac{dv}{dt} = \frac{d(2t)}{dt} = 2 \text{ m/s}$$

The tangential acceleration is a constant. Now, let us determine centripetal acceleration,

$$a_R = \frac{v^2}{r} = \frac{(2t)^2}{1} = 4t^2$$

The option (d) is correct.

Non – uniform circular motion(application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on problem solving

- 1: Calculation of acceleration as time rate of change of speed gives tangential acceleration.
- 2: Calculation of acceleration as time rate of change of velocity gives total acceleration.
- 3: Tangential acceleration is component of total acceleration along the direction of velocity. Centripetal acceleration is component of acceleration along the radial direction.
- 4: We exchange between linear and angular quantities by using radius of circle, "r", as multiplication factor. It is helpful to think that linear quantities are bigger than angular quantities. As such, we need to multiply angular quantity by "r" to get corresponding linear quantities and divide a linear quantity by "r" to get corresponding angular quantity.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to non-uniform circular motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Velocity
- Average total acceleration
- Total acceleration

Velocity

Example:

Problem : A particle, tied to a string, starts moving along a horizontal circle of diameter 2 m, with zero angular velocity and a tangential acceleration given by " $\frac{d}{dt}$ ". If the string breaks off at $t = 5$ s, then find the speed of the particle with which it flies off the circular path.

Solution : Here, tangential acceleration of the particle at time " t " is given as :

We note here that an expression of tangential acceleration (an higher attribute) is given and we are required to find lower order attribute i.e. linear speed. In order to find linear speed, we need to integrate the acceleration function :

$$\frac{d}{dt}$$

Integrating with appropriate limits, we have :

$$\frac{d}{dt} \quad \frac{d}{dt} \quad \frac{d}{dt}$$

Average total acceleration

Example:

Problem : A particle is executing circular motion. The velocity of the particle changes from $(0.1\mathbf{i} + 0.2\mathbf{j})$ m/s to $(0.5\mathbf{i} + 0.5\mathbf{j})$ m/s in a period of 1 second. Find the magnitude of average total acceleration.

Solution : The average total acceleration is :

The magnitude of acceleration is :

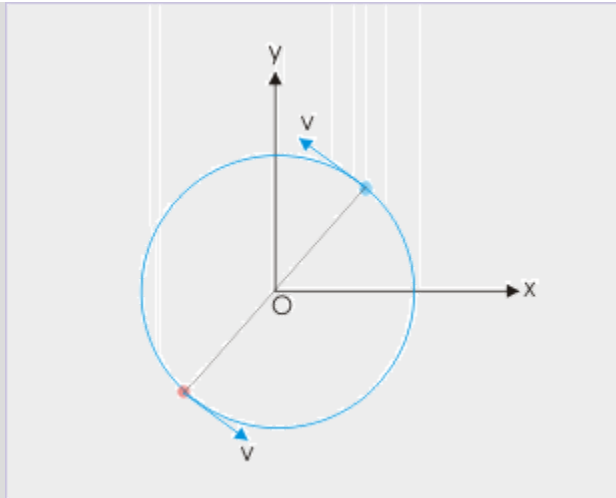
Example:

Problem : A particle starting with a speed “v” completes half circle in time “t” such that its speed at the end is again “v”. Find the magnitude of average total acceleration.

Solution : Average total acceleration is equal to the ratio of change in velocity and time interval.

From the figure and as given in the question, it is clear that the velocity of the particle has same magnitude but opposite directions.

Circular motion



The speeds of the particle are same at two positions.

Putting in the expression of average total acceleration, we have :

$$\frac{\vec{v}_2 - \vec{v}_1}{t} = \frac{v_2 \hat{i} - v_1 \hat{j}}{t}$$

The magnitude of the average acceleration is :

$$= \frac{v}{t}$$

Total acceleration

Example:

Problem : The angular position of a particle (in radian), on circular path of radius 0.5 m, is given by :

At $t = 1$ s, find (i) angular velocity (ii) linear speed (iii) angular acceleration (iv) magnitude of tangential acceleration (v) magnitude of centripetal acceleration and (vi) magnitude of total acceleration.

Solution : Angular velocity is :

$$\frac{d}{dt} \theta$$

The angular speed, therefore, is dependent as it is a function in "t". At $t = 1$ s,

The magnitude of linear velocity, at $t = 1$ s, is :

Angular acceleration is :

$$\frac{d}{dt} \omega$$

Clearly, angular acceleration is constant and is independent of time.

The magnitude of tangential acceleration is :

Tangential acceleration is also constant and is independent of time.

The magnitude of centripetal acceleration is :

At $t = 1$ s,

The magnitude of total acceleration, at $t = 1$ s, is :

Example:

Problem : The speed (m/s) of a particle, along a circle of radius 4 m, is a function in time, "t" as :

Find the total acceleration of the particle at time, $t = 2$ s.

Solution : The tangential acceleration of the particle is obtained by differentiating the speed function with respect to time,

$$\frac{dv}{dt} = \frac{dv}{dt}$$

The tangential acceleration at time, $t = 2$ s, therefore, is :

The radial acceleration of the particle is given as :

In order to evaluate this expression, we need to know the velocity at the given time, $t = 2$ s :

Putting in the expression of radial acceleration, we have :

$$\frac{v^2}{r} - \frac{v^2}{r}$$

The total acceleration of the particle is :

$$\frac{v^2}{r} - \frac{v^2}{r} = 0$$

Circular motion with constant acceleration

The circular motion with constant acceleration allows constructing an equivalent linear system with the help of angular quantities.

The description of non-uniform circular motion of a particle is lot simplified, if the time rate of the change in angular velocity (angular acceleration) is constant. We have seen such simplification, while dealing with motion along a straight line. It is observed that the constant acceleration presents an additive frame work for motion along straight line. The velocity of the particle changes by a fixed value after every second (i.e. unit time). In translational motion along a straight line, we say, for example, that :

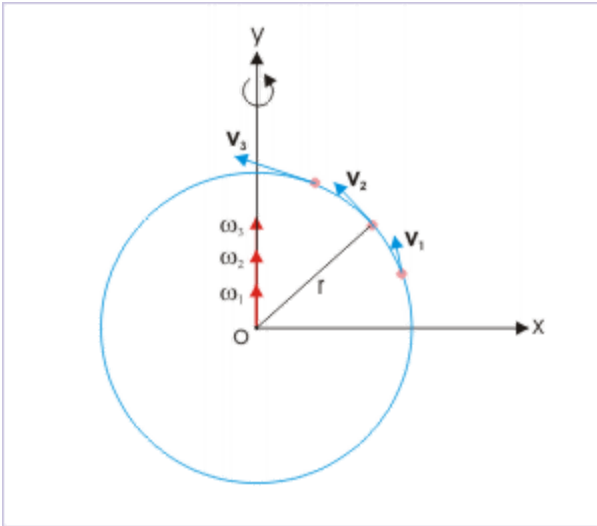
$$v = v_0 + at$$

where “v”, “ v_0 ” and “a” are attributes of motion along straight line with directional property. We can recall here that we are able to treat these vector quantities as signed scalars, because there are only two possible directions along a straight line. Now the moot question is : are these simplifications also possible with non-uniform circular motion, when angular acceleration is constant ?

The answer is yes. Let us see how does it work in the case of circular motion :

1: First, the multiplicity of directions associated with the velocity of the particle rotating about an axis is resolved to be equivalent to one directional angular velocity, " ω ", acting along the axis of rotation. See the figure. If we stick to translational description of motion, then velocity vector is tangential to path. The direction of velocity vector changes as the particle moves along the circular path. On the other hand, angular velocity vector aligns in a single direction.

Angular velocity



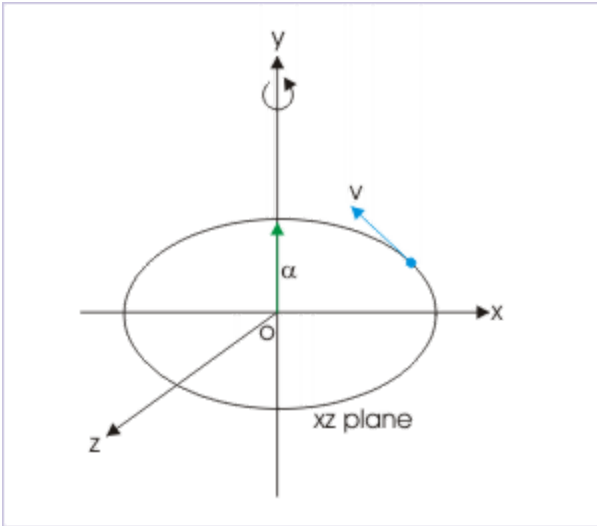
Angular velocity acts along axis of rotation.

2: Second, the acceleration resulting from the change in the speed of the particle is represented by angular acceleration, " α ". Like angular velocity, it is also one directional.

3: Third, there are only two possible directions as far as directional angular quantities are considered. The direction can be either clockwise (negative) or anti-clockwise (positive). This situation is similar to linear consideration where motion in the reference direction is considered positive and motion in opposite direction is considered negative.

4: Fourth, the change in either angular displacement or angular speed is basically translated to a change in the magnitude of one directional angular quantities. This allows us to treat the change in angular quantities as a consideration along straight line as in the case of motion along straight line. The direction of angular quantities

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There are two possible directions with angular quantities.

Clearly, it is possible to convert a motion along a circular path into an equivalent system along a straight line. As the angular quantities are represented along a straight line, it is then logical that relations available for linear motion for constant acceleration are also available for circular motion with appropriate substitution of linear quantities by angular quantities.

For example, the equation " $v = v_0 + at$ " has its corresponding relation in the circular motion as :

$$\omega = \omega_0 + \alpha t$$

As a matter of fact, there exists one to one correspondence between two types of equation sets. Importantly, we can treat angular vector quantities as signed scalars in the equations of motion, dispensing with the need to use vector notation. The similarity of situation suggests that we need not derive equations of motion again for the circular motion. We, therefore, proceed to simply write equation of angular motion with appropriate substitution.

Note: In this section of circular motion kinematics, our interest or domain of study is usually limited to the motion or acceleration in tangential direction. We may not refer to the requirement of motion in the radial direction in the form of centripetal acceleration, unless stated specifically.

What corresponds to what ?

The correspondence goes like this : the angular position is “ θ ” for linear “ x ”; the angular displacement is “ $\Delta\theta$ ” for linear “ Δx ”; the angular velocity is “ ω ” for linear “ v ” and the angular acceleration is “ α ” for linear “ a ”.

Angular quantities

The different angular quantities corresponding to their linear counterparts are listed here for ready reference :

Quantities	Linear variables	Angular variables
Initial position	x_0	θ_0
Final position	x	θ
Displacement	Δx	$\Delta\theta$
Initial velocity	v_0	ω_0
Final velocity	v	ω
Acceleration	a	α
Time interval	t	t

Basic equations

The corresponding equations for the two types of motion are :

S.N.	Linear equation	Angular equation
1:	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
2:	$v_{\text{avg}} = \frac{(v_0+v)}{2}$	$\theta\omega_{\text{avg}} = \frac{(\omega_0+\omega)}{2}$
3:	$\Delta x = x - x_0 = v_0t + \frac{1}{2}at^2$	$\Delta\theta = \theta - \theta_0 = \omega_0t + \frac{1}{2}\alpha t^2$

Derived equations

The derived equations for the two types of motion are :

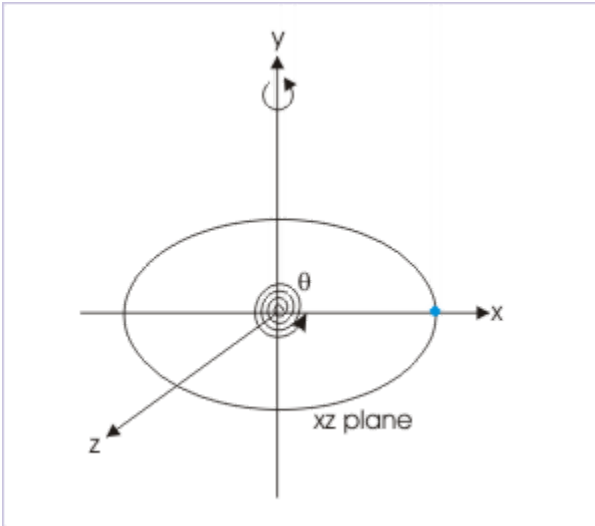
S.N.	Linear equation	Angular equation
1:	$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
2:	$\Delta x = (x - x_0) = \frac{(v_0+v)t}{2}$	$\Delta\theta = (\theta - \theta_0) = \frac{(\omega_0+\omega)t}{2}$
3:	$\Delta x = x - x_0 = vt - \frac{1}{2}at^2$	$\Delta\theta = \theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

Sign of angular quantities

The sign of angular quantities represents direction. A positive sign indicates anti-clockwise direction, whereas a negative sign indicates clockwise direction.

In the measurement of angle, a typical problem arises from the fact that circular motion may continue to rotate passing through the reference point again and again. The question arises, whether we keep adding angle or reset the measurement from the reference point ? The answer is that angle measurement is not reset in rotational kinematics. This means that we can have measurements like 540° and 20 rad etc.

Measurement of angle



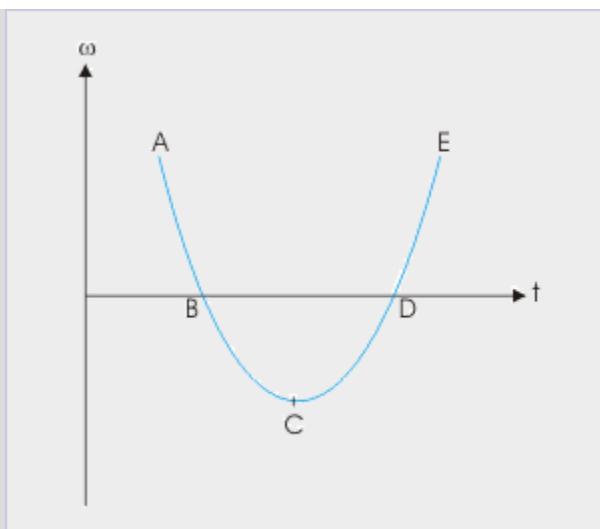
The measurement of angle is not reset.

This convention is not without reason. Equations of motion of circular motion with constant acceleration treats motion in an equivalent linear frame work, which considers only one reference position. If we reset the measurements, then equations of motion would not be valid.

Example:

Problem : The angular velocity – time plot of the circular motion is shown in the figure. (i) Determine the nature of angular velocity and acceleration at positions marked A, B, C, D and E. (ii) In which of the segments (AB, BC, CD and DE) of motion, the particle is decelerated and (iii) Is angular acceleration constant during the motion ?

Angular velocity – time plot



Solution :

(i) Angular velocity :

The angular velocities at A and E are positive (anti-clockwise). The angular velocities at B and D are each zero. The angular velocities at C is negative (clockwise).

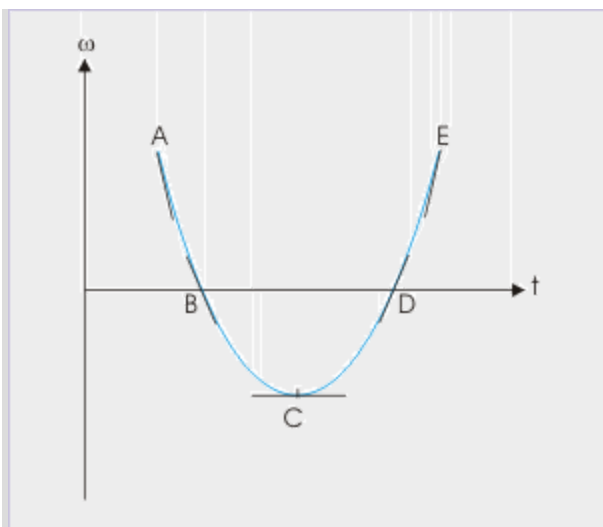
Angular acceleration :

The angular acceleration is equal to the first differential of angular velocity with respect to time.

$$\alpha = \frac{d\omega}{dt}$$

The sign of the angular acceleration is determined by the sign of the slope at different positions. The slopes at various points are as shown in the figure :

Slopes at different points



The angular accelerations at point A and B are negative (angular speed decreases with the passage of time). The angular accelerations at C is zero. The angular accelerations at point D and E are positive (angular speed increases with the passage of time).

(ii) Deceleration :

In the segments AB and CD, the magnitude of angular velocity i.e. angular speed decreases with the passage of time. Thus, circular motions in these two segments are decelerated. This is also confirmed by the fact that angular velocity and angular acceleration are in opposite directions in these segments.

(iii) The slopes on angular velocity - time plot are different at different points. Thus, angular accelerations are different at these points. Hence, angular acceleration of the motion is not constant.

Angular velocity

The angular velocity increases by a constant value at the end of every unit time interval, when angular velocity and acceleration act in the same direction. On the other hand, angular velocity decreases, when angular velocity and acceleration act in the opposite direction.

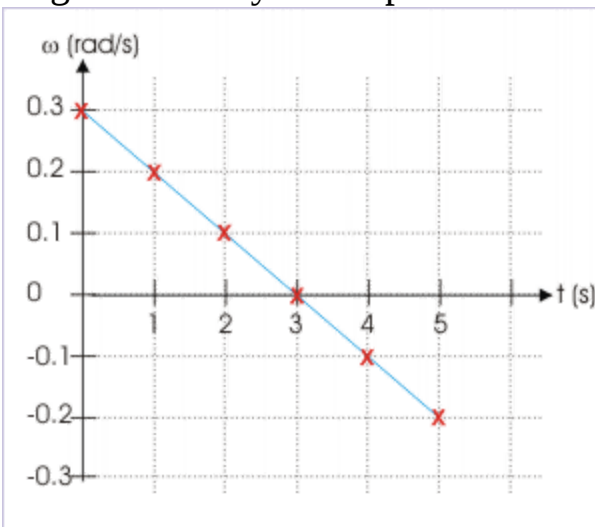
To appreciate this, we consider a circular motion of a particle whose initial angular velocity is 0.3 rad/s . The motion is subjected to an angular acceleration of magnitude $0.1 \frac{\text{rad}}{\text{s}^2}$ in the opposite direction to the initial

velocity. In the table here, we calculate angular velocity of the particle, using relation, $\omega = \omega_0 + \alpha t$, at the end of every second and plot the data (for first 5 seconds) to understand the variation of angular velocity with time.

Time (s)	Angular acceleration (rad/s.s)	Angular velocity (rad/s)
0	-0.1	0.3
1	-0.1	0.2
2	-0.1	0.1
3	-0.1	0.0
4	-0.1	-0.1
5	-0.1	-0.2

Here, the particle stops at the end of 3 seconds. The particle then reverses its direction (clockwise from anti-clockwise) and continues to move around the axis. The angular velocity – time plot is as shown here :

Angular velocity - time plot



We observe following aspects of the illustrated motion with constant acceleration :

1. The angular velocity decreases at uniform rate and the angular velocity – time plot is straight line. Positive angular velocity becomes less positive and negative angular velocity becomes more negative.
2. The slope of the plot is constant and negative.

Example:

Problem : A particle at the periphery of a disk at a radial distance 10 m from the axis of rotation, uniformly accelerates for a period of 5 seconds. The speed of the particle in the meantime increases from 5 m/s to 10 m/s. Find angular acceleration.

Solution : The initial and final angular velocities are :

$$\omega_0 = \frac{5}{10} = 0.5 \text{ rad / s}$$

$$\omega = \frac{10}{10} = 1 \text{ rad / s}$$

$$t = 5 \text{ s}$$

Using equation of motion, $\omega = \omega_0 + \alpha t$, we have :

$$\Rightarrow 1 = 0.5 + \alpha \times 5$$

$$\Rightarrow \alpha = \frac{0.5}{5} = 0.1 \text{ rad / s}^2$$

Angular displacement

The angular displacement is given as :

$$\Rightarrow \Delta\theta = \theta - \theta_0 = \omega_0 t + \frac{1}{2} \alpha t^2$$

If we start observation of motion as $t = 0$ and $\theta^\circ = 0$, then displacement (θ) is :

$$\Rightarrow \theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

The particle covers greater angular displacement for every successive time interval, when angular velocity and acceleration act in the same direction and the particle covers smaller angular displacement, when angular velocity and acceleration act in the opposite direction.

To appreciate this, we reconsider the earlier case of a circular motion of a particle whose initial angular velocity is 0.3 rad/s. The motion is subjected

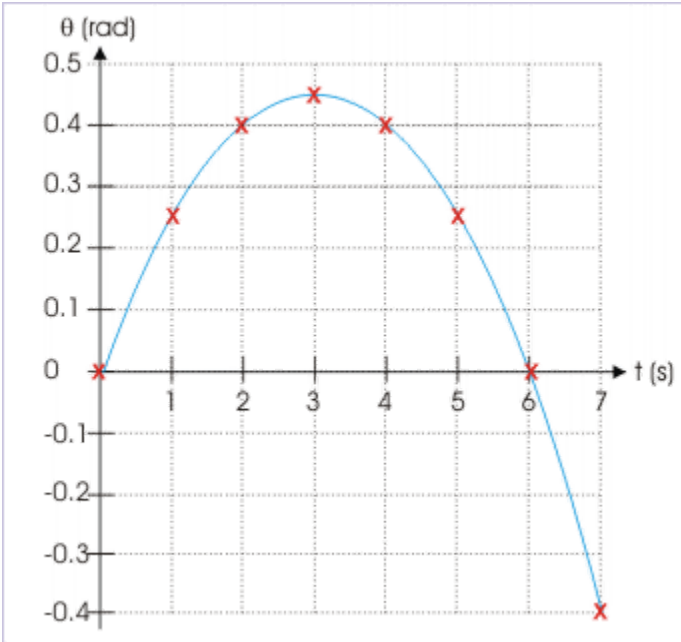
to an angular acceleration of magnitude 0.1 rad/s^2 in the opposite direction to the initial velocity. In the table here, we calculate angular displacement of the particle, using relation, $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$, at the end of every second and plot the data (for first 7 seconds) to understand the variation of angular displacement with time.

----- Time wo t									
Angular displacement (θ) (s) (rad) (rad) -----									

----- 0 0.0 0.00 1 0.3 x 1 0.3									
- 0.5 x 0.1 x 1 = 0.3 - 0.05 = 0.25 2 0.3 x 2 0.6									
- 0.5 x 0.1 x 4 = 0.6 - 0.20 = 0.40 3 0.3 x 3 0.9									
- 0.5 x 0.1 x 9 = 0.9 - 0.45 = 0.45 4 0.3 x 4 1.2									
- 0.5 x 0.1 x 16 = 1.2 - 0.80 = 0.40 5 0.3 x 5 1.5									
- 0.5 x 0.1 x 25 = 1.5 - 1.25 = 0.25 6 0.3 x 6 1.8									
- 0.5 x 0.1 x 36 = 1.8 - 1.80 = 0.00 7 0.3 x 7 2.1									
- 0.5 x 0.1 x 49 = 2.1 - 2.50 = -0.40 -----									

We see here that the particle moves in anti-clockwise direction for first 3 seconds as determined earlier and then turns back (clockwise) retracing the path till it reaches the initial position. Subsequently, the particle continues moving in the clockwise direction.

The angular displacement – time plot is as shown here :
Angular displacement – time plot



Example:

Problem : A particle at the periphery of a disk at a radial distance 10 m from the axis of rotation, uniformly accelerates for a period of 20 seconds. The speed of the particle in the meantime increases from 5 m/s to 20 m/s. Find the numbers of revolutions the particle completes around the axis.

Solution : We need to find angular displacement to know the numbers of revolutions made. Here,

$$\omega_0 = \frac{5}{10} = 0.5 \text{ rad/s}$$

$$\omega = \frac{20}{10} = 2 \text{ rad/s}$$

$$t = 20 \text{ s}$$

To calculate angular displacement, we need to know angular acceleration. Here, we can calculate angular acceleration as in the earlier exercise as :

$$\alpha = \frac{\omega - \omega_0}{\Delta t} = \frac{2.0 - 0.5}{20} = \frac{1.5}{20} = 0.075 \text{ rad/s}^2$$

Now, using equation of motion, $\Delta\theta = \omega_0 t + \frac{1}{2} \alpha t^2$, we have :

$$\Rightarrow \Delta\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\Delta\theta = 0.5 \times 20 + \frac{1}{2} \times 0.075 \times 20^2 = 10 + 15 = 25 \text{ rad}$$

The number of revolutions (nearest integer), n ,

$$\Rightarrow n = \frac{\Delta\theta}{2\pi} = \frac{25}{6.28} = 3.97 = 3 \text{ (integer)}$$

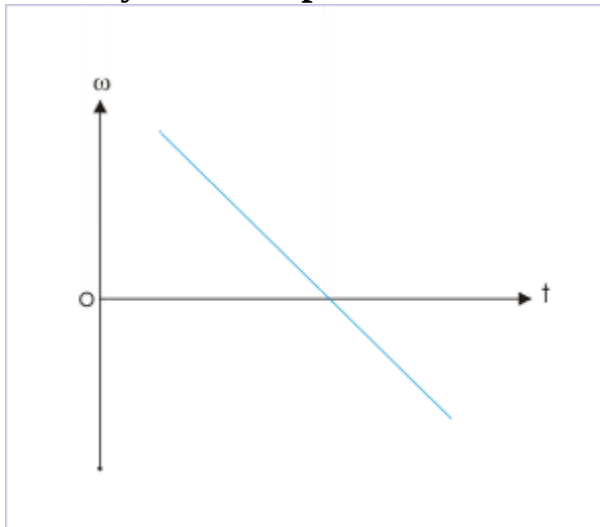
Exercises

Exercise:

Problem:

The angular velocity .vs. time plot of the motion of a rotating disk is shown in the figure. Then,

Velocity .vs. time plot



Velocity .vs. time plot

angular acceleration is constant.

angular acceleration is negative.

disk comes to a stop at a particular instant.

disk reverses direction during the motion.

Solution:

The slope of the straight line is a negative constant. Therefore, options (a) and (b) are correct. The line crosses time axis, when angular velocity is zero. Thus option (c) is correct. The angular velocities have opposite sign across time axis. It means that the disk reverses its direction. Thus option (d) is correct.

Hence, options (a),(b),(c) and (d) are correct.

Exercise:

Problem:

A point on a rotating disk, starting from rest, achieves an angular velocity of 40 rad/s at constant rate in 5 seconds. If the point is at a distance 0.1 meters from the center of the disk, then the distance covered (in meters) during the motion is :

(a) 5 (b) 10 (c) 20 (d) 20

Solution:

We can find the distance covered, if we know the angular displacement. On the other hand, we can find angular displacement if we know the average angular speed as time is given.

$$\omega_{\text{avg}} = \frac{\omega_i + \omega_f}{2} = \frac{40 + 0}{2} = 20 \text{ rad/s}$$

We can consider this accelerated motion as uniform motion with average speed as calculated above. The angular displacement is :

$$\Rightarrow \theta = \omega_{\text{avg}} \times t = 20 \times 0.5 = 100 \text{ rad}$$

The distance covered is :

$$\Rightarrow s = \theta \times r = 100 \times 0.1 = 10 \text{ m}$$

Exercise:

Problem:

A point on a rotating disk completes two revolutions starting from rest and achieves an angular velocity of 8 rad/s. If the angular velocity of the disk is increasing at a constant rate, the angular acceleration (rad/s^2) is :

$$(a) \ 8 \quad (b) \ 8\pi \quad (c) \ \frac{8}{\pi} \quad (d) \ \frac{16}{\pi}$$

Solution:

We can use the equation of motion for constant acceleration. Here,

$$\theta = 2 \text{ revolution} = 2 \times 2\pi = 4\pi$$

$$\omega_i = 0$$

$$\omega_f = 8 \text{ rad/s}$$

Looking at the data, it is easy to find that the following equation will serve the purpose,

$$\omega_f^2 = \omega_i^2 = 2\alpha\theta$$

Solving for " α " and putting values,

$$\Rightarrow \alpha = \frac{\omega_f^2 - \omega_i^2}{2\theta} = \frac{8^2 - 0^2}{2 \times 4\pi} = \frac{8}{\pi} \text{ rad/s}^2$$

Hence, option (c) is correct.

Exercise:**Problem:**

A point on a rotating disk is accelerating at a constant rate 1 rad/s^2 till it achieves an angular velocity of 10 rad/s . What is the angular displacement (radian) in last 2 seconds of the motion?

- (a) 18 (b) 24 (c) 28 (d) 32
-

Solution:

Here, final angular velocity, angular acceleration and time of motion are given. We can find the angular displacement using equation of motion for angular displacement that involves final angular velocity :

$$\theta = \omega_f t - \frac{1}{2} \alpha \frac{t^2}{2}$$

Putting values, we have :

$$\Rightarrow \theta = 10 \times 2 - \frac{1}{2} \times 1 \times \frac{2^2}{2} = 20 - 2 = 18 \text{ rad}$$

Hence, option (a) is correct.

Circular motion with constant acceleration (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Check understanding

An online examination, based on the basic subject matter, is available at the link given here. The examination session is designed only for the basic level so that we can ensure that our understanding of subject matter is satisfactory.

[Circular motion with constant acceleration](#)

Hints for problem solving

1: Visualize the circular motion as if we are dealing with straight line (pure translational) motion. Write down formula with substitution of linear quantities with angular quantities.

2: Use formula in scalar form. Stick to anticlockwise measurement as positive and clockwise measurement as negative.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to circular motion with constant acceleration. The questions are categorized in terms of the characterizing features of the subject matter :

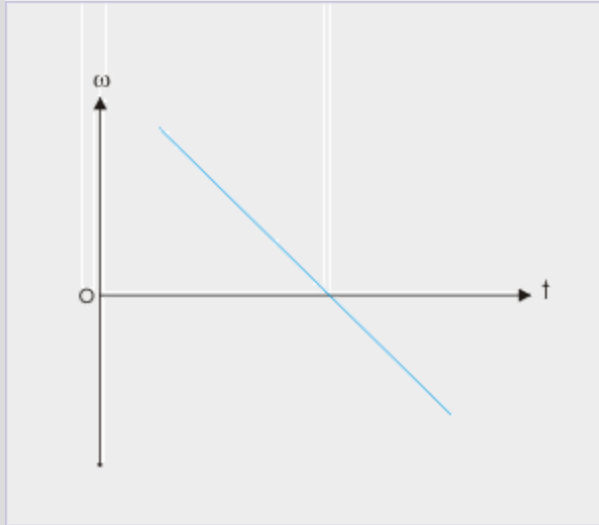
- Nature of angular motion
- Time interval
- Angular displacement

Nature of angular motion

Example:

Problem : The angular velocity .vs. time plot of the motion of a rotating disk is shown in the figure.

angular displacement .vs. time plot



Determine (i) nature of angular velocity (ii) nature of angular acceleration and (iii) whether the disk comes to a standstill during the motion?

Solution : The angular velocity is anticlockwise (positive) above time axis and clockwise (negative) below time axis.

The slope of angular velocity - time plot indicates nature of acceleration. Since the plot is a straight line, motion is accelerated/ decelerated at constant rate. Further, the slope of the straight line (angular velocity - time plot) is negative all through out.

Recall that it is easy to determine the sign of the straight line. Just move from left to right in the direction of increasing time along the time - axis. See whether the angular velocity increases or decreases. If increases, then slope is positive; otherwise negative. The angular velocity, here, becomes less positive above time axis and becomes more negative below time axis. Hence, slope is negative all through out.

It means that acceleration (negative) is opposite to angular velocity (positive) above time axis. Therefore, the disk is decelerated and the angular speed of disk decreases at constant rate. Below time - axis, the angular acceleration is still negative. However, angular velocity is also

negative below the time axis. As such, disk is accelerated and the angular speed increases at constant rate.

We see here that the plot intersects time - axis. It means that the disk comes to a stand still before changing direction from anticlockwise rotation to clockwise rotation.

Time interval

Example:

Problem : The angular position of a point on a flywheel is given by the relation :

$$\theta(\text{rad}) = -0.025t^2 + 0.01t$$

Find the time (in seconds) when flywheel comes to a stop.

Solution : The speed of the particle is :

$$\omega = \frac{d\theta}{dt} = -0.025 \times 2t + 0.1$$

When flywheel comes to a standstill, $\omega = 0$,

$$\Rightarrow 0 = -0.025 \times 2t + 0.1$$

$$\Rightarrow t = \frac{0.1}{0.025} = \frac{100}{25} = 4 \text{ s}$$

Example:

Problem : The magnitude of deceleration of the motion of a point on a rotating disk is equal to the acceleration due to gravity (10 m/s^2). The point is at a linear distance 10 m from the center of the disk. If initial speed is 40 m/s in anti-clockwise direction, then find the time for the point to return to its position.

Solution : The disk first rotates in anti-clockwise direction till its speed becomes zero and then the disk turns back to move in clockwise direction. In order to analyze the motion, we first convert linear quantities to angular quantities as :

$$\omega_i = \frac{v_i}{r} = \frac{40}{10} = 4 \quad \text{rad / s}$$

$$\alpha = \frac{a_T}{r} = \frac{10}{10} = 1 \quad \text{rad / s}^2$$

When the point returns to its initial position, the total displacement is zero. Applying equation of motion for angular displacement, we have :

$$\theta = \omega_i t + \frac{1}{2} \alpha t^2$$

$$\Rightarrow 0 = 4 \times t - \frac{1}{2} \times 1 \times t^2$$

$$\Rightarrow t^2 - 8t = 0$$

$$\Rightarrow t(t - 8) = 0$$

$$\Rightarrow t = 0 \quad \text{or} \quad 8s$$

The zero time corresponds to initial position. The time of return to initial position, therefore, is 8 seconds.

Angular displacement

Example:

Problem : A disk initially rotating at 80 rad/s is slowed down with a constant deceleration of magnitude 4 rad / s² . What angle (rad) does the disk rotate before coming to rest ?

Solution : Initial and final angular velocities and angular acceleration are given. We can use $\omega = \omega_0 + \alpha t$ to determine the time disk takes to come to stop. Here,

$$\omega_0 = 80 \text{ rad / s}, \alpha = 4 \text{ rad / s}^2$$

$$\Rightarrow 0 = 80 - 4 t$$

$$\Rightarrow t = 20 \text{ s}$$

Using equation, $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$, we have :

$$\Rightarrow \theta = 80 \times 20 - \frac{1}{2} \times 4 \times 20^2 = 1600 - 800 = 800 \text{ rad}$$

Example:

Problem : The initial angular velocity of a point (in radian) on a rotating disk is 0.5 rad/s. The disk is subjected to a constant acceleration of 0.2 rad / s² . in the direction opposite to the angular velocity. Determine the angle (in radian) through which the point moves in third second.

Solution : Here, we need to be careful as the point reverses its direction in the third second ! For $\omega = 0$,

$$\omega = \omega_0 + \alpha t$$

$$\Rightarrow 0 = 0.5 - 0.2t$$

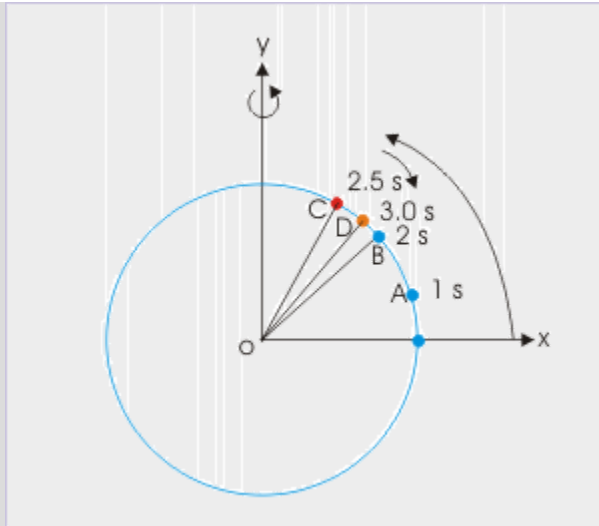
$$\Rightarrow t = \frac{0.5}{0.2} = 2.5 \text{ s}$$

Thus the disk stops at $t = 2.5$ second. In this question, we are to find the angle (in rad) through which the point moves in third second – not the displacement. The figure here qualitatively depicts the situation. In the third second, the point moves from B to C and then from C to D. The displacement in third second is BOD, whereas the angle moved in the third second is $|\angle BOC| + |\angle DOC|$. Where,

$\angle BOC$ = displacement between 2 and 2.5 seconds.

$\angle DOC$ = displacement between 2.5 and 3 seconds.

Angle measurement



The angular velocity at the end of 2 seconds is :

$$\omega = 0.5 - 0.2 \times 2 = 0.1 \text{ rad /s}$$

$$\angle BOC = \omega t - \frac{1}{2} \alpha t^2$$

$$\Rightarrow \angle BOC = 0.1 \times 0.5 - \frac{1}{2} \times 0.2 \times 0.5^2$$

$$\Rightarrow \angle BOC = 0.05 - 0.1 \times 0.25 = 0.05 - 0.025 = 0.025 \text{ rad}$$

The angular velocity at the end of 2.5 seconds is zero. Hence,

$$\Rightarrow \angle DOC = 0 - \frac{1}{2} \times 0.2 \times 0.5^2$$

$$\Rightarrow \angle BOC = -0.1 \times 0.25 = -0.025 \text{ rad}$$

It means that the point, at $t = 3$ s, actually returns to the position where it was at $t = 2$ s. The displacement is, thus, zero.

The total angle moved in third second = $|0.025| + |-0.025| = 0.05 \text{ rad}$

Example:

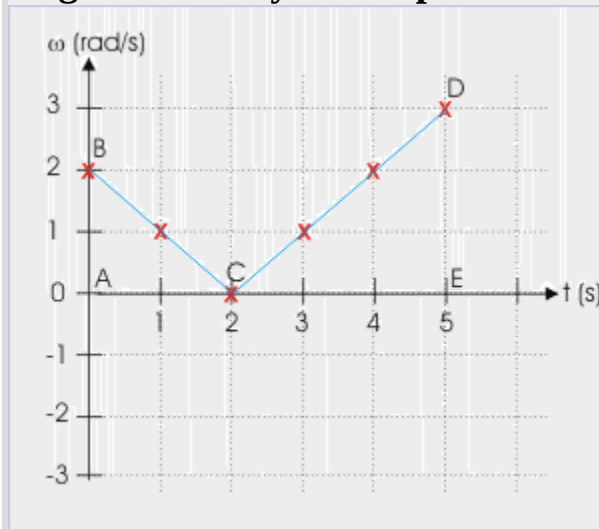
Problem : The angular velocity of a point (in radian) on a rotating disk is given by $|t - 2|$, where “t” is in seconds. If the point aligns with the reference direction at time $t = 0$, then find the quadrant in which the point falls after 5 seconds.

Problem : The area under the angular velocity – time plot and time axis is equal to angular displacement. As required, let us generate angular velocity data for first 5 seconds to enable us draw the requisite plot :

Time (t)	Angular velocity (ω) (rad/s)
0	2
1	1
2	0
3	1
4	2
5	3

The angular velocity – time plot is as shown in the figure :

Angular velocity – time plot



The displacement is equal to the area of two triangles :

$$\Rightarrow \theta = \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 3 \times 3 = 6.5 \text{ rad}$$

Thus, the point moves 6.5 rad from the reference direction. Now, one revolution is equal to $2\pi = 2 \times 3.14 = 6.28$ rad. The particle is, therefore, in the fourth quadrant with respect to the reference direction.